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Ratcheting and constitutive patterns of rate-form defined in Preferred Reference Frames

Dedicated to the legacy of W. Ślebodziński
as a token of scientific gratitude

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A PATTERN of Rate Form allowing the modelling of the thermomechanical features of the cyclic plasticity has proved to be effective in order to take into account various isotropic behaviours, ranging from those of granular media exhibiting isotropic-deviatoric coupling effects, to those distinguished by shape memory effects or ferrohysteresis ([1] to [8]). Under its basic form, of pure hysteresis type, this discrete memory pattern is able, by definition of a formalism brought into play in the fixed frame and in the A.A. Iliushin space of stress, to describe the behaviour involving perfectly closed cycles. Owing to this fact, it is also unable, in its basic form, to take into account second order effects such as those of multiaxial effects of ratchet type. In order to introduce a well founded and straightforward approach to the continuum thermomechanics of the isotropic solid-like inelastic behaviours, *including* rate-independent and hardening-independent *second order* tensorial coupling effects of *ratchet* type, a new approach is introduced. It is founded on the following assumptions: firstly, the formalism of the initial pattern is not modified, maintaining its role in the A.A. Iliushin spaces of stress and strain (of power and geometry); secondly, the unaltered initial formalism is no longer brought into play in the fixed frame and works now in the Preferred Reference Frame (PRF); thirdly, the definition of this PRF is basically thermomechanical and may be dependent on a mesoscale of Nél-Krumhansl type. The formalism of the general multiaxial isotropic case is introduced and the approach is shown to be straightforward even when the pattern is (easily) generalized to the viscoelastoplastic case. It is shown that the usual multiaxial effect of ratchet type obtained through cyclic torsion is not necessarily hardening-dependent and is existing under zero axial load, in accordance with experimental evidence.

Notations

$(\mathbf{0}, \mathbf{e}_i)$	orthonormal fixed reference frame ($i = 1, 2, 3$),
t	absolute time,
• or $\partial/\partial t$	partial time derivative, the only time derivative which is implemented,
$M(x^k)$	material point of invariant convected co-ordinates x^k, (following the old-fashioned classical terminology of J.A. Schouten type, the x^k are the dragged along co-ordinates),
$(M, \mathbf{H}_i)$	initial ($t = 0$) field of orthonormal Preferred Reference Frame (PRF),
$(M, \mathbf{h}_i)$	current ($t > 0$) field of PRF,
$(M, \mathbf{G}_i)$	initial ($t = 0$) field of Convected Reference Frame (DARF),
$(M, \mathbf{g}_i)$	current ($t > 0$) field of DARF,
$\mathbf{OP}(M, 0)$	initial position vector of $M(x^k)$ in $(O, \mathbf{e}_i)$,
$\mathbf{Op}(M, t)$	current position vector of $M(x^k)$ in $(O, \mathbf{e}_i)$,
$Z_i(M)$	components of $\mathbf{OP}$ in $(O, \mathbf{e}_i)$,
$z_i(M, t)$	components of $\mathbf{Op}$ in $(O, \mathbf{e}_i)$,
$x_i(M)$	invariant co-ordinates of M in the convected co-ordinate system defined at $t = 0$,
$\mathbf{G}, g_{ii}, g$	current metric tensor and associated covariant components and determinant,
$\rho(M, t)$	current specific mass ($\rho\sqrt{g}$ = constant masse at M),
$\mathbf{V}(M, t)$	current velocity vector defined in $(O, \mathbf{e}_i)$ ($\mathbf{e}_i V_i = \mathbf{e}_i \partial z_i/\partial t = \partial\,\mathbf{Op}/\partial t$),
$\nu^i, \nu_i(M, t)$	current components of $\mathbf{V}$ expressed in the DARF at M,
Lv	generic symbol of the four Lie derivatives: L .. , L ·· , L . · , L · . ,
$\nu, \mathbf{N}$	relative and absolute normal vector ($Lv\nu = 0$; N $\mathbf{N} \cdot \mathbf{N} = 1$),
$\mathbf{D}(M, t)$	current strain rate tensor of $M(2D_{ij} = \partial g_{ij}/\partial t = Lv..g_{ij} = \nabla_j \nu_i + \nabla_i \nu_j)$,
$T(M, t)$	current absolute temperature of the material point $M(x^k)$,
$\mathbf{D}_T(M, t)$	strain rate associated with the usual (isotropic) thermal dilatation $\mathbf{D}_T = \alpha(T)\dot{T}\mathbf{G}$,
$K_n(M, t)$	current stretches along $\mathbf{e}_n$ for the material point $M(x^k)$,
$J_n(t)$	$1 + K_n(M, t)$,
$\gamma_1, \gamma_2, \gamma_3$	current shears in the directions of $\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2$ for the material point $M(x^k)$,
$\mathbf{d}_n(M, t)$	current unit vectors of the principal directions (PPD) of $\mathbf{D}$, resulting in an orthonormal frame $(M, \mathbf{d}_n(M, t))$,
$\mathbf{I}_\phi(M, t)$	current inertia-like tensor of the material point $M(x^k)$,
$\mathbf{i}_n(M, t)$	current unit vectors of the PPD of $\mathbf{I}_{\phi,}$, resulting in the frame $(M, \mathbf{i}_n(M, t))$,
$\psi_\delta, \theta_\delta, \phi_\delta$	current Euler's angles (precession, nutation and proper rotation) of $(M, \mathbf{d}_n(M, t))$,
$\psi_\phi, \theta_\phi, \phi_\phi$	current Euler's angles (precession, nutation and proper rotation) of $(M, \mathbf{i}_n(M, t))$,
$\Omega_\delta, \Omega_\phi$	current angular velocity pseudo-vectors in $(0, \mathbf{e}_i)$ of $(M, \mathbf{d}_n(M, t))$ and $(M, \mathbf{i}_n(M, t))$,
$\Omega(M, t)$	current angular velocity pseudo-vector in $(0, \mathbf{e}_i)$ of the PRF,
$\dot{\alpha}_\delta(M, t)$	scalar used instead of Ω_δ in the special simple kinematics with only one shear γ_3,

$\dot{\theta}_\phi(M,t)$	scalar used instead of Ω_ϕ in the special simple kinematics with only one shear γ_3,
$\dot{\alpha}(M,t)$	scalar used instead of Ω in the special simple kinematics with only one shear γ_3,
${}^t_R\mathbf{G}$	current Cauchy strain tensor convected from $t_R < t$ to the current state at t,
$\Delta^t_R\varepsilon$	current Almansi strain tensor defined as the $\mathbf{G}$ variation $\Delta^t_R\varepsilon = (\mathbf{G} - {}^t_R\,\mathbf{G})/2$ on $]t_R, t]$,
$\mathbf{S}, \sigma(M,t)$	current Cauchy stress tensor ($\sigma = \mathbf{S}\sqrt{g}$),
${}^t_R\mathbf{S}, {}^t_R\sigma$	current Cauchy stress tensor convected from $t_R < t$ to the current state at t,
$\Delta^t_R\mathbf{S}$	current "variation" stress tensor defined as the $\mathbf{S}$ variation $\mathbf{S} - {}^t_R\mathbf{S}$ on $[t_R,\, t]$,
$\overline{\mathbf{S}}, \mathrm{I_s}, \overline{\mathrm{II}}_\mathrm{s}, \overline{\mathrm{III}}_\mathrm{s}$	deviator of $\mathbf{S}$, trace of $\mathbf{S}$, trace of $(\mathrm{tr}\overline{\mathbf{S}}^2)/2$, trace of $(\mathrm{tr}\overline{\mathbf{S}}^3)/3$, respectively,
$\hat{P}_i, P_i, p_i$	current internal power received by M per unit extent of x^k, of volume, of mass, respectively ($\hat{P}_i = P_i\sqrt{g} = p_i\rho\sqrt{g}$; $P_i(M,t) = \rho(M,t)p_i(M,t)$),
$\hat{\Pi}_i, \Pi, \pi$	current reversible power received by M,
$\hat{\Phi}, \Phi, \phi$	current intrinsic dissipation received by $M (\Phi = -P_i - \Pi$; etc.),
$\hat{E}, E, e$	current internal energy received by M,
$\hat{Q}_i, Q_b, q_i$	current internal intrinsic heat supply received by M,
$\hat{Q}_e, Q_e, q_e$	current reversible heat supply received by $M (Q_{ek}$ for Kelvin effect),
$\hat{K}, K, k$	current kinetic energy received by M,
S_0	limit stress of the Huber-von Mises criterion $\overline{II}_s = S_0^2$ (associated cylinder of radius $Q_0 = \sqrt{2}S_0$),
λ, μ	Lamé parameters or similar parameters,
θ_ν	relaxation time parameter of the Oldroyd pattern,
η_1, η_2	viscosity parameters of the Oldroyd pattern,
$\stackrel{\mathrm{def}}{=}$	equality giving a definition,
$\stackrel{\mathrm{ident}}{=}$	identity,
δ_{ij}	Kronecker delta,
ε_{ijk}	relative alternator ($e_{ijk} = \sqrt{g}\varepsilon_{ijk}$),

1. Introduction

THE MAIN RESTRICTIVE assumptions of the study are as follows:

1. The behaviour of the material is supposed to be *isotropic*.

2. *Fatigue* effects as well as ageing effects are neglected.

3. Rate-independent hardening effects are neglected for the sake of simplicity, in spite of the fact that they can be taken into account, to a great extent, through an effective modelling defined previously [4].

4. Rate-dependent hardening effects are taken into account through a thermomechanical modelling which have proved to be effective in the case of a solid polymer [14].

5. The temperature is supposed to be constant for the sake of simplicity, but it is worth noting that the temperature-dependent case can be taken into account through effective modelling, including the cases of shape memory effects and of thermomechanical coupling effects [4, 14].

6. The limit plastic behaviour of the material is supposed to be of Huber – von Mises type for the sake of simplicity, but the Coulomb-like case has been previously taken into account through effective modelling able to describe the isotropic-deviatoric coupling effects exhibited by granular materials [5].

7. The cyclic loading paths which are actually considered in this preliminary paper are "rather simple", namely *not* of demagnetization type [7]. Owing to the above restrictive assumptions, the study is devoted mainly to a simultaneous modelling of the first and second order effects which are specific of the (elastoplastic) *pure hysteresis* solid behaviour [1, 3] in the isothermal, isotropic and without hardening case (cf. Fig. 1a). Moreover, the definition of a rate-dependent thermomechanical modelling is up to now introduced (cf. Fig. 1b, d) in order to compare the theory with thermomechanical experimental results obtained through the study of a solid polymer [14].

It is possibly useful to give some hints concerning some of the features of the presentation. First, the study is mainly confined to the case of shear-type kinematics, a feature which may suggest that one neglects the use of the classical way, going from a general framework to a special case, or conversely. This feature of the presentation is connected with the fact that the case of *rotational* kinematics is, in essence, the only one which is of fundamental *theoretical* interest in the framework of the proposed approach. On the contrary, in the framework of this approach, the case of irrotational kinematics, of course being of a fundamental *technological* interest, may be investigated (in the isotropic case) through a much simpler study than that of the rotational case.

Second, the (idealized) homogeneous kinematics is the only one introduced in the study, giving *only* the opportunity to define some extremely special initial-value problems. Moreover, with regard to the matter of the Lagrangian-Eulerian dilemma, the approach is old-fashioned for it follows the line of LODGE [15]. Consequently, a feeling of doubt may arise concerning the effectiveness of such an approach with respect to the matter of boundary-value problem. It is worth noting that a comprehensive study of implicit numerical strategies has been given elsewhere, concerning the step-by-step advancement of quasi-static elastic-plastic solutions of large size problems discretized through finite elements such as, for example, notched bar problems, plate with hole problems, necking problems with shear-band localization and shell problems [16 to 21].

Third, this preliminary paper does not provide any hints concerning a *comparison* between our proposal and the set of current approaches to cyclic plasticity and ratchetting effects. On the one hand, it is well known that many papers on cyclic plasticity appear every year. On the other hand, it is worth noting that one of the most impressive efforts ever achieved in the field of fundamental technological research based on continuum mechanics, has been indeed applied to the matter of (quasi-isothermal) second order effects of ratchetting type (see Fig. 7 of [22] and [23 to 25]). Consequently, it should be useful to give, in a concise form, a clue allowing to imagine the sketch of the comparison mentioned above. In order to be concise, it is convenient to avoid the introduction of various theoretical elements concerning the notions of reference frame $(0, \mathbf{e}_i)$ [26], of motion $z^i(M(x^k), t)$, of current metric tensor $\mathbf{G}(M(x^k), t)$, of current strain tensor $\Delta^t_R \varepsilon(M(x^k), t)$, of current strain rate tensor $\mathbf{D}(M(x^k), t)$ and of discrete memory of a previously selected metric tensor (Cauchy tensor ${}^t_R\mathbf{G}$), and of a previously selected stress tensor (owing to [27] let us say the "Ślebodzinski-Cauchy" tensor ${}^t_R\mathbf{S}$). Secondly, it is also convenient to make use of the fact that the various approaches of Armstrong-Frederick-Ohno-Wang type are well known. Hence the introduction of the following elements allowing a comparison between the proposed pattern and the usual set of theories:

(1) The *strain rate* tensor $\mathbf{D}$ involved in the constitutive pattern of irreversible behaviour is implemented without the aid of some splitting up (of $\mathbf{D}^e + \mathbf{D}^p$ type).

(2) From (1) it follows that the hardening notion is not involved *before* the definition of a hardening-independent and rate-independent basic behaviour (pure hysteresis [1 to 3]) which is *always* irreversible, independly of how small are the cycle amplitudes of the loading-unloading cyclic path. With the aid of this basic theory, the effects of transient rate-independent hardening and of rate-dependent hardening are more easily distinguished and classified.

(3) Following Gibbs (plasticity cannot be described in the framework of the thermostatics), one drops the idea of some "T.I.P.- like" generalization of classical thermodynamics.

(4) Following the old-fashioned sketch of Prager allowing continuous transition between quasi-elasticity to quasi-perfect plasticity, the basic classical notions involved in the pure hysteresis pattern are *only* those of infinitesimal Hookean behaviour and of *single* limit surface of plasticity. Moreover, owing to the fact that manufacturing processes often involve large strains and large temperature variations on which depends the future material behaviour, the theory must be, from the outset, defined in the finite strain form.

(5) From (3) and (4), the current Cauchy stress tensor $\mathbf{S}(M(x^k), t)$ of the material point M is the only stress tensor involved in the constitutive definition and, owing to the *perfectly closed cycle* problem, one admits the use of

$$ {}^t_R\mathbf{S} = \mathbf{S}(t_R) \qquad t_R < t; \qquad \text{rate of} \quad {}^t_R\mathbf{S} = 0, $$

an assumption involving the use of $\Delta^t_R\mathbf{S} = \mathbf{S} - {}^t_R\mathbf{S}$ and of rate of $\Delta^t_R\mathbf{S}$ equal to the rate of $\mathbf{S}$ along a branch of cycle. Consequently, the use of the *discrete* *memory* notion is accepted, as well as the implementation of differential-difference equations in which the delay ${}^{t}_{R}\mathbf{S}$ is a piecewise constant (tensorial) functional of the loading history.

(6) From (1) to (5), the pattern cannot be based mainly on differential-difference equations specific to a given material. It must be founded, to a great extent, on general rules involving explicitly both several ordered discrete sets of pure scalars and the *intrinsic dissipation functional Φ of discrete memory form.* Thereby a process is allowed of *comparison* between elements of ordered discrete sets of energy levels or, still better, between elements of ordered discrete sets of intensities of rates of Φ.

(7) From (5) and (6), the pattern is rate-form and the *stress rate* definition is a basic issue. Hence the role of Preferred Reference Frames.

(8) The generalization to second order effects must make invariant the basic ingredients of the first order pattern.

(9) The implementation of Preferred Reference Frames is accepted as relevant [3].

Fourth, a comment may be useful regarding the "rather simple loading path" restrictive assumption. It means that the features of the ratchet effect during demagnetization-like cyclic loading processes is not studied in this paper. This is the main theoretical gap of the study, because the proposed approach already covers, in the three-dimensional case, the first order effects of plastic hysteresis and of ferrohysteresis including application to electromechanical coupling effects [28 – 30]. Owing to this gap, it should be useful to find some clues in the field of the current ferrohysteretic modelling.

However, the *three-dimensional first order* effects of ferrohysteresis, including the intricate case of demagnetization processes, remain difficult to describe through the usual non-mechanical approaches [31 to 35]. Hence the difficulty to find from now on, concerning *second order effects involved in three-dimensional* situations, some useful clues in order to guide the study to come.

1.1. From a constitutive pattern of the first order effects to taking into account the second order effects

Up to now, only simple hints have been given about the Preferred Reference Frame definition (cf. [6], Sec. 3.3, 3.5.4. and Fig. 16, for example). This definition seems to provide an effective and rather general theory in order to deal with the problem of a *simultaneous* treatment of the first order behaviour and of the second order effects. Consequently, it is useful to give immediately a short reminder concerning the treatment of the first order effects (such as those involved in the field of multiaxial cyclic plasticity and viscoelastoplasticity). Then, it will be easy to state how it is possible, with the aid of the relevant PRF, to take into account the second order ratchet effects. At first, the two-fold foundation of the initial constitutive modelling is suggested (Sec.1.2). Next, some simple intuitive hints (Sec. 1.3) are given regarding the modelling of the second order effects.

1.2. From stress splitting approach and preponderant role of the pure hysteresis stress contribution to the associated viscoelastoplastic theory

1.2.1. A clue of special interest is as follows: *owing to the status of the strain rate notion and to experimental evidences (at microscopic and at macroscopic scale), it is not reasonable to introduce a splitting process of this tensor (cf. [6], Sec. 2.1); consequently, the first assumption which must be introduced in order to respect the thermodynamical distinction between reversibility and irreversibility, is that of a splitting up of the* **stress**. Let us suppose that (cf. Figs. 1, 2):

$$\mathbf{S} \overset{\text{def}}{=} \mathbf{S}_a + \mathbf{S}_\nu + \mathbf{S}_r; \qquad P \overset{\text{def}}{=} P_a + P_\nu + P_r, \tag{1.1}$$

so that the internal powers associated with an *always irreversible* process, with a viscous-viscoelastic process and with the *reversible* (elastic) process, are respectively:

$$P_a \overset{\text{ident}}{=} \hat{P}_a/\sqrt{g} \overset{\text{def}}{=} -\operatorname{tr}(\mathbf{S}_a\mathbf{D});$$

$$P_\nu \overset{\text{ident}}{=} \hat{P}_\nu/\sqrt{g} \overset{\text{def}}{=} -\operatorname{tr}(\mathbf{S}_\nu\mathbf{D});$$

$$P_r \overset{\text{ident}}{=} \hat{P}_r/\sqrt{g} \overset{\text{def}}{=} -\operatorname{tr}(\mathbf{S}_r\mathbf{D}).$$

Then we postulate the following associated splitting:

$$\dot{E} = \left[-P_a + \dot{Q}_{ia}\right] + \left[-P_\nu + \dot{Q}_{i\nu}\right] + \left[-P_r + \dot{Q}_{ek}\right],$$

$$\dot{E} = (\Pi_a - P_r) + \dot{I}_a + \dot{I}_\nu + \dot{Q}_{ek}),$$

of the first principle and of the fundamental equation of Gibbs. Note that the internal intrinsic rate of heat supply has a non-viscous part $\dot{Q}_{ia}$ which is completely non-classical [1]. This splitting allows to preserve the role of the fundamental equation of Gibbs, namely the role of a hinge between the first principle and its "entropic" form. Such a formalism results indeed in the following equations of rate form:

$$\Phi \overset{\text{def}}{=} \Phi_a + \Phi_\nu = (\dot{I}_a + \dot{I}_\nu) + (-\dot{Q}_{ia} - \dot{Q}_{i\nu})$$

$$= -P - (\Pi_a - P_r) = (-P_a - \Pi_a) - P_\nu.$$

The main implicit equations of the rate form of the stress splitting approach (1.1) are:

$$
\begin{aligned}
&\dot{E}_a = -P_a + \dot{Q}_{ia}, \qquad \dot{E}_a = \Pi_a + \dot{I}_a,\\
(1.2)\qquad &\dot{E}_\nu = -P_\nu + \dot{Q}_{i\nu}, \qquad \dot{E}_\nu = 0 + \dot{I}_\nu,\\
&\dot{E}_r = -P_r + \dot{Q}_{ek}, \qquad \dot{E}_r = -P_r + 0 + \dot{Q}_{ek},
\end{aligned}
$$

and

$$
(1.3)\qquad \dot{\mathbf{S}} = \dot{\mathbf{S}}_v + \dot{\mathbf{S}}_r + \dot{S}_a; \qquad \dot{\mathbf{S}}_c(M,t) = \mathbf{f}_c(\mathbf{D}(M,t),...);
$$
$$
\mathbf{S}_c(M,0) = 0; \qquad c = v, r, a.
$$

Let us suppose that the explicit forms of the reversible stress rate $\dot{S}_r$ and of the pure hysteresis stress rate $\dot{S}_a$ are well defined. Then, the above stress splitting approach becomes obviously a strongly constraining approach. The only remaining "degree of freedom" of the modelling is that concerning the viscous stress contribution. Consequently, the whole theory is not flexible and has two fundamental epistemological advantages: firstly, it may be easily invalidated through some experimental evidence; secondly, the definitions of the explicit forms of viscous type are almost necessarily simple and straightforward as regards the physical readings of the formalism.

Let us consider that a promising modelling may be founded on the most effective viscoelastic modelling currently available, namely that of Oldroyd. One obtains immediately the one-dimensional purely mechanical pattern of the form:

$$
S_\nu \overset{\text{def}}{=} S_1 + S_2; \qquad S_2 \overset{\text{def}}{=} \eta_2 D; \qquad S_1/(\theta_\nu \mu_1) + \dot{S}_1/\mu_1 \overset{\text{def}}{=} D
$$

and an associated global mechanical rate form:

$$
(1.4)\qquad \theta_\nu \dot{S}_\nu + S_\nu = (\eta_1 + \eta_2) D + \theta_\nu \eta_2 \dot{D},
$$

which is easily generalized to the tensorial case (Sec. 3.1). It is then obvious that the associated itemized thermodynamical forms are:

$$
\begin{aligned}
\dot{E}_2 &= 0; \qquad -\dot{Q}_2 = S_2 D = \eta_2 D^2,\\
\dot{E}_1 &= S_1(\dot{S}_1/\mu_1) = (\theta_\nu/\eta_1)(S_\nu - S_2)(\dot{S}_\nu - \dot{S}_2);\\
&\qquad -\dot{Q}_1 = S_1(S_1/\mu_1) = (1/\eta_1)(S_\nu - S_2)^2,
\end{aligned}
$$

resulting in the global forms:

$$
(1.5)\qquad m\dot{E}_\nu = (\theta_\nu/\eta_1)(S_\nu - \eta_2 D)(\dot{S}_\nu - \eta_2 \dot{D});
$$
$$
-\dot{Q}_\nu = (1/\eta_1)(S_\nu - \eta_2 D)^2 + \eta_2 D^2.
$$

Consequently, the set of basic forms defining the viscoelastoplastic theory involves, regarding the viscous features, a tensorial generalized split into its isotropic and deviatoric parts:

$$\dot{E}_\nu = \dot{E}_{\nu 0} + \dot{E}_{\nu d}, \qquad \dot{Q}_\nu = \dot{Q}_{\nu 0} + \dot{Q}_{\nu d},$$

with

$$\begin{aligned} \dot{E}_{\nu d} &= (\theta_\nu / 2\eta_{\delta 1}) \mathrm{tr}(\overline{S}_\nu - 2\eta_{\delta 2}\overline{D})(\dot{\overline{S}}_\nu - 2\eta_{\delta 2}\dot{\overline{D}}) \\ -\dot{Q}_{\nu d} &= (1/2\eta_{\delta 1}) \mathrm{tr}(\overline{S}_\nu - 2\eta_{\delta s}\overline{D})^2 + 2\eta_{\delta 2}\mathrm{tr}\overline{D}^2, \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} \dot{E}_{\nu 0} &= (\theta_\nu/3)[1/(3\eta_{01} + 2\eta_{\delta 1})][I_\nu - (3\eta_{02} + 2\eta_{\delta 2})][\dot{I}_\nu \\ &\qquad - (3\eta_{02} + 2\eta_{\delta 2})\dot{I}_D, \\ -\dot{Q}_{\nu 0} &= (1/3)[1/(3\eta_{01} + 2\eta_{\delta 1})][I_\nu - (3\eta_{02} + 2\eta_{\delta 2})I_D]^2 \\ &\qquad + (1/3)(3\eta_{02} + 2\eta_{\delta 2})I_D^2. \end{aligned} \tag{1.7}$$

Regarding the pure hysteresis contribution one has always the thermomechanical equations:

$$\begin{aligned} \omega \dot{E}_a &= -\omega P_{ia} - \Phi_a + C_a; \qquad -\omega \dot{Q}_a = \Phi + C_a; \qquad \omega = 1 \text{ or } 2; \\ -P_{ia} &= S_a D; \qquad C_a \approx \dot{S}_a \Delta_R^t G, \qquad {}_R^t\dot{S}_a = 0; \\ \Delta_R^t \dot{S}_a &= \dot{S}_a = \mu_r D + \beta \Phi_a \Delta_R^t S_a; \\ &\beta \approx -2\mu/(\omega Q_0)^2; \qquad \Phi_a = \Delta_R^t S_a D, \end{aligned}$$

but the tensorial generalizations of Φ_a, of C_a and of the mechanical rate form giving $\dot{S}_a$, already suggested elsewhere (cf. [6], Sec. 3.5.1, for example), cannot be easily recalled in a comprehensive form in the present short section (cf. Sec. 3.2).

A heuristic illustration may be obtained through the one-dimensional numerical simulation of a viscoelastoplastic cyclic behaviour. This illustration is inspired by the famous experimental study of MADELUNG [9]. More exactly, the qualitative modelling regards the results obtained by Madelung at relatively moderate rates and exhibiting therefore moderate viscous effects. The parameters of the pure hysteresis stress contribution (Fig. 1a) are: $\mu = 150$ GPa and $S_o = 200$ MPa. Those concerning the viscous effects (Fig. 1b) are: $\eta_1 = \eta_2 = 100$ GPa s, $\theta_\nu = 100$ s. The reversible pattern (Fig. 1c) is of power type (cf. Sec. 3.1) in order to obtain the qualitative features of ferrohysteresis (Fig. 1d). The rates $\dot{E}_a, \dot{E}_\nu, -\dot{Q}_a, -\dot{Q}_\nu$ are shown in Fig. 2 (parts a, b, c, d, respectively).

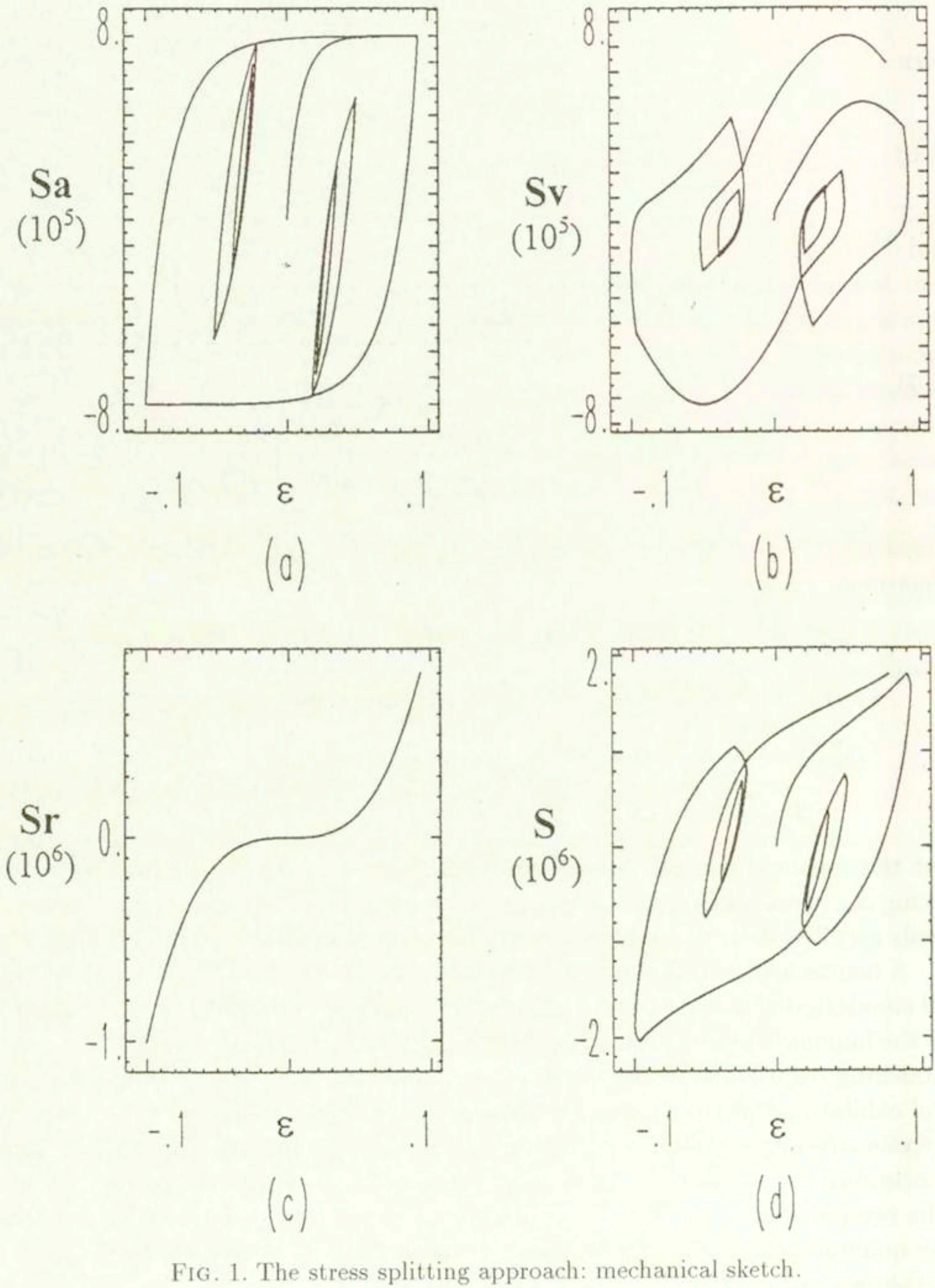

FIG. 1. The stress splitting approach: mechanical sketch.

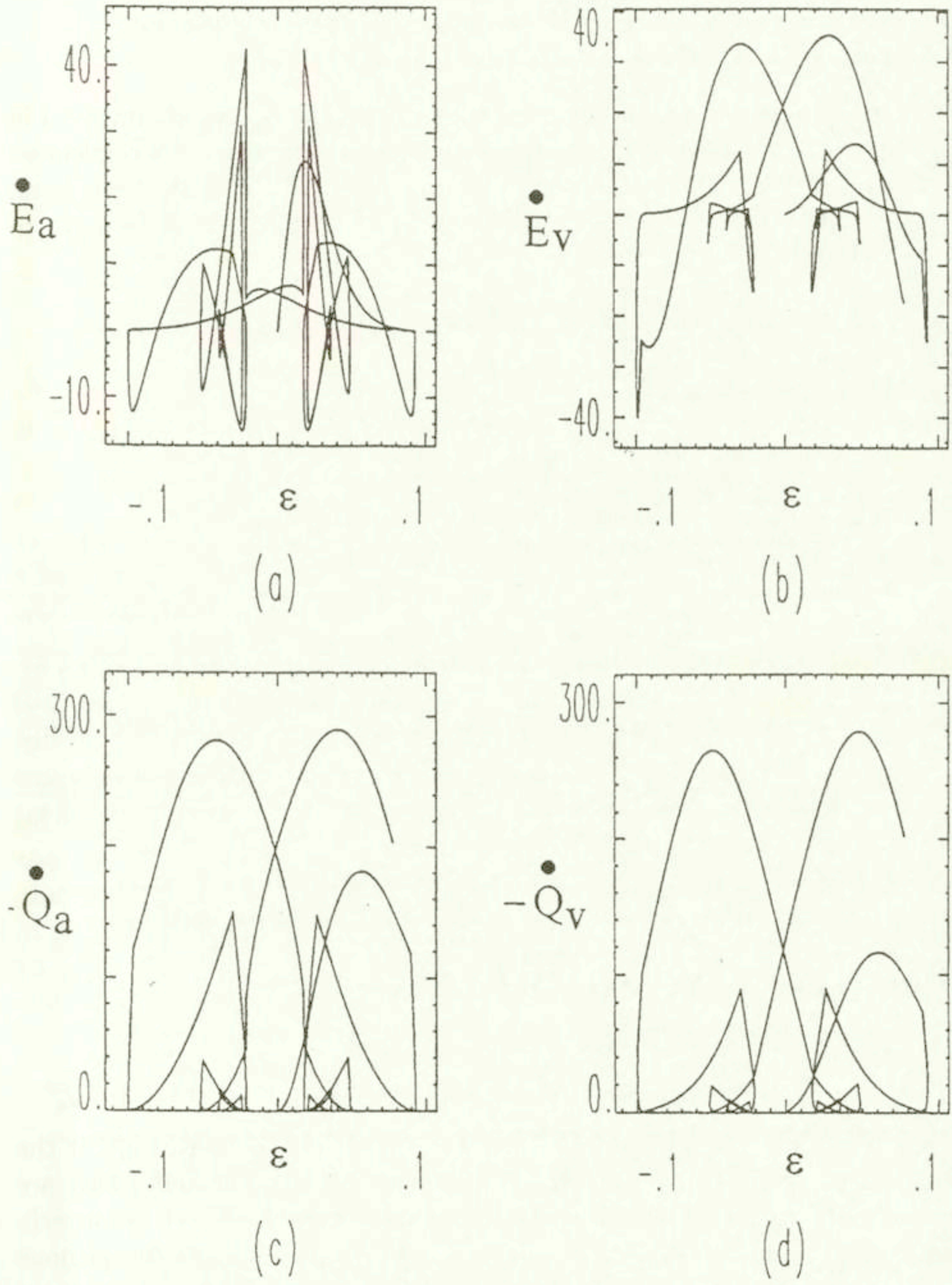

FIG. 2. The stress splitting approach: thermomechanical sketch.

1.2.2. *The preponderant role of the pure hysteresis stress contribution* $\mathbf{S}_a$ has been suggested previously (cf. [1] to [8], particularly in [6] and [7]). It is made conspicuous through the above illustration.

1.3. Short reminder intending to suggest the modelling through the heuristic case of the rotational triaxial kinematics with only one shear

1.3.1. Let us consider one of the three stress contributions $\mathbf{S}_r$, $\mathbf{S}_\nu$, $\mathbf{S}_a$ involved in the splitting of the actual stress $\mathbf{S}$. For the sake of formal simplicity it is convenient and not restrictive to choose the reversible contribution $\mathbf{S}_r$ in a Hookean form. The obvious generalization of the thermomechanical definition of $\mathbf{S}_r$ is done "in the PRF", namely

$$[\partial S_r^{ij}/\partial t]\mathbf{hr}_i \otimes \mathbf{hr}_j \overset{\text{def}}{=} Br^{ij}\mathbf{hr}_i \otimes \mathbf{hr}_j = \mathbf{Br};$$

$$Br^{ij} = \lambda_r I_d \delta^{ij} + 2\mu_r D^{ij}; \qquad \dot{Q}_r \overset{\text{def}}{=} 0,$$

$$\dot{E}_r \overset{\text{def}}{=} \text{tr}(\mathbf{SD}) = (1/3)I_{sr}I_d + \overline{S}_r^{ij}\overline{D}^{ij}$$

$$= (1/3)[1/(3\lambda_r + 2\mu_r)]I_{sr}\dot{I}_{sr} + (1/2\mu_r)\overline{S}_r^{ij}\dot{\overline{S}}_r.$$

It is formally similar to the infinitesimal Hookean form, except that it implies firstly, a time derivative in the PRF, secondly, an appropriate definition – presented below – of the base vectors $\mathbf{hr}_n(x^k, t)$ at the material point $M(x^k)$ and thirdly, no energetical drawbacks (cf. [10], for example). One of the obvious consequences of the new constitutive definition is that its expression in the fixed frame $(0, \mathbf{e}_i)$ involves a complementary term $[\Sigma_r]$. Starting from

$$\partial S_r/\partial t \overset{\text{ident}}{=} \partial\left[\Sigma_r^{kl}\mathbf{e}_k \otimes \mathbf{e}_1\right]/\partial t \overset{\text{ident}}{=} \partial\left[S_r^{kl}\mathbf{hr}_i \otimes \mathbf{hr}_j\right]/\partial t$$

and substituting the above stress rate definition, one obtains:

$$\dot{\Sigma}_r^{kl}\mathbf{e}_k \otimes \mathbf{e}_l = B_r^{ij}\mathbf{hr}_i \otimes \mathbf{hr}_j + [\mathbf{\Sigma}_r] = B_r^{kl}\mathbf{e}_k \otimes \mathbf{e}_l + [\mathbf{\Sigma}_r],$$

$$[\mathbf{\Sigma}_r] = S_r^{ij}(\Sigma_r^{kl})(\dot{\mathbf{hr}}_i \otimes \mathbf{hr}_j + \mathbf{hr}_i \otimes \dot{\mathbf{hr}}_j); \qquad Br_e^{kl} = \lambda_r I_d \delta^{kl} + 2\mu_r D_e^{kl}.$$

1.3.2. From the hints given above, one may conclude that the derivation of the additional tensors $[\mathbf{\Sigma}]$ results in forms which are not working and which are similar to that of Jaumann regarding the role of the stress, but which are utterly *different* regarding the role of the *kinematics*. These points are made conspicuous below (Sec. 1.3.4 and 1.3.5) in a special case and are studied further in the main part of the paper (Sec. 2.5).

1.3.3. However, a special comment is required concerning the third additional term generally involved by the stress splitting up approach, namely concerning the tensor $[\Sigma_a]$. Let us consider the deviatoric part of the contribution of pure hysteresis type defined as follows:

$$\left[{}^t_R\dot{\overline{S}}^{ij}_a\right] \mathbf{ha}_i \otimes \mathbf{ha}_j = 0;$$

$$\left[\overline{\Delta}^t_R\dot{\overline{S}}^{ij}_a\right] \mathbf{ha}_i \otimes \mathbf{ha}_j = \left[\dot{\overline{S}}^{ij}\right] \mathbf{ha}_i \otimes \mathbf{ha}_j$$

$$= \left[2\mu_r \overline{D}^{ij} + \overline{\beta}\,\overline{\Phi}_a\,\overline{\Delta}^t_R\,\overline{S}^{ij}\right] \mathbf{ha}_i \otimes \mathbf{ha}_j.$$

It is necessary to know if such a definition results in the relation

$$\partial\overline{\Sigma}^{kl}_a/\partial t = \overline{B}^{kl}_a + [\overline{\Sigma}_a]^{kl}; \qquad [\overline{\Sigma}_a] = \overline{S}^{ij}_a(\overline{\Sigma}^{kl}_a)(\dot{\mathbf{h}}\mathbf{a} \otimes \mathbf{ha}_j + \mathbf{ha}_i \otimes \dot{\mathbf{h}}\mathbf{a}_j)$$

or

$$\partial\Delta^t_R\overline{\Sigma}^{kl}_a/\partial t = \overline{B}^{kl}_a + [\Delta^t_R\overline{\Sigma}_a]^{kl}; \qquad [\Delta^t_R\overline{\Sigma}_a]$$

$$= \Delta^t_R\overline{S}^{ij}_a(\overline{\Delta}^t_R\overline{\Sigma}^{kl}_a)(\dot{\mathbf{h}}\mathbf{a}_i \otimes \mathbf{ha}_j + \mathbf{ha}_i \otimes \dot{\mathbf{h}}\mathbf{a}_j).$$

A short derivation shows that it results in the former expression involving $[\overline{\Sigma}_a]$. Consequently, there is no amalgamation of the *intrinsic* notions of discrete memory (${}^t_R\overline{S}^{ij}_a$) and of stress variation ($\Delta^t_R S^{ij}_a$) with a spin effect which cannot be entirely intrinsic, due to its partial dependence upon the loading process (a feature suggested below in Sec. 1.3.5), and studied further in the main part of the paper.

1.3.4. Before dealing with the definition of the base vectors $\mathbf{hr}_i$, $\mathbf{ha}_i$ and $\mathbf{hv}_i$, let us consider once again the case where $\mathbf{S}$ may be represented by only one of the three possible types of stress contribution (for example: $\mathbf{S} = \mathbf{S}_r; \mathbf{h} = \mathbf{h}_r$, like above, Sec. 1.3.1). The formal consequences of the approach are easily introduced in a simple kinematical case, namely that of a simple or pure shear defined in $(0, \mathbf{e}_k)$ through the following forms:

$$0 = -z^1 + J_1 Z^1 = -z^2 + J_2 Z^2 + 2\gamma_3(t) Z^3 = -z^3 + J_3 Z^3,$$

$$(J_n = 1 + K_n, \qquad n = 1, 2, 3),$$

$$\mathbf{D} = \sum_1^3 D_n \mathbf{e}_n \otimes \mathbf{e}_n + D_4(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2), \qquad D_n = \dot{K}_n/J_n,$$

$$J_3 D_4 = \dot{\gamma} + \gamma D_2,$$

$$\mathbf{S} = \sum_l^3 \Sigma_n \mathbf{e}_n \otimes \mathbf{e}_n + \Sigma_4 (\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2),$$

$$0 = -\mathbf{h}_1 + \mathbf{e}_1 = -\mathbf{h}_2 + \mathbf{e}_2 \cos\alpha + \mathbf{e}_3 \sin\alpha = -\mathbf{h}_3 - \mathbf{e}_2 \sin\alpha + \mathbf{e}_3 \cos\alpha,$$

$$\alpha(0) = \gamma_3(0) = 0; \qquad \alpha(\gamma_3(t)) \ll \pi/2,$$

where the subscript r is omitted. One obtains:

$$\begin{aligned}
\dot{\Sigma}_1 &= (\lambda + 2\mu)D_1 + \lambda D_2 + \lambda D_3,\\
\dot{\Sigma}_2 + 2\Sigma_4\dot{\alpha}(\gamma_3, \dot{\gamma}_3) &= \lambda D_1 + (\lambda + 2\mu)D_2 + \lambda D_3,\\
\dot{\Sigma}_3 - 2\Sigma_4\dot{\alpha}(\gamma_3, \dot{\gamma}_3) &= \lambda D_1 + \lambda D_2 + (\lambda + 2\mu)D_3,\\
\dot{\Sigma}_4 - (\Sigma_2 - \Sigma_3)\dot{\alpha}(\gamma_3, \dot{\gamma}_3) &= (\mu\gamma_3/J_3)D_2 + (2\mu/J_3)\dot{\gamma}_3.
\end{aligned}$$

The complementary tensor is deviatoric and does no work ($\dot{E} = 2(\Sigma_2 - \Sigma_3)\Sigma_4\dot{\alpha}$ $-2\Sigma_4(\Sigma_2 - \Sigma_3)\dot{\alpha} = 0$). The right-hand sides of these equations are 0, 0, 0 and $2\mu\dot{\gamma}$ respectively, in the simple shear case ($D_n = 0,\ n = 1, 2, 3$). Consequently, one may imagine firstly, that the second order effects are exhibited through such a system of differential equations (giving the relationships between $\dot{S}_1 \ldots \dot{S}_4$ and $\dot{K}_1, \dot{K}_2, \dot{K}_3, \dot{\gamma}_3$) and that, secondly, ratchet effects are involved in the pure hysteresis case, namely in the case where the system of equations is of differential-difference type.

1.3.5. It remains to suggest the definition of the base vectors $\mathbf{h}_i$, *both* in the case of a unique contribution *and* in the case where several contributions are simultaneously involved, in order to define a relevant pattern of the actual behaviour. It is convenient to consider at first the more interesting contribution, namely that of pure hysteresis, making the assumption: $\mathbf{S} = \mathbf{S}_a$.

Let us suppose that the problem of the initial position of the PRF is solved. The definition of the current motion of the base vectors $\mathbf{ha}_i$ is then given by the following set of rules:

1) The motion of the PRF is *continuous* and defined by the angular velocity which is generally the sum of two terms, kinematical and thermomechanical, respectively.

2) The kinematical angular velocity $\dot{\alpha}_\delta$ of the PRF is given by that of the principal directions of the strain rate tensor $\mathbf{D}$. Consequently, $\dot{\alpha}_\delta$ is obtained through the time derivative of

$$\operatorname{tg} 2\alpha_\delta = 2(J_2\dot{\gamma} - \gamma J_2)/(J_3\dot{J}_2 - J_2\dot{J}_3), \tag{1.8}$$

so that the second derivatives of the stretches are involved in the $\dot{\alpha}\delta$ form:

$$\dot{\alpha}_\delta = \Big[(\dot{J}_3\gamma - J_3\dot{\gamma})\ddot{J}_2 - (\dot{J}_2\gamma - J_2\dot{\gamma})\ddot{J}_3 + \Big((J_3\dot{J}_2 - J_2\dot{J}_3)\ddot{\gamma}\Big]\Big[J_2 \cos 2\alpha\delta/(J_3\dot{J}_2 - J_2\dot{J}_3)\Big]. \tag{1.9}$$

It is worth noting that, for the isotropic continua, pure shear kinematics $K_2 = K_3 = 0$ result in $\dot{\alpha}_\delta = 0$, like in the case of triaxial kinematics with fixed principal directions of stress and strain.

3) The (complementary) thermomechanical angular velocity of the PRF, $\dot{\alpha}_{\phi a}$, is defined by three-fold approach.

a) In the first step one considers the angular velocity $\dot{\theta}_\varphi$ of the principal directions of the "Φ_a-inertia" of the deformable material point endowed with a homogeneous field of intrinsic dissipation Φ_a. The role of Φ_a is prominent because it allows the cases of homogeneous mesostructures to be distinguished from those of heterogeneous types.

b) Secondly, the explicit relationships between $\dot{\theta}_\varphi$and the kinematics is obtained (in the homogeneous case), by means of the time derivative of

$$\operatorname{tg} 2\theta_\varphi = 4\gamma_3 J_3/[(J_2)^2 - (J_3)^2 + (2\gamma_3)^2], \tag{1.10}$$

(cf. Fig. 8), so that $\dot{\theta}_\varphi$ is a linear form:

$$\begin{aligned} \dot{\theta}_\varphi &= A_1\dot{K}_1 + A_2\dot{K}_2 + A_3\dot{K}_3 + A_4\gamma; \qquad A_1 = 0; \\ A_2 &= -4\gamma J_2 J_3 A^{-1}; \\ A_3 &= 2\gamma(4\gamma^2 + J_2^2 + J_3^3)A^{-1}, \\ A_4 &= 2J_3(J_2^2 - J_3^2 - 4\gamma^2)A^{-1}; \qquad A = 16\gamma^2 J_3^2 + (4\gamma^2 + J_2^2 - J_3^2)^2, \end{aligned} \tag{1.11}$$

with respect to the rate of stretches, the coefficients A being however non-linear functions.

c) In the third step, the thermomechanical angular velocity is defined as a "small" and constitutive part of the spin $\dot{\theta}_\varphi$, defined by the general condition

$$\dot{\alpha}_{\phi a}\dot{\theta}_\phi < 0, \tag{1.12}$$

and by a constitutive definition of pure hysteresis type:

$$\begin{aligned} {}^t_R\dot{\alpha}_{\phi a} &= {}^t_R\dot{\theta}_\phi = 0, \\ \dot{\alpha}_{\phi a} &= \Delta^t_R\alpha_{\phi a} = F'((\Delta^t_R\theta_\phi)/\omega)\dot{\theta}_\phi, \\ \Delta^t_R\alpha_{\phi a} &= \omega F((\Delta^t_R\theta_\phi)/\omega), \end{aligned} \tag{1.13}$$

suggested in Fig. 3 (where the $\alpha_{\phi a} - \theta_\varphi$ and $\alpha_{\phi r} - \theta_\varphi$ diagrams are able to make conspicuous the relationship between $\dot{\alpha}_\phi$ and $\dot{\theta}_\varphi$ as soon as the reading is done under the assumption: $\dot{\theta}_\varphi = 1$). This definition may be obtained through the time derivative of a form such as:

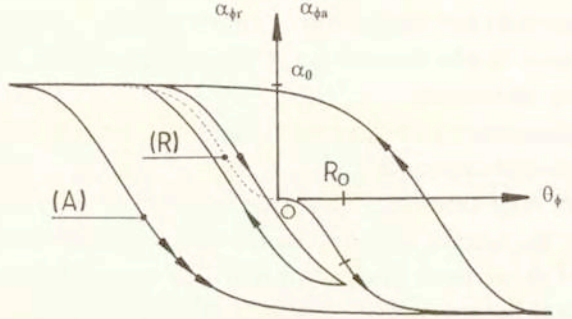

FIG. 3. A sketch suggesting the definition of the thermomechanical angular velocity involved in the definition of the PRF.

$$(1.14) \qquad F(\theta) = -\alpha_0(1 - \exp -(-\theta^2/R_0^2))\mathrm{th}(\theta/R_0);$$

$$0 < \alpha_0 \ll 1; \qquad 0 < R_0 \ll 1,$$

where the constitutive parameters are α_0 (intensity parameter, $\sim S_0/\mu$) and R_0 (location parameter, $\sim S_0/\mu$).

4) Finally, following the splitting rule 1:

$$(1.15)_1 \qquad \dot{\alpha}_a = \dot{\alpha}_\delta + \dot{\alpha}_{\phi a}; \qquad \dot{\mathbf{h}}_{a2} = (-\sin\alpha_a \mathbf{e}_2 + \cos\alpha_a \mathbf{e}_3)\dot{\alpha}_a;$$

$$\dot{\mathbf{h}}_{a3} = (\cos\alpha_a \mathbf{e}_2 - \sin\alpha_a \mathbf{e}_3)\dot{\alpha}_a$$

and, if a reversible contribution is involved ($\mathbf{S} = \mathbf{S}_a + \mathbf{S}_r$), one has also (cf. Fig. 3):

$$(1.15)_2 \qquad \dot{\alpha}_r = \dot{\alpha}_\delta + \dot{\alpha}_{\varphi r}; \qquad \dot{\alpha}_{\varphi r} = F'_{\theta\varphi}(\theta_\varphi)\dot{\theta}\varphi; \qquad \alpha_{\phi r} = F(\theta_\varphi)$$

in order to define the base vectors $\mathbf{h}_{rn}$. Consequently, it is not necessary to make use of a specific set of constitutive parameters α_{or} and R_{or}.

5) It is worth noting that it is not effective to modify the definitions of $\dot{\alpha}_a$ and $\dot{\alpha}_r$ making use of a very simple form of $\mathrm{tg}\, 2\theta_\varphi$, such as $2\gamma_3/J_3$, resulting in

$$(1.16) \qquad \dot{\theta}_\varphi = (2\dot{\gamma}_3 J_3 - 2\gamma_3 \dot{K}_3)/(J_3^2 + 4\gamma_3^2).$$

1.3.6. An example of illustration is given in Fig. 4 a,b where the "axial stress-shear strain" diagram is obtained by a numerical simulation of the above pattern under the following conditions: $\mathbf{S} = \mathbf{S}_a$; $\gamma_3 > 0$, $K_n = 0$, $n = 1, 2, 3$, resulting in fixed principal directions ($\dot{\alpha}_\delta = 0$) of the strain rate tensor $\mathbf{D}$. The constitutive parameters are: $\lambda = 2\mu = 150$ GPa; $S_0 = 200$ MPa; $\alpha_0 = (1/2)\ S_0/\mu$; $R_0 = 4\ (S_0/\mu)$. Note that the ratchet effects are also obtained using the intuitive definition (1.16) of the spin $\dot{\theta}_\varphi$ but that these effects are not physically relevant (Fig. 4 c, d). The intuitive approach is no more satisfying in the case of "not symmetrical" sets of cycles of different constant amplitudes (cf. Fig. 5 a, b for the thermomechanical approach and Fig. 5 c, d for the intuitive approach, both obtained under a small initial compression $-S_0/1000$).

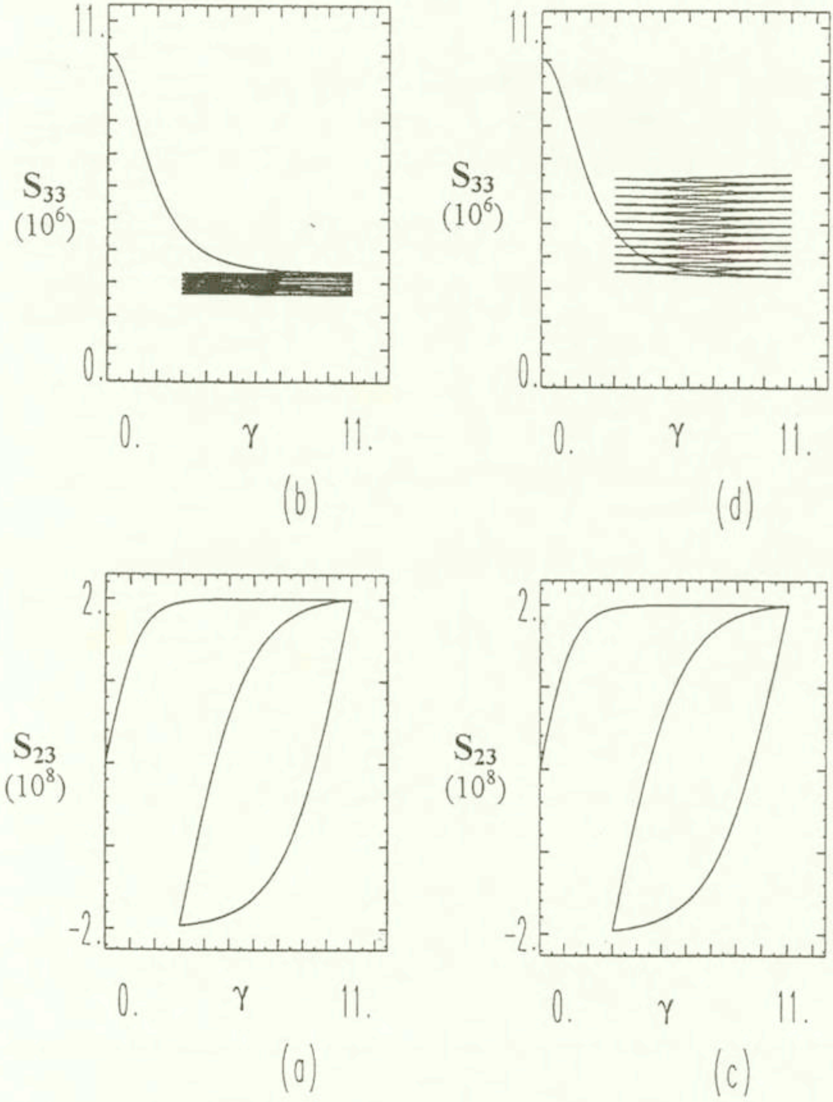

FIG. 4. Ratchet effect (b) exhibited after a small traction through the cumulative increase of the axial stress compression S_{33} during the cyclic simple shear (a).

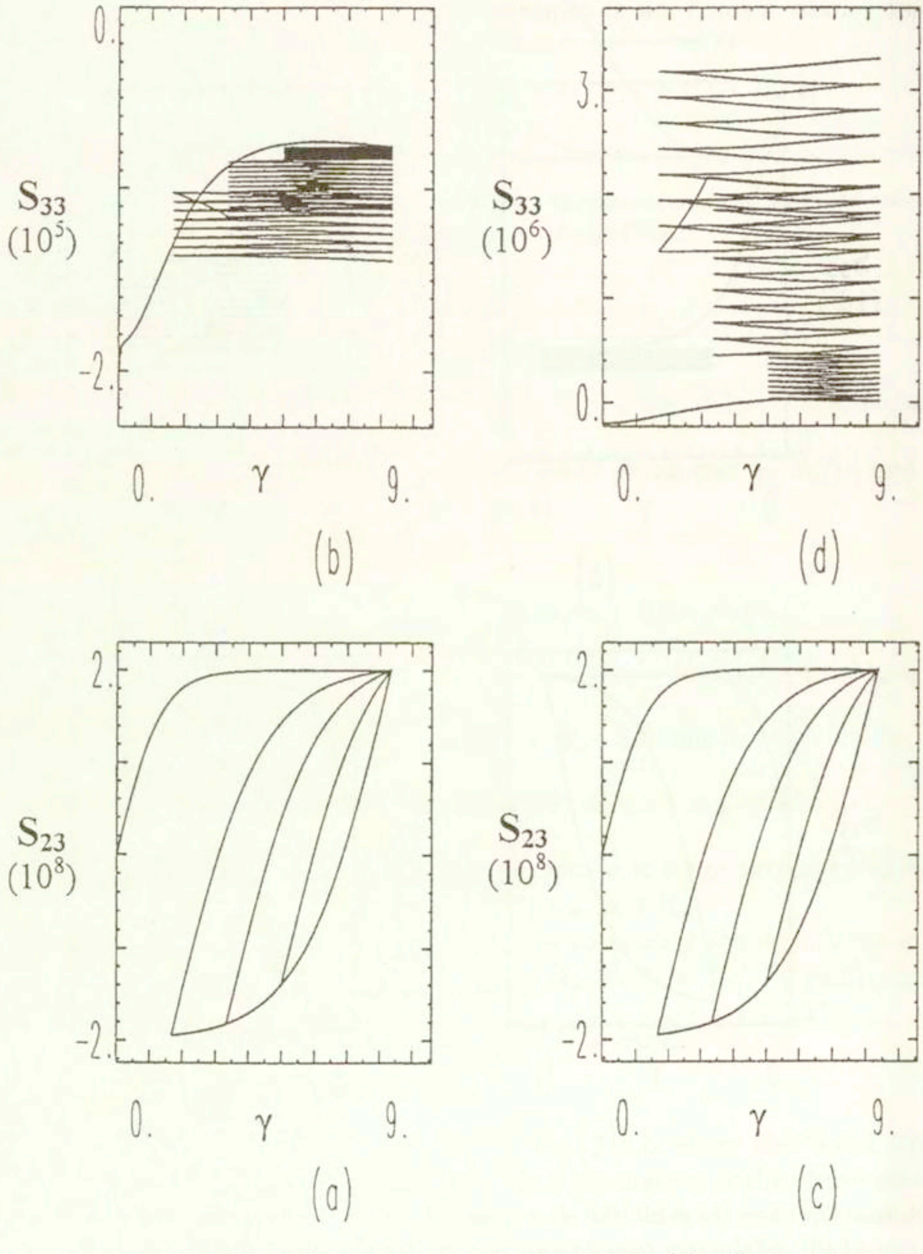

FIG. 5. Ratchet effect (b) in the case of several sets of cycles (a). The intuitive approach based on (1.16) results in (d) for a first order behaviour (c) identical to (a).

1.4. Towards a rather general pattern of rate form defined in the PRF

The first part of the paper is devoted to the *definition* of the PRF (Sec. 2). Some remarks are added concerning the three basic *stress contributions* of the viscoelastoplastic pattern (Sec. 3). In order to supplement briefly the illustration given by Figures 4 and 5, comments are added concerning the results available from now on through various numerical *simulations* (Sec. 4).

2. Definition of the Preferred References Frames

In this section the clue of the presentation is that suggested above (Sec. 1.3.5), namely a series of relationships, firstly between frames and spins, secondly between spins and spins, finally between spins and frames. In the first part of the paragraph (a generalization of point 2, Sec. 1.3.5), one starts from the frame $(M, \mathbf{d}_n)$ of $\mathbf{D}$ to derive the "kinematical" spin $\mathbf{\Omega}_\delta$. The cumbersome formalism is two-fold: a) that of giving, through $\dot{\mathbf{d}}_n$, the notions of the principal directions; b) that associated with the basic "direct" form $\Sigma_3^1 \mathbf{d} \wedge \dot{\mathbf{d}}$. The second part (a generalization of points 3 a,b, Sec. 1.3.5) is devoted only to the definition of the "inertia-like" tensor $\mathbf{I}$ (the derivation of the relationships between the frame $(M, \mathbf{i}_n)$ and the spin $\mathbf{\Omega}_\phi$ is omitted because it is similar to that concerning $(M, \mathbf{d}_n)$ and $\mathbf{\Omega}_\delta$). The third part of the paragraph (a generalization of point 3c, Sec. 1.3.5) is the only one which may be interesting from the constitutive point of view. One defines a constitutive relationship between the spin $\mathbf{\Omega}_\phi$ and the thermomechanical spin $\mathbf{\Omega}_\mu$, in order to obtain the spin of the PRF through the spin splitting, similar to (1.15):

$$\mathbf{\Omega} = \mathbf{\Omega}_\delta + \mathbf{\Omega}_\mu(\mathbf{\Omega}_\phi); \qquad \mathbf{\Omega}_\mu = f\mathbf{\Omega}_\phi.$$

Note that the proposed relationship between $\mathbf{\Omega}_\mu$ and $\mathbf{\Omega}_\phi$ is isotropic, and that the definition of the involved scalar factor f is a basic issue. The fourth paragraph (a generalization of point 4, Sec. 1.3.5) deal with the relationship between the spin $\mathbf{\Omega}$ and the PRF. The cumbersome formalism is that expressing the rates of the base vectors in terms of the basic "inverse" form $\mathbf{\Omega} \wedge \mathbf{d}$.

2.1. The "kinematical" term involved in the splitting of the angular velocity of the PRF

2.1.1. In spite of the fact that, in the isotropic case, simple kinematics is sufficient to study the constitutive functional relationships between the space of strain and the space of stress, the anisotropic generalization must be kept in view. Therefore, it is not suitable to consider an oversimplified form of the strain rate tensor. In $(0, \mathbf{e}_a)$, an interesting expression of

$$\mathbf{D} = \Sigma_3^1 D_n \mathbf{e}_n \otimes \mathbf{e}_n + D_4(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2) + D_5(\mathbf{e}_3 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_3) + D_6(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$$

may be, for example, that obtained in the case of a six-parameter deformation in the reference planes of the fixed frame $(0, \mathbf{e}_a)$:

$$0 = z^1 + J_1 Z^1 + 2\gamma_2 Z^2, \qquad x^1 = Z^1 \quad \text{for example,}$$

$$0 = -z^2 + J_2 Z^2 + 2\gamma_3 Z^3, \quad x^2 = Z^2 \quad \text{for example,}$$

$$0 = -z^3 + J_3 Z^3 + 2\gamma_1 Z^1, \quad x^3 = Z^3 \quad \text{for example.}$$

One obtains immediately:

$$\begin{aligned}
\mathbf{D} = (1/2)\partial \mathbf{G}/\partial t = [\mathbf{g}^1 \otimes \mathbf{g}^1][J_1 \dot{J}_1 + 4\gamma_1\dot{\gamma}_1] + \ldots \\
+ [\mathbf{g}^2 \otimes \mathbf{g}^3 + \mathbf{g}^3 \otimes \mathbf{g}^2][\dot{J}_2\gamma_3 + J_2\dot{\gamma}_3] + \ldots \\
= [\mathbf{e}_1 \otimes \mathbf{e}_1][(\dot{J}_1/J_1 + 8\gamma_1\dot{\gamma}_2\gamma_3/J_1J_2J_3)(1+K)^{-1}] + \ldots \\
+[\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2][(\dot{\gamma}_3/J_3 - 2\gamma_2\dot{\gamma}_1/J_2J_1 - \dot{J}_2\gamma_3/J_2J_3 \\
+2\gamma_2\gamma_1\dot{J}_3/J_1J_2J_3)(1+K)^{-1}] + \ldots \\
= [\mathbf{e}_1 \otimes \mathbf{e}_1][\dot{J}_1/J_1 + 8\gamma_1\dot{\gamma}_2\gamma_3/J_1J_2J_3](1+K)^{-1} + \ldots \\
[\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2][\dot{C}_3/J_3 - (2\gamma_2/J_2)(\dot{C}_1/J_1)]
\end{aligned}$$

$$\dot{C}_n = \dot{\gamma}_n - (\dot{J}_m/J_m)\gamma_n; \qquad m(n) = 3,1,2; \qquad n = 1,2,3;$$

$$K = 8\gamma_1\gamma_2\gamma_3/J_1J_2J_3.$$

This form make conspicuous the difference between 3-shear and 1 or 2-shears. In the current study T is supposed to be constant and $\mathbf{D}$ is not modified by $\mathbf{D}_T$. However, it is worth noting that the stress splitting approach may be generalized to the variable temperature case [4].

2.1.2. In order to introduce an explicit overview of the algorithm giving the kinematical spin $\mathbf{\Omega}_\delta$ for a given $\mathbf{D}$ (and giving also $\mathbf{\Omega}_\phi$ for a given $\mathbf{I}$), it is useful to set the notations needed for the derivation of the components in $(0, \mathbf{e}_a)$ of the unit vectors $\mathbf{d}_n$. The notation of the determinant equation and of its associated roots (supposed to be unequal) are:

$$\det[D_{rs} - d\delta_{rs}] = e^{ijk}[D_{1i} - d\delta_{1i}][D_{2j} - d\delta_{2j}][D_{3k} - d\delta_{3k}] = 0,$$

$$d^3 - I_D d^2 + \mathit{\Pi}_D d - \det D = d^3 + pd^2 + qd + r = 0;$$

$$I_D = -p = D_1 + D_2 + D_3,$$

$$\Pi_D = (1/2)(I_D^2 - D_{ij}D_{ij}) = q = D_2D_3 + D_3D_1 + D_1D_2$$
$$-D_4^2 - D_5^2 - D_6^2; \qquad III_D = \det\mathbf{D} = -r,$$

$$d_m = 2\sqrt{-a/3}\cos[y/3 + (m-1)2\pi/3)], \qquad m = 1, 2, 3,$$

$$\cos y = (-b/2)/(a^3/27)^{1/2}, \qquad 3a = 3q - p^2, \qquad 27b = 2p^3 - 9pq + 27r.$$

The notations concerning the principal directions are:

$$(D_{rs} - d_m\delta_{rs})d_m^s = 0; \qquad \mathbf{d}_m = d_m^s\mathbf{e}_s$$

giving, in $(0, \mathbf{e}_a)$, the three unit vectors $\mathbf{d}_m$ (associated with the above roots d_m and giving the principal directions of $\mathbf{D}$); it is also possible to derive the explicit forms giving the components d_m^s of $\mathbf{d}_m$ in $(0, \mathbf{e}_a)$. The derivation is obtained through the implementation of a relevant set of minors:

$$M_n^{ir} = (1/2)e^{ijk}e^{rst}(D_{js} - d_n\delta_{js})(D_{kt} - d_n\delta_{kt}) \quad \text{of} \quad (\mathbf{D} - d_n\delta).$$

Explicitly:

$$M_n^{11} = (D_2 - d_n)(D_3 - d_n) - (D_4)^2; \qquad M_n^{12} = D_4D_5 - (D_3 - d_n)D_6;$$
$$M_n^{13} = D_6D_4 - (D_2 - d_n)D_5;$$

$$M_n^{22} = (D_3 - d_n)(D_1 - d_n) - (D_5)^2; \qquad M_n^{23} = D_5D_6 - (D_1 - d_n)D_4;$$

$$M_n^{33} = (D_1 - d_n)(D_2 - d_n) - (D_6)^2$$

$$\mathbf{d}_n = (M_n^{11}\mathbf{e}_1 + M_n^{12}\mathbf{e}_2 + M_n^{13}\mathbf{e}_3)/[(M_n^{11})^2 + (M_n^{12})^2 + (M_n^{13})^2]^{1/2};$$
$$n, i = 1, 2, 3,$$

$$\mathbf{d}_n = (A_n\mathbf{e}_1 + B_n\mathbf{e}_2 + C_n\mathbf{e}_3)/E_n; \qquad E_n^2 = A_n^2 + B_n^2 + C_n^2;$$
$$A_n = M_n^{n1}; \qquad B_n = M_n^{n2}; \qquad C_n = M_n^{n3},$$

associated with the convention $i = n$, namely with the implementation of the set of minors $[M^{n1}, M^{n2}, M^{n3}]$ with the root d_n, for the derivation of the vector $\mathbf{d}_n$.

2.1.3. It remains necessary to obtain the explicit form of the components Ω_δ^n of the angular velocity pseudo-vector:

$$\Omega_\delta = \Omega_\delta^a\mathbf{e}_a = (1/2)\Sigma_1^3\mathbf{d}_n \wedge \dot{\mathbf{d}}_n; \qquad \mathbf{d}_n = d_n^m\mathbf{e}_m \tag{2.1}$$

The rate $\dot{d}_n^m$ involves the rates $\dot{A}_n, \ldots, \dot{E}_n$, namely the rates of the products $D_a D_b$ and $D_a d_n$. Hence, the second order time derivative $\ddot{K}$ and $\ddot{\gamma}$ are involved. One has indeed:

$$\dot{A}_1 = \partial[(D_2 - d_1)(D_3 - d_1) - (D_4)^2]/\partial t;$$

$$E_3\dot{E}_3 = A_3\dot{A}_3 + B_3\dot{B}_3 + C_3\dot{C}_3, \ldots$$

$$(1+K)\dot{D}_n = \ddot{J}_1/J_1 - (\dot{J}_1/J_1)^2 + [2\ddot{\gamma}_1/J_1 - 2\dot{\gamma}_1\dot{J}_1/J_1^2][4\gamma_2\gamma_3/J_2J_3]$$
$$+(\dot{\gamma}_1/J_1)[[\partial(\gamma_2/J_2)/\partial t](\gamma_3/J_3) + [\partial(\gamma_3/J_3)/\partial t]]$$
$$(\gamma_2/J_2)] - D_n\partial K/\partial t; \ldots$$

$$\dot{d}_n = \partial[2(-a/3)^{1/2}\cos\Phi/3]/\partial t;$$

$$-\sin\Phi\dot{\Phi} = \partial[(b/2)/(-a^3/27)^{1/2}]/\partial t; \ldots$$

$$27\dot{b} = 6p^2\dot{p} - 9\dot{p}q - 9p\dot{q} + 27\dot{r}; \qquad \dot{a} = \dot{q} - (2/3)p\dot{p}, \ldots$$

$$-\dot{p} = \dot{D}_1 + \dot{D}_2 + \dot{D}_3;$$

$$\dot{q} = \dot{D}_2D_3 + \dot{D}_3D_2 + \dot{D}_3D_1 + \dot{D}_1D_3 + \dot{D}_1D_2 + \dot{D}_2D_1$$
$$-2\Sigma_4^6 D_n\dot{D}_n; \ldots$$

The Euler's angles of the frame $(M, \mathbf{d}_n)$ can be obtained by the integration of the usual forms involving the components Ω_δ^a ($\dot{\theta}_\delta = \Omega_\delta^1\cos\psi_\delta + \Omega_\delta^2\sin\psi_\delta$; $\phi_\delta\sin\theta_\delta$; $\phi_\delta\sin\theta_\delta = \Omega_\delta^1\sin\psi_\delta - \Omega_\delta^2\cos\psi_\delta$; ...).

2.2. The "power inertia" tensor I

2.2.1. In order to follow the clue given above (Sec. 1.3.5, point 3a,b), it is necessary:

i. To define **I** giving the inertia-like frame $(M, \mathbf{i}_n(M,t))$ and $\Omega_\phi(M,t)$;

ii. To define the scalar function (or functional) f involved in the splitting giving the spin of the PRF. The second issue is studied below (Sec. 3). Here the attention is focused on the definition of **I**.

2.2.2. As soon as a rather general approach is needed, the relevant method of definition of $\mathbf{I}_\phi(M,t)$ must be, in principle, introduced working "on the chart of the convected co-ordinates x^k, in the field of the DARF, $(M, \mathbf{g}_i)$". The two main steps of the approach are, first, to define the relevant field of power (say Φ) and, second, to define the inertia-like tensor of the material point in the four relevant cases, namely: a) the Φ – homogeneous and isotropic case; b) the

Φ – homogeneous and anisotropic case; c) the Φ – heterogeneous and isotropic case; d) the Φ – heterogeneous and anisotropic case. In this preliminary study it is reasonable to deal only with the isotropic cases a) and c). Let us consider, in the Φ – homogeneous isotropic case, the pseudo-scalar field $\Phi(t)$ of intrinsic dissipation. The problem under consideration being kinematically homogeneous, the co-ordinates: $x^k = \text{ident} = Z^k$; $k = 1,2,3$ are rectilinear at the initial time $t = 0$, and remain rectilinear during the deformation. At the current time:

$$\begin{aligned}(2.2)\qquad \mathbf{I}(M,t) &= I^{ij}\mathbf{g}_i \otimes \mathbf{g}_j \\ &\overset{\text{def}}{=} \left[\int\int\int x^i(M)x^j(M)\phi(t)\rho(g)^{1/2}dx^i dx^j dx^k\right]\mathbf{g}_i \otimes \mathbf{g}_i \\ &\overset{\text{ident}}{=} \left[\int\int\int x^i(M)x^j(M)\hat{\Phi}(t)dx^i dx^j dx^k\right]\mathbf{g}_i \otimes \mathbf{g}_j.\end{aligned}$$

Performing the integration from $-1/2$ to $1/2$ one obtains

$$[12/\Phi(t)]\mathbf{I}(t) = \mathbf{g}_1 \otimes \mathbf{g}_1 + \mathbf{g}_2 \otimes \mathbf{g}_2 + \mathbf{g}_3 \otimes \mathbf{g}_3.$$

The frame $(M, \mathbf{i}_n)$ is therefore defined by the principal directions of $\mathbf{I} = \delta^{ij}\mathbf{g}_i \otimes \mathbf{g}_j$. Instead, making use of the covariant form I_{ij} and of the associated determinant equation $\det[(I-d)g_{ii} - I_{ij}] = 0$, $I = g_{ij}I^{ij}$, it is relevant, in a theoretical study of "homogeneous problems", to express the tensor $\mathbf{I}$ in the fixed frame $(0, \mathbf{e}_a)$:

$$\begin{aligned}\mathbf{I} &= (J_1\mathbf{e}_1 + 2\gamma_1\mathbf{e}_3) \otimes (J_1\mathbf{e}_1 + 2\gamma_1\mathbf{e}_3) + (2\gamma_2\mathbf{e}_1 + J_2\mathbf{e}_2) \\ &\quad \otimes (2\gamma_2\mathbf{e}_1 + J_2\mathbf{e}_2) + (2\gamma_3\mathbf{e}_2 + J_3\mathbf{e}_3) \otimes (2\gamma_3\mathbf{e}_2 + J_3\mathbf{e}_3) \\ &= [(J_1)^2 + (2\gamma_2)^2]\mathbf{e}_1 \otimes \mathbf{e}_1 + [(J_2)^2 + (2\gamma_3)^2]\mathbf{e}_2 \otimes \mathbf{e}_2 \\ &\quad + [(J_3)^2 + (2\gamma_1)^2]\mathbf{e}_3 \otimes \mathbf{e}_3 + [J_3 2\gamma_3](\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2) \\ &\quad + [J_1 2\gamma_1](\mathbf{e}_3 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_3) + [J_2 2\gamma_2](\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1).\end{aligned}$$

It is worth noting firstly, that in the heuristic one-shear case, one obtains immediately not only the usual results: $\operatorname{tg} 2\theta_\phi = 0/0$ or $\operatorname{tg} 2\theta_\phi = 0$ and $J_3/2\gamma_3$ for the cube, the rectangle and the rhombus, but also the previously introduced expression (1.10) of $\operatorname{tg} 2\theta_\phi$; secondly, that the initial orientation of the PRF $(M, \mathbf{H}_k)$ may be chosen arbitrarily (0/0 form). From the above expression of $\mathbf{I}$, one obtains the frame $(M, \mathbf{i}_n)$ using the previous forms (cf. Sec. 2.1). The only difference with the treatment concerning $\mathbf{D}$ comes from the fact that the time derivative of the components in $(0, \mathbf{e}_i)$ of $\mathbf{I}$ does not involve the second derivatives $\ddot{J}_n$ and $\ddot{\gamma}_n$ of the stretches.

2.2.3. Let us now consider, in the heterogeneous isotropic case, a material point endowed with a characteristic sub-structure at a physically relevant mesoscale of Néel-Krumhansl type. For example, the field of power (intrinsic dissipation Φ or elastic power $\dot{E}_r$) is so strongly heterogeneous that it is almost homogeneously located near the faces of the material cube. Owing to the definition of **I**, two basic types of mesostructural processes must be distinguished, according to the existence or absence of a perfect dragging along of the power field. In the first case the principal directions $\mathbf{i}_n$ are insensitive with respect to the mesostructure. On the contrary, if the microprocesses taking place in the walls involve a typical invariant micro-length e_0 and an associated typical meso-length J_0 (and if the process is sufficiently slow to allow the homogeneity of the power in the walls), then the vectors $\mathbf{i}_n$ are dependent on the nondimensional parameter: $e = e_0/J_0$. For example, in the two-dimensional case, one obtains the $\mathbf{i}_n$ from:

$$\begin{aligned}\mathbf{I}^* = \mathbf{I} - \mathbf{I}(\text{core}) = [(J_2)^2 + (2\Gamma_3)^2]\mathbf{e}_2 \otimes \mathbf{e}_2 + [(J_3)^2]\mathbf{e}_3 \otimes \mathbf{e}_3 \\ +[J_3 2\Gamma_3](\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2) - [(J_2)^2 + (2\gamma_3)^2]\mathbf{e}_2 \otimes \mathbf{e}_2 \\ -[(J_3)^2]\mathbf{e}_3 \otimes \mathbf{e}_3 - [J_3 2\gamma_3](\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2).\end{aligned}$$

In spite of the fact that sophisticated mesoscopic modelling [11, 12] are not currently included in the theory, it is worth noting that the definition of the Pattern of Rate Form in PRF may, in principle, be mesoscale-dependent, through the first order effects (for example those of anisotropy) as well as through the second order effects, as suggested above: the theory is not "incomplete" in the sense supported by Bunge.

2.3. The definitions of the thermomechanical spins through spin-spin isotropic constitutive relationships

2.3.1. Pure hysteresis ($\mathbf{S} = \mathbf{S}_a$) and Φ – homogeneity. Let us consider the current amplitude of rotation and the current amplitude by branch:

$$\Delta_0^t R_\phi = \int_0^t (\mathrm{tr}\boldsymbol{\Omega}_\phi^2)^{1/2} d\tau, \qquad \Delta_R^t R_\phi = \int_R^t (\mathrm{tr}\boldsymbol{\Omega}_\phi^2)^{1/2} d\tau \tag{2.3}$$

given by the integration in $(0, \mathbf{e}_i)$ of the history of the spin $\boldsymbol{\Omega}_\phi$. The thermomechanical spin $\boldsymbol{\Omega}_\mu$ is defined as equal to $\boldsymbol{\Omega}_\phi$ up to a scalar functional f_i of R_ϕ:

$$\boldsymbol{\Omega}_\mu = f\boldsymbol{\Omega}_\phi \tag{2.4}$$

where f is defined by taking as previously (Sec. 1.5.5, point 3c) the derivative

$$f_a(R_\phi/\omega) = \partial F_a[R_\phi/\omega]/\partial R_\phi \tag{2.5}$$

with respect to R_ϕ of the constitutive scalar functional of pure hysteresis type:

$$(2.6)\qquad \begin{aligned} &F_a(\Delta_R^t R_\phi) = -a_0[1 - \exp -(\Delta_R^t R_\phi / R_0)^2]\mathrm{th}(\Delta_R^t R_\phi / \omega R_0);\\ &0 < a_0 \ll 1; \qquad 0 < R_0 \ll 1; \qquad \omega = 1 \text{ or } 2, \end{aligned}$$

The resulting global spin components Ω^k in $(0, \mathbf{e}_a)$ of the PRF are such as

$$\Omega^k = \Omega_\delta^k(J_n, \gamma_n; \dot{J}_n, \dot{\gamma}_n; \ddot{J}_n, \ddot{\gamma}_n) + f[\Delta_R^t R_\phi]\Omega_\phi^k(J_n, \gamma_n; \dot{J}_n, \dot{\gamma}_n).$$

2.3.2. Reversibility ($\mathbf{S} = \mathbf{S}_r$) and homogeneity with respect to the reversible power. The approach is similar, but now f is defined as a function:

$$(2.7)\qquad f_r(R_\phi) = \partial F_r(R_\phi)/\partial R_\phi,$$

$$(2.8)\qquad F_r(R_\phi) = -a_0[1 - \exp -(R_\phi/R_0)^2]\mathrm{th}(R_\phi/R_0); \qquad 0 < a_0 \ll 1; \qquad 0 < R_0 \ll 1.$$

2.3.3. The coupled cases $\mathbf{S} = \mathbf{S}_r + \mathbf{S}_a$, or $\mathbf{S} = \mathbf{S}_v + \mathbf{S}_r$, or $\mathbf{S} = \mathbf{S}_v + \mathbf{S}_a$, or $\mathbf{S} = \mathbf{S}_v + \mathbf{S}_r + \mathbf{S}_a$, with associated "homogeneity" with respect to the inertia of the powers. In the first case the approach is similar ($F_a = F_r$) to that suggested in the Introduction (Sec. 1.3.5). But in the three cases including $\mathbf{S}_v$, the theory cannot provide reasonable physical arguments to choose between the two following types of definitions: firstly, a partial spin splitting up obtained through the amalgamation of $\mathbf{h}_r$ and $\mathbf{h}_v$ leading to the unique PRF of base vectors $\mathbf{h}_{rv}$; secondly, a perfect splitting leading to separate definitions of $\mathbf{h}_v$, $\mathbf{h}_r$ and $\mathbf{h}_a$. The partial splitting associated with the first definition may be defined as above. On the contrary, the total splitting associated with the second definition imply that the base vectors $\mathbf{h}_v$ are defined by a viscous pattern.

2.4. From the differential-difference equations and/or differential equations of the PRF to the constitutive equations in the fixed frame

i. The base vectors $\mathbf{h}_n$ of the PRF (in fact each type, $\mathbf{h}(a)_n$ and $\mathbf{h}(r\nu)_n$, of base vectors of the PRF(a) and of the PRF($r\nu$)) are obtained by the integration of the differential-difference system (cf. Eq. (2.7)) defined above (cf. Sec. 2.1.3).

ii. In $(0, \mathbf{e}_m)$:

$$(2.9)\qquad \begin{aligned} \dot{\mathbf{h}}_n &= \dot{H}_n^m \mathbf{e}_m = \Omega \wedge \mathbf{h}_n = \Omega \wedge (H_n^m \mathbf{e}_m) = [\Omega]_n^m \mathbf{e}_m\\ [\Omega]_n^l &= H_n^3 \Omega^2 - H_n^2 \Omega^3; \ldots; \quad H_1^1 = \cos\phi\cos\psi - \sin\psi\cos\theta\sin\phi; \ldots; \\ &\qquad F_3^3 = \cos\theta. \end{aligned}$$

The generalization of (1.11) results in intricate forms of the complementary deviatoric tensor $[\Sigma]$. It is indeed linear with respect to the components Σ^{kl} and Ω^i:

$$[\Sigma]^{11} = 2(\Omega^2\Sigma^{13} - \Omega^3\Sigma^{12}); ...; [\Sigma]^{23} = \Omega^1(\Sigma^{22} - \Sigma^{33}) -\Omega^2\Sigma^{12} + \Omega^3\Sigma^{13}; ...,$$

but Ω^i are given by (2.7).

2.5. Remark concerning the comparison between the PRF spin term and the usual spin terms

Instead of the constitutive definition such as:

$$[\partial S^{ij}/\partial t]\mathbf{h}_i \otimes \mathbf{h}_j \overset{\text{def}}{=} B^{ij}\mathbf{hr}_i \otimes \mathbf{hr}_j = \mathbf{B}$$

given in the relevant PRF, let us consider similar usual definitions of Oldroyd type, of Rivlin type and of Jaumann type:

$$(L\nu^{\cdot\cdot}S)_0 \overset{\text{def}}{=} [\partial S^{ij}/\partial t]\mathbf{g}_i \otimes \mathbf{g}_j \overset{\text{def}}{=} B^{ij}\mathbf{gr}_i \otimes \mathbf{gr}_j = \mathbf{B},$$

$$(L\nu_{..}S)_r \overset{\text{def}}{=} [\partial S_{ij}/\partial t]\mathbf{g}^i \otimes \mathbf{g}^j \overset{\text{def}}{=} B_{ij}\mathbf{gr}^i \otimes \mathbf{gr}^j = \mathbf{B},$$

$$(L\nu^{\cdot\cdot}S)_j \overset{\text{def}}{=} (1/2)[(\partial S^{ij}/\partial t)\mathbf{g}_i \otimes \mathbf{g}_j + (\partial S_{ij}/\partial t)\mathbf{g}^i \otimes \mathbf{g}^j] \overset{\text{def}}{=} B^{ij}\mathbf{gr}_i \otimes \mathbf{gr}_j = \mathbf{B}.$$

In $(0, \mathbf{e}_i)$ the constitutive definition involves the complementary terms of Oldroyd, Rivlin and Jaumann types:

$$\begin{aligned}
[\Sigma]_o &= S^{ij}(\Sigma^{kl}_{re})(\dot{\mathbf{g}}_i \otimes \mathbf{g}_j + \mathbf{g}_i \otimes \dot{\mathbf{g}}_j),\\
[\Sigma]_r &= S_{ij}(\Sigma^{kl}_{re})(\dot{\mathbf{g}}^i \otimes \mathbf{g}^j + \mathbf{g}^i \otimes \dot{\mathbf{g}}^j),\\
[\Sigma]_j &= (1/2)([\Sigma]_o + [\Sigma]_r),
\end{aligned}$$

respectively, instead of

$$[\Sigma]_{\text{PRF}} = S^{ij}(\Sigma^{kl})(\dot{\mathbf{h}}_i \otimes \mathbf{h}_j + \mathbf{h}_i \otimes \dot{\mathbf{h}}_j).$$

The linearity with respect to the stress components Σ^{kl}_r in $(0, \mathbf{e}_i)$ is a feature the four constitutive definitions have in common. But the main difference between the PRF definition and the usual ones lies in the fact that the PRF definition is not directly linked to the kinematics. A constitutive ingredient may be involved through the definition of a *vectorial* function or functional of the power-inertia spin Ω_ϕ. Moreover, *it is even possible to make use of a scalar* function or functional, if one addmits, *like in this paper*, that the orientation of Ω_ϕ needs not to be modified. The point deserves thinking about with the aid of a formally simple illustration. In the special case of the one-shear kinematics, for example, the explicit forms of the $[\Sigma\ ]$ tensors are:

$$[\Sigma]_o = (2\Sigma_1 D_1)(\mathbf{e}_1 \otimes \mathbf{e}_1) + (2\Sigma_2 D_2 + 4\Sigma_4 D_4)(\mathbf{e}_2 \otimes \mathbf{e}_2)$$
$$+(2\Sigma_3 D_3)(\mathbf{e}_3 \otimes \mathbf{e}_3) + [2\Sigma_3 D_4 + \Sigma_3(D_2 + D_3)](\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2),$$
$$[\Sigma]_r = (-2\Sigma_1 D_1)(\mathbf{e}_1 \otimes \mathbf{e}_1) + (-2\Sigma_2 D_2)(\mathbf{e}_2 \otimes \mathbf{e}_2)$$
$$+(-2\Sigma_3 D_3 - 4\Sigma_4 D_4)(\mathbf{e}_3 \otimes \mathbf{e}_3) + [-2\Sigma_2 D_4$$
$$-\Sigma_3(D_2 + D_3)](\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2),$$
$$[\Sigma]_{\mathrm{PRF}} = (\Sigma_4 \dot{\alpha})(\mathbf{e}_2 \otimes \mathbf{e}_2) + (-\Sigma_4 \dot{\alpha})(\mathbf{e}_3 \otimes \mathbf{e}_3)$$
$$+((\Sigma_3 - \Sigma_2)\dot{\alpha}(\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2).$$

If $D_4 = (\dot{\gamma}_3 + \dot{\gamma}_3)D_2/J_3$ may be reduced to $\dot{\gamma}_3$, the components of $[\Sigma]_j$ are factorised by $\dot{\gamma}_3$, but that of $[\Sigma]_{\mathrm{PRF}}$ are factorised by a function or functional, as it has been stressed in Sec. 1.3.5.

3. Remarks on the definitions of the three basic sets of thermomechanical rates, of viscous type, of reversible type and of elastoplastic type, respectively

3.1. The viscous and reversible stress contributions

i. An Oldroyd-like viscous stress contribution is defined through the usual covariant pattern:

$$\mathbf{S}_\nu = \mathbf{S}_1 + \mathbf{S}_2; \qquad \mathbf{S}_2 = (\theta_2\lambda_2)I_D\boldsymbol{\delta} + 2(\theta_2\mu_2)\mathbf{D} = \eta_{02}I_D\boldsymbol{\delta} + 2\eta_{\delta 2}\mathbf{D}$$

associated with the unusual rate form:

$$[\partial S_1^{ij}/\partial t]\mathbf{hv}_i \otimes \mathbf{hv}_j = (1/\theta_\nu)[-S_1^{ij} + \eta_{01}I_D\delta^{ij} + 2\eta_{\delta 1}D^{ij}]\mathbf{hv}_i \otimes \mathbf{hv}_j$$

defined in the relevant PRF. Eliminating S_1 and S_2 one obtains the global form:

$$[\partial S_\nu^{ij}/\partial t] - \partial(\eta_{02}I_D\delta^{ij} + 2\eta_{\delta 2}D^{ij})/\partial t]\mathbf{hv}_i \otimes \mathbf{hv}_j$$
$$= (1/\theta_\nu)] - S_\nu^{ij} + (\eta_{01} + \eta_{02})D^{ij}]\mathbf{hv}_i \otimes \mathbf{hv}_j$$

and the associated splitting into isotropic and deviatoric forms:

$$\theta_\nu \dot{I}_\nu + I_\nu = [3(\eta_{01} + \eta_{02}) + 2(\eta_{\delta 1} + \eta_{\delta 2})I_D + \theta_\nu(3\eta_{02} + 2\eta_{\delta 2})\dot{I}_D,$$

$$\left[\theta_\nu \dot{\overline{S}}_\nu^{ij} + \overline{S}_\nu^{ij}\right] \mathbf{hv}_i \otimes \mathbf{hv}_j = \left[2(\eta_{\delta 1} + \eta_{\delta 2})\overline{D}^{ij} + \theta_\nu 2\eta_{\delta 2}\dot{\overline{D}}^{ij}\right] \mathbf{hv}_i \otimes \mathbf{hv}_j .$$

The thermomechanical rates have been already given (Sec. 1.2). Owing to the fact that the definition of the PRF $(M(x^k), \mathbf{hv}_i(M(x^k,t))$ is obtained through integration of an angular velocity with respect to the *fixed frame* $(0, \mathbf{e}_i)$, the only possible interesting invariance of the above constitutive pattern is that under *constant* rotation of the co-ordinates system z^k referred to the *fixed material reference frame* associated with $(0, \mathbf{e}_i)$.

Moreover, the boundary conditions are specified in the fixed frame $(0, \mathbf{e}_i)$ of the experimental machinery, whatever may be the underlying clue of the strategy of the loading. This specified program, actually "defined" in $(0, \mathbf{e}_i)$, is endowed with the above invariance.

The set [constitutive equations ; boundary conditions] is then endowed with the invariance introduced by W.A. FOCK [13].

ii. The simplest differential definition of the generalized Hooke law is:

$$\dot{S}_r^{ij} \mathbf{hr}_i \otimes \mathbf{hr}_j = (\lambda_r I_D \delta^{ij} + 2\mu_r D^{ij}) \mathbf{hr}_i \otimes \mathbf{hr}_j; \qquad \dot{I}_r = (3\lambda_r + 2\mu_r) I_D;$$

$$\dot{\overline{S}}_r \mathbf{hr}_i \otimes \mathbf{hr}_j = 2\mu_r \overline{\mathbf{D}}$$

In order to obtain a diversity of qualitative simulations, it may be supplemented with:

$$\dot{S}_r^{ij} \mathbf{hr}_i \otimes \mathbf{hr}_j = [[\lambda_r + (2\mu_r/3)(1 - (Q/Q_0)^n)] I_D \delta^{ij} + 2\mu_r (Q/Q_0)^n D^{ij}] \mathbf{hr}_i \otimes \mathbf{hr}_j,$$

$$\dot{I}_r = (3\lambda_r + 2\mu_r) I_D; \qquad \dot{\overline{S}}_r^{ij} \mathbf{hr}_i \otimes \mathbf{hr}_j = 2\mu_r (Q/Q_0)^n \overline{\mathbf{D}},$$

$$Q^2 = \mathrm{tr}[(\overline{\Delta}_R^t \overline{\varepsilon})^2]; \qquad \overline{\Delta}_R^t \overline{\varepsilon} = \Delta_R^t \varepsilon - (1/3)\mathbf{G}\,\mathrm{tr}(\Delta_R^t \varepsilon);$$

$$2\Delta_R^t \varepsilon = \mathbf{G} -_R^t \mathbf{G},$$

for example, or with

$$\dot{S}_r^{ij} \mathbf{hr}_i \otimes \mathbf{hr}_j = \left(\lambda_r I_D \delta^{ij} + 2\mu_r D^{ij} + \overline{\beta}_r \overline{P}_r \overline{S}_R^{ij}\right) \mathbf{hr} \otimes \mathbf{hr}_j$$

$$\Rightarrow \dot{I}_r = (3\lambda_r + 2\mu_r) I_D;$$

$$\dot{\overline{S}}_r^{ij} \mathbf{hr}_i \otimes \mathbf{hr}_j = 2\mu_r \dot{\mathbf{D}} + \overline{\beta}_r \overline{P}_r \overline{\mathbf{S}}_r; \qquad \overline{P}_r = \mathrm{tr}(\overline{\mathbf{S}}_r \overline{\mathbf{D}});$$

$$\overline{\beta}_r = 2\mu_r / Q_0^2.$$

in the case where bounded stress states have to be simulated. The last form may be easily modified in order to obtain a classical "three stages" behaviour along the first loading paths. The thermomechanical rates have been already given (Sec. 1.3.1). Concerning the question of invariance, the useful remarks are similar to those given above.

3.2. The pure hysteresis stress contribution

Some rather detailed analysis of the formal features of the tensorial pattern have been introduced recurrently (cf. particularly [5, 6] and [7]).

The basic features which are to be considered here are the following:

i. The definition is *built* in the Iliushin space, making use of the components in the PRF of the relevant tensors.

ii. For each stress (or strain) path it gives the corresponding strain (or stress) path.

iii. To a fixed rotation of the co-ordinate system referred to the testing machinery, is associated a rotation in the Iliushin space. Consequently, the question of invariance arises as stated above.

3.3. Remark concerning the boundary conditions specified in the fixed frame in order to study a pattern defined in the PRF

Regarding the complexity of the loading processes it is worth noting that the constitutive patterns defined in PRF do not result in more intricate problems than those following classical approaches. Let us consider an example of one-shear test, namely that performed in order to obtain, in the stress space, a deviatoric path along the circle: $I_s = Q_s - Q_0 = 0$. The path definition involving an invariant, one can make use of the components in the PRF or of the components in the fixed frame: the representations are identical. The complexity of the control stems from the classical geometrical non-linearity associated with finite strains: the rotation of the PRF is not involved.

Let us now consider a much more intricate case, namely the case of a path defined not only by relations which are invariant in the ordinary sense (such as $I_s = 0$, for example) but also by non-invariant relations between components in the PRF. In such a case, the rotation of the PRF is explicitly involved in order to obtain the relations of the above components with the components in the fixed frame and finally, with the forces on the facets of the material point. The complexity of the problem is of a standard level: well-founded approximations of the spin forms are useful. The last issue is all the more important that the strains are large because the classical geometrical nonlinearity is able to hide the genuine constitutive features. In the one-shear case, for example, the forces on the faces $n (n = 1, 2, 3)$ of the "cube" of dimensions $0 \leqslant x^k \leqslant 1$, $k = 1, 2, 3$,

$$\mathbf{F}^{(n)} = \mathbf{g}^i S_i^j \sqrt{g} \nu_j^{(n)} = \mathbf{g}^i S_i^j \sqrt{g} \delta_j^n,$$

the associated forces per unit area $\mathbf{f}^{(n)}$, and the normal and tangential components $p^{(n)}$ and $t^{(n)}$associated with the forces $\mathbf{f}^{(n)}$are indeed connected with the mixed stress components S_i^j in $(M, \mathbf{g}^i, \mathbf{g}_j)$, and with the stress components Σ_{kl} in $(0, \mathbf{e}_k)$ through the following equations:

$$\begin{aligned}
\mathbf{f}^{(1)} &= \mathbf{F}^{(1)}/J_2 J_3 = \mathbf{g}^1 S_1^1 J_1; \qquad p^{(1)} = S_1^1 = \Sigma_{11}; \qquad t^{(1)} = 0,\\
\mathbf{f}^{(3)} &= \mathbf{F}^{(3)}/J_1 J_2 = \mathbf{g}^2 S_2^3 J_3 + \mathbf{g}^3 S_3^3 J_3;\\
p^{(3)} &= S_3^3 - (2\gamma/J_2) S_2^3 = \Sigma_{33}; \qquad t^{(3)} = S_2^3 J_3/J_2 = \Sigma_{23},\\
\mathbf{f}^{(2)} &= \mathbf{F}^{(2)}/J_1\sqrt{J_3^2 + 4\gamma^2} = (\mathbf{g}^2 S_2^2 + \mathbf{g}^3 S_3^2)(J_2 J_3/\sqrt{J_3^2 + 4\gamma^2}),\\
p^{(2)} &= S_2^2 - [(2\gamma J_2)/(J_3^2 + 4\gamma^2)] S_3^2; \qquad t^{(2)} = S_3^2 [J_3 J_2/(J_3^2 + 4\gamma^2)],\\
S_2^2 &= \Sigma_{22} - \Sigma_{23}(2\gamma/J_3); \qquad S_3^3 = \Sigma_{33} + \Sigma_{23}(2\gamma/J_3); \qquad S_2^3 = \Sigma_{23} J_2/J_3,\\
S_3^2 &= (\Sigma_{22} - \Sigma_{33})((2\gamma/J_3) + \Sigma_{23}[(J_3^2 - 4\gamma^2)/J_2 J_3].
\end{aligned}$$

The geometry is involved (with $2\gamma/J_3$) in the relationships between $\Sigma_{22}, \Sigma_{33}, \Sigma_{23}, p^{(2)}$ and $t^{(2)}$, resulting in a noticeable experimental difficulty if one tries to perform accurately a path selected in the stress space of Iliushin associated with the stress definition in PRF. For the facets $n = 2$, intricate rate-form equation are involved.

4. Concluding remarks

A number of numerical simulations has been performed. It may be useful to give a brief account of the features of this study and of the results from now on available.

4.1. Features of a short span numerical study of homogeneous problems

i. The numerical study has been restricted as follows.

In order to suppress the effect of hardening one considers the pure hysteresis case such as $\mathbf{S} = \mathbf{S}_a$. Moreover, the global form of the pattern is strongly simplified, firstly because only its deviatoric part is of pure hysteresis form (its isotropic part being reversible), secondly because the plastic limit is as simple as possible, namely of Huber – von Mises type. The physical parameters are always $\lambda = 2\mu = 150$ GPa, $S_0 = 200$ MPa for the first order pattern, and $\alpha_0 = S_0/2\mu$, $R_0 = 4S_0/\mu$ concerning the definition of the PRF. Regarding the kinematics, only the one-shear case is implemented. Generally, the loading programmes are either such as $J_1 = J_2 = J_3 = 1$ or such as $S_1 = S_2 = 0$.

ii. All the loading programmes have been two-fold:

a) firstly, a preliminary irrotational loading is specified through $\gamma_3 = 0$ and Σ_3 increasing from zero towards a small fraction of $S_0 : \Sigma_3 = \pm S_0/n$ (in fact, $n = 2$, or 10, or 20, or 1000, or 10000);

b) secondly, a cyclic loading is performed (under the constant axial stress $\Sigma_3 = \pm S_0/n$ previously reached) through a cyclic shear $\gamma_3(\tau)$ on constant intervals, "symmetrical" or "not symmetrical" with respect to the origin of the shear axis.

iii. In order to compare the *thermomechanical* definition of the thermomechanical spin Ω_ϕ with an intuitive definition, each simulation has been performed two times, making use first of (1.11) and then of (1.16).

iv. The numerical study being not devoted to the identification of a well-specified actual material, a last simplification has been implemented trough the assumption $\dot{\alpha}_\delta = 0$, an assumption which is exactly verified only in the case $J_1 = J_2 = J_3 = 1$.

v. The case of loading involving "nonsymmetrical" cycles has been studied also under the stress control, namely under the conditions $S_1 = S_2 = 0$, and for the small traction $S_3 = S_0/10000$. Simulations have been performed for several values of the axial stress S_3 (for a unique set of cycles). The result is as follows: for large axial traction $S_3(S_0/2, S_0/10$, for example), the two definitions of the spin give the same ratchet, but for small $S_3(10^{-3}S_0$ or $10^{-4}S_0$, for example) the intuitive definition of the spin results in irrelevant effects.

vi. The same conclusion has been obtained in the case of a special stress control, of tri-traction type, which may be a priori able to reduce a ratchet effect existing under small axial traction (and even under small axial compression, of the order of $-10^{-4}S_0$). The stress control $S_1 = S_2 = S_0/100$, $S_3 = 10^{-4}S_0$ has been studied only in the case of a unique set of "symmetrical" cycles. The intuitive definition of the spin results in a "ratchet" effect of the first order magnitude, which is irrelevant to experimental evidence.

4.2. Provisional results

i. With the aid of the proposed pattern, the ratchet effect is hardening-independent and existing under zero axial load, or even under a small compression, in accordance with experimental evidence [14, 15].

ii. Saturation of the ratchet effect may be obtained after a suitable number of cycles (50 cycles; $\Sigma_3 = S_0/10$, cf. Fig. 6).

iii. In order to define the thermomechanical spin Ω_ϕ, a *thermomechanical* approach has proved to be effective up to now. Moreover, it allows to avoid the implementation of intricate approaches extracted from a general tensorial formalism, or the implementation of non-effective approaches based on some intuitive sketches.

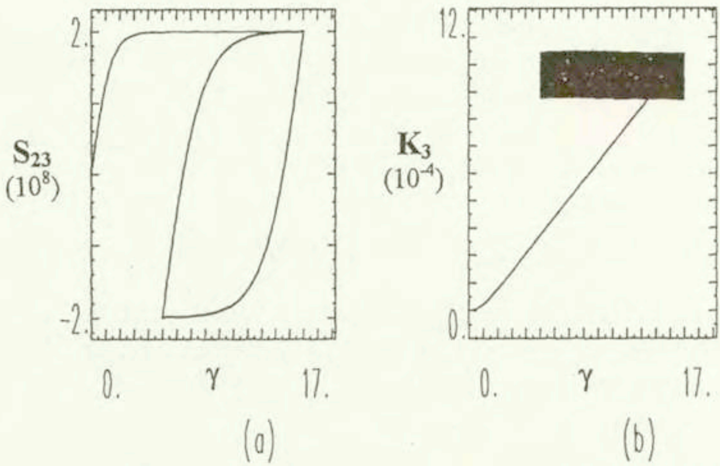

FIG. 6. Saturation of the ratchet after 50 cycles (initial traction $S_3 = S_0/10$).

4.3. Remark

One of the main gaps in the study is that concerning the features of the ratchet effect during demagnetization-like processes [7]. Some hints are already obtained, but this interesting issue has not been studied as yet (following, for example, the method proposed in [7]).

References

1. P. GUELIN, *Remarques sur l'hysteresis mécanique*, J. Mec. Th. et Appl., **19**, 2, 217–247, 1980.
2. P. GUELIN, W.K. NOWACKI and P. PEGON, *Etude des schémas thermomécaniques á mémoire discréte*, Arch. Mech. 37, 343–363, 1985.
3. P. PEGON and P. GUELIN, *On thermomechanical Zaremba schemes of hysteresis*, Res. Mech. Letters, 21–34, 1987.
4. D. FAVIER, P. GUELIN, A. TOURABI, B. WACK and P. PEGON, *Ecrouissages-Schémas thermomécaniques et á variables internes*, Arch. Mech., **40**, 5-6, 611–640, 1988.
5. P. PEGON, P. GUELIN and D. FAVIER, B. WACK and W.K. NOWACKI, *Constitutive scheme of discrete memory form for granular materials*, Arch. Mech., **43**, 1, 3–27, 1991.
6. A. TOURABI, P. GUELIN and D. FAVIER, *Towards the modelling of deformable ferromagnetics and ferroelectrics*, Arch. Mech., **47**, 3, 437–483, 1995.
7. A. TOURABI, P. GUELIN, D. FAVIER and W.K. NOWACKI, *From material discrete memory patterns to the study of demagnetization-like processes*, Arch. Mech., **49**, 4, 737–766, 1997.
8. A. TOURABI, P. GUELIN, B. WACK, P. PEGON, D. FAVIER and W.K. NOWACKI, *Useful scalar parameters for multiaxial fatigue deformations studies*, Fatigue Fract. Engng. Mater. Struct., 19-10, 1181–1195, 1996.

9. E. MADELUNG, *Über Magnetisierung durch Schnellverlaufende Ströme und die Wirkungweise des Rutherford-Marconischen Magnetdetektors*, Annalen der Physik, **17**, 861–890, 1905.

10. M. KLEIBER, *On errors inherent in commonly accepted rate forms of elastic constitutive law*, Arch. Mech., **38**, 271–279, 1986.

11. D. JEULIN, *Random models for the morphological analysis of powders*, Journal of Microscopy, **172**, 1, 13–21, 1993.

12. J. SERRA, *Image analysis and mathematical morphology*, Academic Press, London 1982.

13. W.A. FOCK, *Les principes physiques de la théorie de la gravitation d'Einstein*, Annales de l'Institut Henri Poincaré, **5**, 3, 205–215, 1966.

14. G. BLES, S.P. GADAJ, P. GUELIN, W.K. NOWACKI and A. TOURABI, *Thermomechanics of viscoplastic large strains and isotropic patterns of solid polymers*, submitted to Arch. Mech., Warsaw 1999.

15. A.S. LODGE, *Body tensor fields in continuum mechanics with applications to polymer rheology*, Academic Press, 1974.

16. P. PEGON and P. GUELIN, *Etude des schémas thermomécaniques á mémoire discréte: problémes aux limites en grandes déformations elastoplastiques*, Arch. Mech. **37**, 4-5, 465–483, 1985.

17. P. PEGON and P. GUELIN, *Finite strain plasticity in dragged along frames*, Int. J. Numer. Meth. in Eng., **39**, 245–285, 1983.

18. P. PEGON and P. GUELIN, *Implicit finite strain analysis in dragged along frames*, 5th Int. Symp. on Numerical Methods in Engineering, Lausanne 1989.

19. P. PEGON, W.K. NOWACKI, P. GUELIN, D. FAVIER and B. WACK, *From the modelling of granular media to the numerical study of the modes of bifurcations*, 2nd Int. Workshop on Num. Meth. for localization and bifurcation of granular bodies, 141–159, Eds. Dembicki, Gudehus, Sikora, Gdańsk 1989.

20. G. RIO, P.Y. MANACH and D. FAVIER, *Finite element simulation of shell and 3D mechanical behaviour of NiTi shape memory alloys*, Arch. Mech., **47**, 3, 537–556, 1995.

21. P.Y. MANACH, D. FAVIER and G. RIO, *Finite element simulations of internal stresses generated during the pseudoelastic deformation of NiTi bodies*, J. de Physique, **C1**, 6, 253–244, 1996.

22. T. INOUE, *Benchmark project on inelastic deformation and life prediction of 21/4 Cr-1 mo steel under multiaxial stresses*, Trans. 10th Int. Conf. on S.M.i.R.T., 141–152, 1989.

23. N. OHNO and J.-D. WANG, *Kinematic hardening rules with critical state of dynamic recovery, Parts* I *an* II. Int. J. of Plasticity, **9**, 375–403, 1993.

24. N. OHNO, *Current state of the art in constitutive modeling for ratchetting*, Trans. 14th Int. Conf. on S.M.i.R.T., LW/1, 201–212, 1997.

25. Z. XIA and F. ELLYIN, *A constitutive model with capability to simulate complex multiaxial ratchetting behaviour of materials*, Int. J. of Plasticity, **13**, 14/2, 127–142, 1997.

26. L. BRILLOUIN, *Qu'entend-on par systéme de référence en mécanique?* Mémoires de la Section des Sciences, Publications du Comité des Travaux Historiques et Scientifiques, Bibliothéque Nationale, Paris 1968.

27. W. ŚLEBODZIŃSKI, *Sur les Equations de Hamilton*, Bull. Acad. Roy. De Belg., **17**, 864–870, 1931.

28. M. ARIB and J.L. ILLE, *Toward an energetic model of ferromagnetic hysteresis*, Int. J. of Applied Electromagnetics and Mechanics, **8**(3), 273–284, 1997.

29. M. Arib and J.L. Ille, *Magnetoelastic couplings in ferromagnetic solids*, Int. J. of Applied Electromagnetics and Mechanics, **8**(4), 329–341, 1997.

30. M. Arib, *Contribution á l'étude des interactions magnéto-mécaniques dans les ferromagnétiques solides*, Ph D., Univ. of Compiégne 1998.

31. I.D. Mayergoyz, *Mathematical models of hysteresis*, Springer, 1991.

32. I.D. Mayergoyz and G. Friedman, *Generalized preisach model of hysteresis*, I.E.E.E. Trans. Magn. 24, 212–217, 1988.

33. A. Visintin, *Differential Models of Hysteresis*, Springer, 1994.

34. M. Brokate, K. Dressler and P. Krejci, *The Mroz model: a hysteresis operator for rate-independent plasticity*, Proc. of World Congress of Nonlinear Analysis, Tampa 1992.

35. M. Brokate, N. Kenmochi, I. Muller, J.F. Rodriguez and C. Verdi, Phase transitions and hysteresis, Springer-Verlag, 1994.

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