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Frequency dependence on space-time for electromagnetic propagation in dispersive medium

A. CIARKOWSKI

Polish Academy of Sciences
Institute of Fundamental Technological Research
Świętokrzyska 21, 00-049 Warszawa, Poland

PROPAGATION OF AN ELECTROMAGNETIC HIGH-FREQUENCY modulated signal in a lossy dispersive Lorentz medium is reconsidered. A new approximate solution is found to the equation relating complex frequency of the propagating signal and the space-time point. This solution is compared numerically with the solution to the saddle point equation in the special case of the Brillouin medium parameters.

1. Introduction

INTEREST IN PROPAGATION OF ELECTROMAGNETIC SIGNAL through dispersive media dates back to early 1900's. Fundamental works in this field are due to SOMMERFELD [1] and BRILLOUIN [2, 3]. They studied the evolution of a signal propagating in a Lorenz medium that accurately models some real lossy media. On the grounds of asymptotic considerations, the authors showed that the main change in the form of an electromagnetic signal propagating in this medium takes place at the initial stage of propagation, while at higher penetration depths the pulse form is almost unchanged. They revealed that two different precursors precede the steady state signal. At every space-time point, the precursors' speed of oscillations and their dumping are functions of complex frequencies, which in turn are determined by the locations of saddle points in the integral defining the signal. The works of Sommerfeld and Brillouin are very important because they gave insight into the mechanism of electromagnetic pulse evolution as it propagates in a dispersive medium, and they provided approximate analytic description of this evolution in the mature dispersion regime.

More recently, OUGHSTUN and SHERMAN, in a series of papers [4, 5, 6, 7, 8, 9, 10], reexamined the classical theory of Sommerfeld and Brillouin. Equipped with modern asymptotic methods and access to powerful computers, they were able to improve the classical asymptotic results by extending them to special instances of the pulse dynamics, and to provide better understanding of physical phenomena associated with propagation of various signals in a dispersive material.

In short terms, asymptotic analysis of a signal propagation in the dispersive medium reduces to asymptotic evaluation of a contour integral that describes the signal dynamics in the medium. A key step in this approach is to find, at each space point and at each time instant, the complex frequencies that determine the way in which particular parts of the signal oscillate and are dumped in the medium. These frequencies are derived from the equation which determines the locations of saddle points corresponding to the phase function in the integral. Because of the complex form of the equation, it seems to be impossible to solve the equation exactly. Brillouin found four families of saddle points in the complex frequency plane and obtained approximate analytical formulas describing them. Those formulas fail to hold in some instances of pulse evolution. New, better approximation to these frequencies was found by OUGHSTUN and SHERMAN in [4].

In this paper we obtain still other approximate formulas for the complex frequencies that govern the dynamics of the signal propagating in the dispersive medium. By numerical comparison we show that in vast range of the space and time coordinates, our approximation surpasses those obtained by Brillouin and Oughstun and Sherman.

2. Formulation of the problem

Motivated by the works of SOMMERFELD [1], BRILLOUIN [2, 3] and OUGHSTUN and SHERMAN [4], we reconsider the problem of propagation of an arbitrary electromagnetic signal in a linear, homogeneous, isotropic, temporary dispersive medium filling the half-space $z > 0$. Let $A(z,t)$ represent any scalar component of the electromagnetic field or its potential. Assume further that

$$A(0,t) = f(t)$$

is known for all time t, where $f(t)$ is zero for $t < 0$. Then, (see OUGHSTUN and SHERMAN [4]),

$$A(z,t) = \frac{1}{2\pi} \int\limits_C \tilde{f}(\omega) \exp\left[\frac{z}{c}\phi(\omega,\theta)\right] d\omega, \tag{2.1}$$

where

$$\tilde{f}(\omega) = \int\limits_0^\infty f(t) e^{-i\omega t} d\omega$$

is the Laplace transform of the signal evolution at the plane $z = 0$,

$$\phi(\omega,\theta) = i\frac{c}{z}\left[\tilde{k}(\omega)z - \omega t\right] = i\omega[n(\omega) - \theta]$$

is the phase function, and

$$\theta = \frac{ct}{z}$$

is a dimensionless parameter corresponding to a space-time point. The quantity $n(\omega)$ stands for the complex index of refraction of the medium and in the Lorentz model it is equal to

$$n(\omega) = \left(1 - \frac{b^2}{\omega^2 - \omega_0^2 + 2i\delta\omega}\right)^{1/2}. \tag{2.2}$$

Here, b, ω_0 and δ are medium parameters ([12]). Brillouin chose the following numerical values for those parameters:

$$b^2 = 20.0 \times 10^{32}\ s^{-1}, \qquad \omega_0 = 4.0 \times 10^{16}\ s^{-1}, \qquad \delta = 0.28 \times 10^{16}\ s^{-1},$$

which are characteristic of a dispersive, lossy medium. His choice has become a standard in works by many other authors.

Asymptotic evaluation of the integral in (2.1) for large values of the phase in the integrand requires that the saddle points of $\phi(\omega, \theta)$ are known. At those points the phase function is stationary, i.e. it satisfies the equation $\phi'(\omega, \theta) = 0$, or equivalently,

$$n(\omega) + \omega n'(\omega) - \theta = 0. \tag{2.3}$$

By using explicit form of $n(\omega)$ in this equation one arrives at

$$\left[\omega^2 - \omega_1^2 + 2i\delta\omega + \frac{b^2\omega(\omega + i\delta)}{\omega^2 - \omega_0^2 + 2i\delta\omega}\right]^2 = \theta^2(\omega^2 - \omega_1^2 + 2i\delta\omega)(\omega^2 - \omega_0^2 + 2i\delta\omega), \tag{2.4}$$

where $\omega_1^2 = \omega_0^2 + b^2$. This equation determines the exact locations of saddle points in the ω-complex plane. It does not seem possible to solve this equation exactly. Instead, approximate solutions were constructed (BRILLOUIN [2, 3], OUGHSTUN and SHERMAN [4]).

In this paper we find still another approximation to the function relating the location of the saddle points to particular values of θ. Below, we summarize the results known in the literature and present the new approximate solution to Eq. (2.3).

3. Approximate solutions to the saddle point equation

3.1. Brillouin's solution

Brillouin's study of the saddle-point Eq. (2.3) led to the conclusion that for each θ there are four distinct saddle points, situated symmetrically with respect to the imaginary axis in the ω complex plane. Two of the four simple saddle points, pertaining to the Brillouin precursor in the signal, move along the imaginary axis as θ increases from 1 to θ_1, one downwards (crossing the real axis) and the other upwards, and coalesce into one second-order saddle point at $\theta = \theta_1$. When θ increases any further, this saddle point divides into two simple saddle points which detach from the axis and move outwards, approaching in the limit, as $\omega \to \infty$, the respective points ω_+ and ω_-. The limiting points are located in the third and the fourth quadrant, respectively, of the complex frequency plane and are defined by

$$\omega_{\pm} = \pm(\omega_0^2 - \delta^2)^{1/2} - i\delta. \tag{3.1}$$

For any $\theta \geqslant 1$ these saddle points vary in the regions $|\omega| \leqslant |\omega_0|$. Therefore in the literature they are referred to as the near saddle points.

The remaining simple saddle points appear in the lower ω half-plane and are related to the Sommerfeld precursor in the signal. As θ increases from 1 to infinity, those points move inwards from $+\infty - 2i\delta$ to ω'_+ and from $-\infty - 2i\delta$ to ω'_-, respectively, where

$$\omega'_{\pm} = \pm(\omega_1^2 - \delta^2)^{1/2} - i\delta. \tag{3.2}$$

Since they vary in the regions $|\omega| \geqslant |\omega_1|$, they are referred to as the distant saddle points.

The points $\omega_{\pm}$ and $\omega'_{\pm}$ are the branch points of $n(\omega)$. In particular, if $\omega = \omega_{\pm}$,

$$\omega^2 - \omega_0^2 - 2i\delta\omega = 0 \tag{3.3}$$

and if $\omega = \omega'_{\pm}$,

$$\omega^2 - \omega_1^2 - 2i\delta\omega = 0. \tag{3.4}$$

In the region $|\omega| \geqslant |\omega_1|$ Brillouin approximated the complex index of refraction by

$$n(\omega) \approx 1 - \frac{b^2}{2\omega(\omega + 2i\delta)}. \tag{3.5}$$

By substituting this approximation into Eq. (2.3) and solving for ω, he obtained the following approximate formula for the distant saddle points:

$$\omega^{\pm}_{SP_D}(\theta) \approx \pm\frac{b}{[2(\theta - 1)]^{1/2}} - 2i\delta. \tag{3.6}$$

According to this formula, when $\theta = 1$, the distant saddle points lie at infinity, their locations being given by $\pm\infty - 2i\delta$. As θ increases to infinity, this formula incorrectly states that they move towards the imaginary ω axis along the line

$\omega = -2i\delta$. As a consequence, Eq. (3.6) does not provide a good approximation for the distant saddle point locations in the vicinity of the limiting points $\omega'_{\pm}$.

In the region $|\omega| \leqslant |\omega_0|$ Brillouin adopted the following approximation for the complex index of refraction:

$$n(\omega) \approx \frac{\omega_1}{\omega_0} + \frac{b^2}{2\omega_1\omega_0^3}\omega(\omega + 2i\delta) - \frac{\delta^2 b^2(4\omega_1^2 - b^2)}{2\omega_1^3\omega_0^5}\omega^2. \tag{3.7}$$

(In fact, the final term in this expression was appended by OUGHSTUN and SHERMAN [4] to assure that the approximation is correct to ω^2). With (3.7) used in Eq. (2.3) its approximate solutions are

$$\omega^{\pm}_{SP_N} \approx \pm\frac{1}{3}\left[6\frac{\theta_0\omega_0^4}{\alpha b^2}(\theta - \theta_0) - 4\frac{\delta^2}{\alpha^2}\right]^{1/2} - i\frac{2\delta}{3\alpha}, \tag{3.8}$$

where

$$\alpha = 1 - \frac{\delta^2(4\omega_1^2 - b^2)}{\omega_0^2\omega_1^2}, \tag{3.9}$$

$$\theta_0 = \frac{\omega_1}{\omega_0} = (1 + b^2/\omega_0^2)^{1/2}. \tag{3.10}$$

According to (3.8) there are two near saddle points which, as θ increases from 1 to θ_1, move towards each other along the imaginary ω axis. The "upper" saddle point crosses the real ω axis at $\theta = \theta_0$. Both saddle points coalesce into a saddle point of the order 2 at $\theta = \theta_1 > \theta_0$. The value of θ_1 follows from equating the square root in (3.8) to zero, which yields

$$\theta_1 = \theta_0 + \frac{2\delta^2 b^2}{3\alpha\theta_0\omega_0^2}. \tag{3.11}$$

The corresponding location of the saddle point of the order 2 is by (3.8) equal to $\omega = -i2\delta/3\alpha$.

For higher values of θ the formula (3.8) incorrectly states that two simple saddle points move off the imaginary ω axis towards infinity along the line $\omega = -i2\delta/3\alpha$. In particular, it does not predict that the points tend to $\omega^{\pm}_{SP_N}$ as $\theta \to \infty$.

3.2. Oughstun and Sherman's solution of the saddle point equation

The starting point in Oughstun and Sherman's approach was the Eq. (2.4). Crucial step in their procedure consisted in expanding the rational term in the square bracket into a sum of integer powers of ω (positive or negative, depending on which region in the complex ω plane was under consideration).

In particular, for the saddle points varying in the distant regions $|\omega| \geqslant |\omega_1|$, the terms of the order w^{-2} and higher were dropped from the expansion of the rational function as $\omega \to \infty$. With that approximation the saddle point equation Eq. (2.4)) was reduced to a cubic equation which could be solved exactly. Appropriate solutions of that equation were further simplified to give finally

$$\omega^{\pm}_{SP_D}(\theta) \cong \pm\xi(\theta) - i\delta[1 + \eta(\theta)], \tag{3.12}$$

where

$$\xi(\theta) = \left(\omega_0^2 - \delta^2 + \frac{b^2\theta^2}{\theta^2 - 1}\right)^{1/2}, \tag{3.13}$$

$$\eta(\theta) = \frac{\delta^2/27 + b^2/(\theta^2 - 1)}{\xi^2(\theta)}. \tag{3.14}$$

If θ is close to unity, $\xi(\theta) \cong b/[2(\theta-1)]^{1/2}$, $\eta(\theta) \cong 1$, and thus the approximation (3.6) is recovered. If, on the other hand, θ is large, $\xi(\theta) \cong (\omega_1^2 - \delta^2)^{1/2}$, $\eta(\theta) \cong \delta^2/[27(\omega_1^2 - \delta^2)] \cong 0$, and hence

$$\omega_{SP_D}(\theta) \cong \pm(\omega_1^2 - \delta^2)^{1/2} - \delta i = \omega'_{\pm} \quad \text{as} \quad \theta \to \infty, \tag{3.15}$$

i.e. present approximation and the Brillouin's one (3.6) are very close to each other.

In case when the saddle points vary in the region $|\omega| \leqslant |\omega_0|$, Oughstun and Sherman replaced the rational functions by their power expansions as $\omega \to 0$, up to the order $O(\omega^3)$. This again reduced the saddle point equation to a cubic equation. Since the cubic term is small in comparison to other terms, at least for ω not too great, the authors deleted that term and finally obtained a further simplified quadratic equation whose solutions are given by

$$\omega^{\pm}_{SP_N} \cong \pm\psi(\theta) - \frac{2}{3}i\delta\zeta(\theta), \tag{3.16}$$

where

$$\psi(\theta) = \left[\frac{\omega_0^2(\theta^2 - \theta_0^2)}{\theta^2 - \theta_0^2 + \dfrac{3b^2}{\omega_0^2}\alpha} - \delta^2\left(\frac{\theta^2 - \theta_0^2 + \dfrac{2b^2}{\omega_0^2}}{\theta^2 - \theta_0^2 + \dfrac{3b^2}{\omega_0^2}\alpha}\right)^2\right]^{1/2}, \tag{3.17}$$

$$\zeta(\theta) = \frac{3}{2}\,\frac{\theta^2 - \theta_0^2 + \dfrac{2b^2}{\omega_0^2}}{\theta^2 - \theta_0^2 + \dfrac{3b^2}{\omega_0^2}\alpha}. \tag{3.18}$$

and

$$\alpha = 1 - \frac{\delta^2}{3\omega_0^2\omega_1^2}(4\omega_1^2 + b^2). \tag{3.19}$$

If θ is close to θ_0, (3.16) reduces to (3.8). On the other hand, if θ tends to infinity then (3.16) approaches $\omega_\pm$, as it happens in the case of the exact solution.

Approximate value of θ_1 as predicted by (3.16) follows from equating $\psi(\theta)$ to zero. With some additional simplifications it appears to be

$$\theta_1 \cong \theta_0 + \frac{2\delta^2 b^2}{\theta_0\omega_0^2(3\alpha\omega_0^2 - 4\delta^2)}. \tag{3.20}$$

Oughstun and Sherman's approximation for the location of the saddle points appeared to be quite satisfactory, both because of its accuracy and its simplicity. However, as it will be shown in the next section, a different approximation can be constructed, which provides better accuracy over a wide range of θ values.

3.3. Alternative solution to the saddle point equation

Unlike in the Oughstun and Sherman approach, our starting point is the original saddle point equation (2.3). We formulate this equation in terms of the complex index of refraction n. If approximate solutions are found for n from that equation, corresponding values for the frequencies can be obtained from (2.2) to give

$$\begin{aligned} \omega^+ &= -i + \sqrt{\omega_0^2 - \delta^2 - \frac{b^2}{n^2-1}}, \\ \omega^- &= -(\omega^+)^* = -i\delta - \sqrt{\omega_0^2 - \delta^2 - \frac{b^2}{(n^*)^2-1}}. \end{aligned} \tag{3.21}$$

Here the asterisk denotes complex conjugate.

With $n'(\omega)$ determined from (2.2), the saddle point equation formulated in terms of n now reads

$$n + \frac{(n^2-1)^2}{b^2 n}\left[-i\delta\left(\omega_0^2 - \delta^2 - \frac{b^2}{n^2-1}\right)^{1/2} + \omega_0^2 - \delta^2 - \frac{b^2}{n^2-1}\right] = \theta. \tag{3.22}$$

As mentioned earlier, for θ increasing from 1 to ∞, the near saddle points tend to $\omega_\pm$, and the distant saddle points tend to $\omega'_\pm$, respectively. On the other hand, $|n(\omega)|$ tends to infinity as ω approaches $\omega_\pm$, and $|n(\omega)|$ decreases to zero as it comes near $\omega'_\pm$. This suggests that in the vicinity of those limiting points, the saddle point equation can be approximated by asymptotically expanding its both sides for $|n| \to \infty$ or $|n| \to 0$, respectively. (The right-hand side of the equation is left intact, as it is independent of n). Now we consider both cases in turn.

First, consider the case of the distant saddle point, i.e. the case of $|n| < 1$. Let the l.h.s. of the Eq. (3.22) be expanded in positive powers of n as $n \to 0$, and terms higher than cubic are neglected. Then the saddle point equation takes the following approximate form

$$an^4 + cn^2 + dn + e \sim 0, \tag{3.23}$$

where

$$a = \left[\omega_0^2 - \delta^2 + \frac{i\delta b^2}{\sqrt{\omega_1^2 - \delta^2}} - i\delta\sqrt{\omega_1^2 - \delta^2} - \frac{i\delta b^2[3b^2 + 4(\omega_0^2 - \delta^2)]}{8(\omega_1^2 - \delta^2)^{3/2}}\right]\frac{1}{b^2}, \tag{3.24}$$

$$c = -\frac{i\delta}{2\sqrt{\omega_1^2 - \delta^2}} - \frac{2}{b^2}\left(\omega_0^2 - \delta^2 - i\delta\sqrt{\omega_1^2 - \delta^2}\right), \tag{3.25}$$

$$d = -\theta, \tag{3.26}$$

and

$$e = \frac{\omega_1^2 - \delta^2 - i\delta\sqrt{\omega_1^2 - \delta^2}}{b^2}. \tag{3.27}$$

Appropriate solution to this equation is not too complicated (presumably because the term proportional to n^3 is missing from the equation) and is given by

$$n \cong g - \sqrt{\frac{1}{4a}\left(\frac{\theta}{g} - 2c\right) - g^2}, \tag{3.28}$$

where

$$g = \frac{1}{2\sqrt{3a}}\sqrt{\frac{2^{-1/3}}{(u + \sqrt{u^2 - v^3})^{1/3}}\left[(u + \sqrt{u^2 - v^3})^{2/3} + v\right] - 2c}, \tag{3.29}$$

$$u = 2c^3 - 72ace + 27a\theta^2, \tag{3.30}$$

$$v = 2^{2/3}(c^2 - 12ae). \tag{3.31}$$

Here, the parameter θ appears directly under the square root sign and in g via the variable u.

As θ approaches infinity, g behaves like $\theta^{1/3}/(2a^{1/3})$ and both components in (3.28) cancel out, thus making n tend to zero. Consequently, by (3.21), as $\theta \to \infty$ the distant saddle points tend to $\omega'_\pm$. This agrees with Brillouin's and Oughstun and Sherman's results. For sufficiently small n, powers of n higher than 1 can be disregarded in Eq. (3.23). If n so approximated is used in (3.21), the following result is obtained:

$$\omega \cong -i\delta \pm \sqrt{(\omega_1^2 - \delta^2)\left(1 + \frac{1}{\theta}\right) - i\delta \frac{\sqrt{\omega_1^2 - \delta^2}}{\theta}}, \tag{3.32}$$

valid for $\theta \to \infty$. It shows the rate at which distant saddle points tend to their limiting values as θ increases to infinity.

Now consider the near saddle points locations corresponding to the case of $|n| > 1$. By expanding the l.h.s. of the Eq. (3.22) asymptotically as $n \to \infty$, and disregarding powers of n higher than n^{-1}, the saddle point equation reduces to the same form as in (3.22), with partly modified coefficients:

$$a = \frac{\omega_0^2 - \delta^2 - i\delta\sqrt{\omega_0^2 - \delta^2}}{b^2}, \tag{3.33}$$

$$c = \frac{i\delta}{2\sqrt{\omega_0^2 - \delta^2}} - \frac{2}{b^2}\left(\omega_0^2 - \delta^2 - i\delta\sqrt{\omega_0^2 - \delta^2}\right), \tag{3.34}$$

$$d = -\theta, \tag{3.35}$$

and

$$e = \left[\omega_1^2 - {}^2 - \frac{i\delta b^2}{\sqrt{\omega_0^2 - \delta^2}} - i\delta\sqrt{\omega_0^2 - \delta^2} + \frac{i\delta b^2[b^2 + 4(\omega_0^2 - \delta^2)]}{8(\omega_0^2 - \delta^2)^{3/2}}\right]\frac{1}{b^2}. \tag{3.36}$$

It is interesting to note that real parts of the coefficients are the same as in the case of the distant saddle points. For the medium parameters proposed by Brillouin, the imaginary parts are only slightly different, as it can be seen from

$$(0.79608 - 0.05586i)n^4 + (-1.59216 + 0.14681i)n^2 - \theta n$$
$$+1.79608 - 0.07993i \sim 0 \tag{3.37}$$

which applies in the case of near saddle points, and

$$(0.79608 - 0.05730i)n^4 + (-1.59216 + 0.14446i)n^2 - \theta n$$
$$+1.79608 - 0.08390i \sim 0 \tag{3.38}$$

appropriate in the case of distant saddle points.

Now, the appropriate solution to the fourth degree Eq. (3.23) is

$$n \cong g + \sqrt{\frac{1}{4a}\left(\frac{\theta}{g} - 2c\right) - g^2}, \tag{3.39}$$

i.e. it differs from (3.28) in the sign preceding the second term. As θ tends to infinity, n behaves like $a^{-1/3}\theta^{1/3}$. Therefore, the last term under the square root sign in (3.21) vanishes in the limit and the near saddle points attain the values $\omega_{\pm}$. This again agrees with Brillouin's and Oughstun and Sherman's results.

As it will be seen in the following section, the approximation of saddle point locations given here are pretty good in a wide range of changing θ, except for very small values of this parameter, about 1.5 and below. In this exceptional subregion we propose a very simple, local approximation obtained by expanding the l.h.s. of the Eq. (3.22) about $n = \sqrt{1 + b^2/\omega_0^2}$, which corresponds to $\omega = 0$. By retaining only the first term in the expansion we obtain

$$n \approx \frac{1}{2}\left(\frac{\omega_1}{\omega_0} + \theta\right), \tag{3.40}$$

and consequently

$$\omega \approx -i\delta + \sqrt{\omega_0^2 - \delta^2 - \frac{4b^2}{(\omega_1/\omega_0 + \theta)^2 - 1}}, \tag{3.41}$$

the approximation being valid for small ω.

3.4. Numerical results

Results obtained in the previous section will now be compared with the exact ones and those obtained by Oughstun and Sherman.

In Fig. 1a,b the real and imaginary parts of the distant saddle points are displayed as functions of the parameter θ. It is seen that the approximate values obtained in this work are almost undistiguishable from the exact ones, even at small values of θ.

Figure 2a,b presents real and imaginary parts of the near saddle points as functions of θ. Again, the accuracy of the present result is very good, except at very small values of θ. For θ around 1 and slightly above this value, Oughstun-Sherman's result is better. This follows, on the one hand, from the fact that their approximation is constructed under the assumption of small values of ω, and hence θ. On the other hand, the approximation method used here is not well suited for small θ because $d\omega/dn$ blows up at θ equal 1 or $\sqrt{b^2/(\omega_0^2 - \delta^2) + 1}$.

In Fig. 3a,b the local approximation proposed in (3.41) is compared with exact solution of the saddle point equation and the Oughstun-Sherman's approximation. It is very simple but not as good as the latter result.

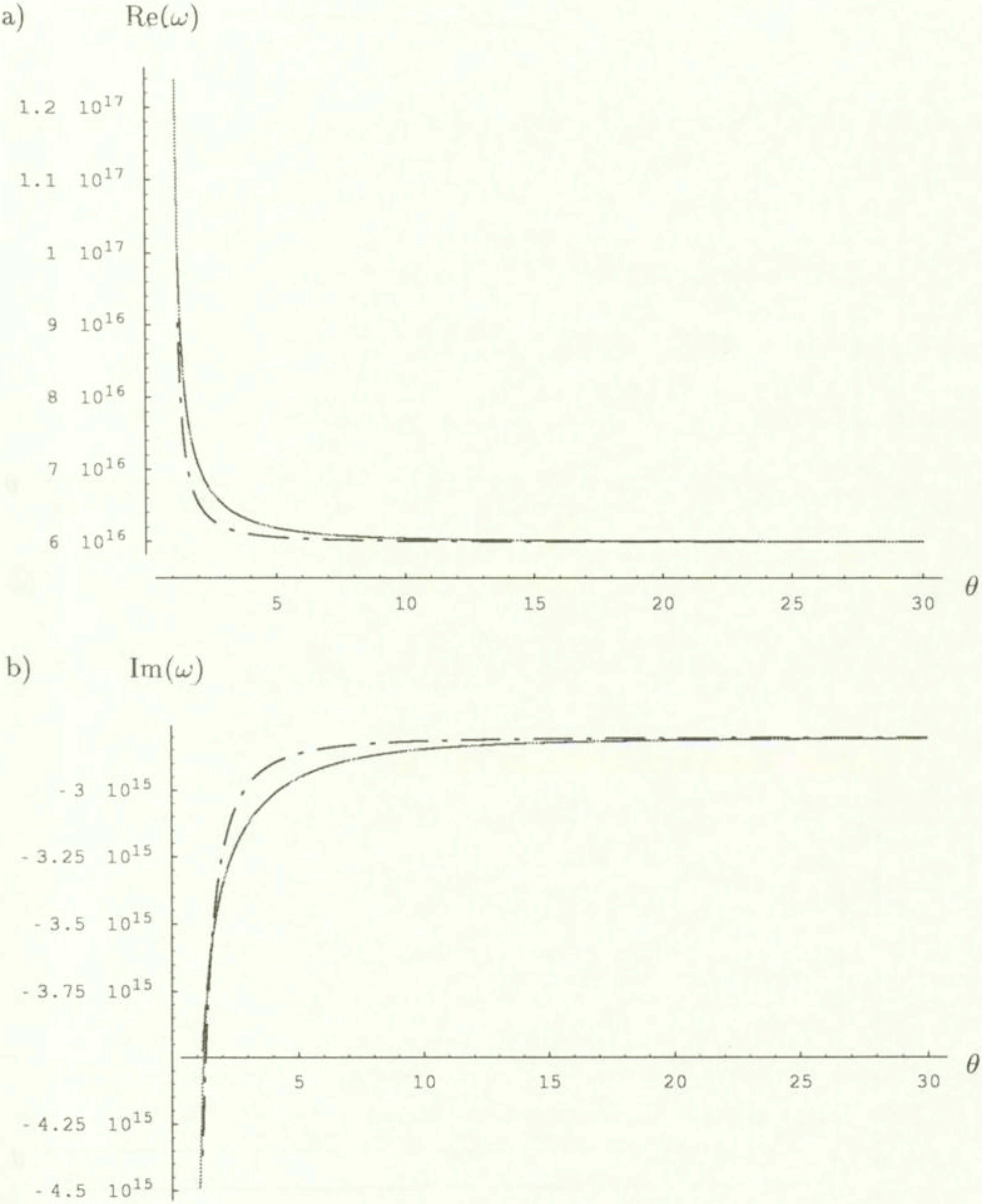

FIG. 1. a) $\mathrm{Re}(\omega)$ versus θ for distant saddle points. —·— Oughstun-Sherman's approximate solution, — exact solution and the approximate solution based on Eq. (3.28) (both plots are undistinguishable except for values of θ below 1.5); b) $\mathrm{Im}(\omega)$ versus θ for distant saddle points. The same caption as in Fig. 1a.

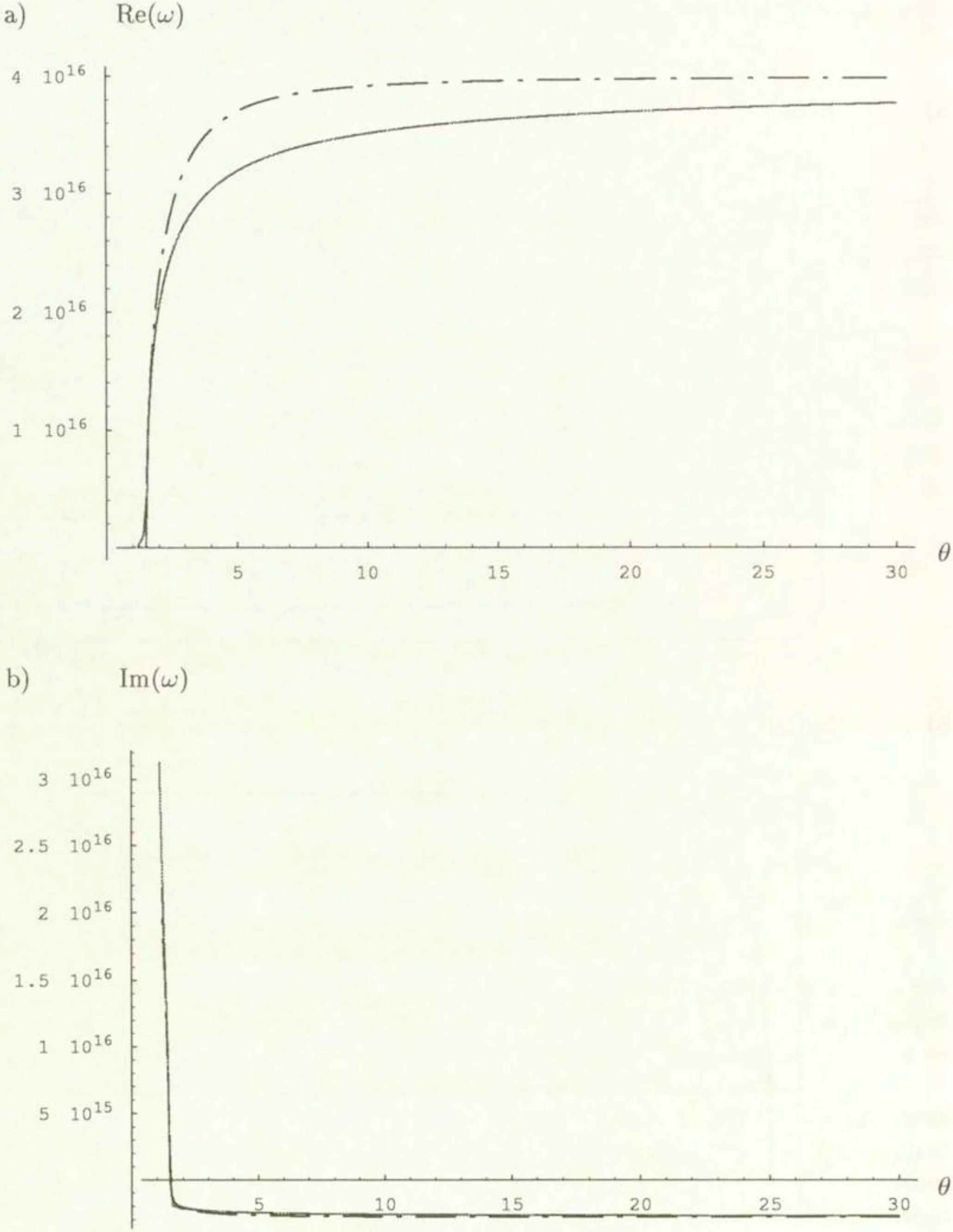

FIG. 2. a) Re(ω) versus θ for near saddle points. — · — Oughstun-Sherman's approximate solution, — exact solution and the approximate solution based on Eq. (3.39) (both plots are undistinguishable except for values of θ below 1.5); b) Im(ω) versus θ for near saddle points. The samecaption as in Fig. 2a.

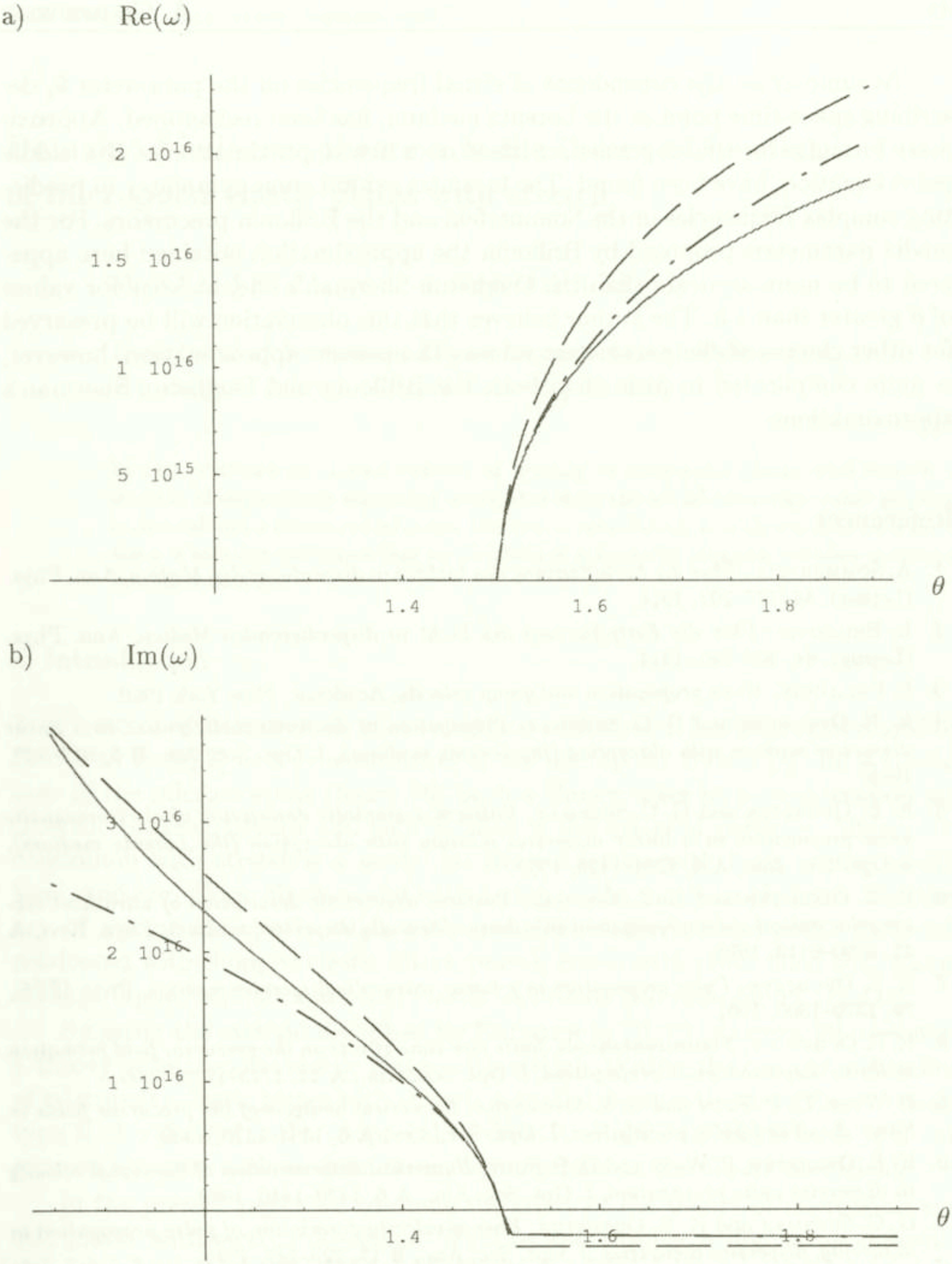

FIG. 3. a) Re(ω) versus θ for near saddle points at small values of θ. — exact solution, —·— Oughstun-Sherman's approximate solution, — — present approximate solution; b) Im(ω) versus θ for near saddle points at small values of θ. Same caption as in Fig. 3a.

To summarize, the dependence of signal frequencies on the parameter θ, describing space-time point in the Lorentz medium, has been reexamined. Approximate formulas for this dependence, based on a new approximation of the saddle point equation, have been found. The formulas exhibit some symmetry in predicting complex frequencies in the Sommerfeld and the Brillouin precursors. For the media parameters proposed by Brillouin the approximation obtained here appeared to be more accurate than the Oughstun-Sherman's one, at least for values of θ greater than 1.5. The author believes that this observation will be preserved for other choices of the parameters values. The present approximation, however, is more complicated in form than both the Brillouin and Oughstun-Sherman's approximations.

References

1. A. SOMMERFELD, *Über die Fortpflanzung des Lichtes in disperdierenden Medien*, Ann. Phys. (Lepzig), **44**, 177–202, 1914.
2. L. BRILLOUIN, *Über die Fortpflanzung des Licht in disperdierenden Medien*, Ann. Phys. (Lepzig), **44**, 203–240, 1914.
3. L. BRILLOUIN, *Wave propagation and group velocity*, Academic, New York 1960.
4. K. E. OUGHSTUN and G. C. SHERMAN, *Propagation of electromagnetic pulses in a linear dispersive medium with absorption (the Lorentz medium)*, J. Opt. Soc. Am. **B 5**, 817–849, 1988.
5. K. E. OUGHSTUN and G. C. SHERMAN, *Uniform asymptotic description of electromagnetic pulse propagation in a linear dispersive medium with absorption (the Lorentz medium)*, J. Opt. Soc. Am., **A 6**, 1394–1420, 1989.
6. K. E. OUGHSTUN and G. C. SHERMAN, *Uniform asymptotic description of ultrashort rectangular optical pulse propagation in a linear, causally dispersive medium*, Phys. Rev., **A 41**, 6090–6113, 1990.
7. K. E. OUGHSTUN, *Pulse propagation in a linear, casusally dispersive medium*, Proc. IEEE, **79**, 1379–1390, 1991.
8. K. E. OUGHSTUN, *Noninstantaneous, finite rise-time effects on the precursor field formation in linear dispersive pulse propagation*, J. Opt. Soc. Am., **A 12**, 1715–1729, 1995.
9. P. WYNS, D. P. FOTY and K. E. OUGHSTUN, *Numerical analysis of the precursor fields in linear dispersive pulse propagation*, J. Opt. Soc. Am., **A 6**, 1394–1420, 1989.
10. K. E. OUGHSTUN, P. WYNS and D. P. FOTY, *Numerical determination of the signal velocity in dispersive pulse propagation*, J. Opt. Soc. Am., **A 6**, 1430–1440, 1989.
11. G. C. SHERMAN and K. E. OUGHSTUN, *Energy-velocity description of pulse propagation in absorbing, dispersive dielectrics*, J. Opt. Soc. Am., **B 12**, 229–247, 1995.
12. J. A. STRATTON, *Electromagnetic theory*, Mc Graw Hill, New York 1941, Sec. 5.14.

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