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Laminated pressurized elastic tube and its homogenization

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In this paper the state of stress and deformation of a pressurized tube consisting of various elastic isotropic or transversely isotropic layers is evaluated. Especially a periodically laminated tube made of many thin layers is analysed by means of a homogenization process, and closed-form solutions are obtained. The results are illustrated by means of a numerical example.

Key words: Theory of elasticity, layered tube, heterogeneous media, homogenization.

1. Introduction

Laminates consisting of a great number of different layers are characterized by a considerable amount of parameters, and an exact analysis seems therefore to be more or less hopeless. However, as shown by the author [1], the method of transfer matrices (possibly in connection with integral transforms) turns out to be a clear and effective procedure for the construction of solutions without simplifying assumptions. The transfer matrix method usually applied to problems which are governed by differential equations with constant coefficients, see Pestel and Leckie [2], has been used for isotropic layered tubes (governed by differential equations with variable coefficients) already by Saumweber [3], however without taking into account the homogenization. The homogenization is studied additionally in the present paper. It applies to a periodically laminated tube made of many thin isotropic or transversely isotropic layers, and consists in the transformation of an original discrete problem (governed by a matrix difference equation for the "state vector" at the boundaries of a finite layer group) into a continuous one (governed by a matrix differential equation) by means of a limiting process. The resulting differential equation is of the same type as that for a homogeneous transversely isotropic tube, and its exact solution is given in closed form. The procedure is similar to that proposed by the author [4] in a recent paper which deals with laminated hollow spheres.

As a numerical example, a pressurized tube consisting of only five double layers is considered. The results for the radial displacement, the radial and the circumferential stresses are evaluated according the the exact (inhomogeneous) model and the approximate (homogenized) one, respectively, and compared to each other. It turns out that the homogenization procedure leads to a sufficiently accurate solution, even for the considered low number of double layers.

2. Basic equations and transfer matrix for a transversely isotropic and homogeneous thick tube

Using cylindrical coordinates r, φ, z, and assuming an axisymmetric state of stress and deformation (without torsion), the remaining equilibrium equation and strain-displacement relations are

$$r\frac{d\sigma_r}{dr} + \sigma_r - \sigma_\varphi = 0; \qquad \varepsilon_r = \frac{du}{dr}; \qquad \varepsilon_\varphi = \frac{u}{r} \tag{2.1}$$

(σ_r radial stress, σ_φ tangential stress, ε_r and ε_φ corresponding strains, and u radial displacement). Since no shear is involved with respect to the cylindrical coordinates, in the case of a transversely isotropic tube we have only the following equations of the generalized Hooke's law:

$$\begin{bmatrix} \varepsilon_r \\ \varepsilon_\varphi \\ \varepsilon_z \end{bmatrix} = \begin{bmatrix} 1/E' & -\nu'/E' & -\nu'/E' \\ -\nu'/E' & 1/E & -\nu/E \\ -\nu'/E' & -\nu/E & 1/E \end{bmatrix} \begin{bmatrix} \sigma_r \\ \sigma_\varphi \\ \sigma_z \end{bmatrix},$$

where E, E', ν and ν' are material parameters such that the strain energy is positive definite. In the case of *plane strain* ($\varepsilon_z = 0$) one obtains

$$\begin{aligned} \varepsilon_r &= \frac{1}{E'}\left(1 - \nu'^2\frac{E}{E'}\right)\sigma_r - \frac{\nu'}{E'}(1+\nu)\sigma_\varphi; \\ \varepsilon_\varphi &= \frac{1}{E}(1-\nu^2)\sigma_\varphi - \frac{\nu'}{E'}(1+\nu)\sigma_r; \\ \sigma_r &= \frac{(1-\nu)E'}{1-\nu-2\nu'^2E/E'}\left(\varepsilon_r + \frac{\nu'}{1-\nu}\frac{E}{E'}\varepsilon_\varphi\right); \\ \sigma_\varphi &= \frac{1-\nu'^2E/E'}{1-\nu-2\nu'^2E/E'}\frac{E}{1+\nu}\left(\varepsilon_\varphi + \frac{(1+\nu)\nu'}{1-\nu'^2E/E'}\varepsilon_r\right). \end{aligned} \tag{2.2}$$

In the case of *plane stress* ($\sigma_z = 0$) there is

$$\varepsilon_r = \frac{1}{E'}\sigma_r - \frac{\nu'}{E'}\sigma_\varphi; \tag{2.2$'$}$$

(2.2)′ [cont.]

$$\varepsilon_\varphi = \frac{1}{E}\sigma_\varphi - \frac{\nu'}{E'}\sigma_r;$$

$$\sigma_r = \frac{E'}{1-\nu'^2 E/E'}(\varepsilon_r + \nu'\frac{E}{E'}\varepsilon_\varphi);$$

$$\sigma_\varphi = \frac{E}{1-\nu'^2 E/E'}(\varepsilon_\varphi + \nu'\varepsilon_r).$$

Eliminating the strains, the basic equations (2.1) and (2.2) for the state of plane strain and (2.1) and (2.2)′ for the state of plane stress can be put into the following common form:

$$\sigma_\varphi(r) = [N \quad L/r]\begin{bmatrix}\sigma_r(r)\\ u^*(r)\end{bmatrix}; \qquad u^* = \frac{E^*}{h^*}u; \tag{2.3}$$

(h^* reference thickness, E^* reference modulus of elasticity), and

$$\frac{d\mathbf{a}(r)}{dr} = \mathbf{A}(r)\mathbf{a}(r) \tag{2.4}$$

with

$$\mathbf{A}(r) = \begin{bmatrix}-K/r & L/r^2\\ M & -N/r\end{bmatrix}; \qquad \mathbf{a}(r) = \begin{bmatrix}\sigma_r(r)\\ u^*(r)\end{bmatrix}. \tag{2.5}$$

In (2.4) and (2.5) $\mathbf{a}$(r) is called *state vector* and $\mathbf{A}$(r) – *fundamental matrix.*

The quantities K, L, M and N contain the material parameters and are defined as follows:

(a) For *plane strain and transversely isotropic material*

$$K = 1 - \frac{\nu'}{1-\nu}\frac{E}{E'}; \qquad L = \frac{1}{1-\nu^2}\frac{E}{E^*}h^*;$$
$$M = \left(1 - \frac{2\nu'^2}{1-\nu}\frac{E}{E'}\right)\frac{E^*}{E'}\frac{1}{h^*}; \qquad N = \frac{\nu'}{1-\nu}\frac{E}{E'}. \tag{2.6}$$

(b) For *plane strain and isotropic material* ($\nu' = \nu$ and $E' = E$)

$$K = 1 - \frac{\nu}{1-\nu}; \qquad L = \frac{1}{1-\nu^2}\frac{E}{E^*}h^*;$$
$$M = \frac{(1+\nu)(1-2\nu)}{1-\nu}\frac{E^*}{E}\frac{1}{h^*}; \qquad N = \frac{\nu}{1-\nu}. \tag{2.7}$$

(c) For *plane stress and transversely isotropic material*

$$K = 1 - \nu'\frac{E}{E'}; \qquad L = \frac{E}{E^*}h^*;$$
$$M = \left(1 - \nu'^2\frac{E}{E'}\right)\frac{E^*}{E'}\frac{1}{h^*}; \qquad N = \nu'\frac{E}{E'}. \tag{2.6'}$$

(d) For *plane stress and isotropic material*

$$(2.7)' \qquad K = 1-\nu; \qquad L = \frac{E}{E^*}h^*; \qquad M = (1-\nu^2)\frac{E^*}{E}\frac{1}{h^*}; \qquad N = \nu.$$

It can be easily seen that the quantities K, L, M and N are not independent of each other. There exist the following compatibility equations:
For *transversely isotropic material* (cases (a) and (c))

$$(2.8) \qquad K + N = 1,$$

and for *isotropic material* (cases (b) and (d))

$$(2.9) \qquad K + N = 1 \quad \text{and} \quad ML + N^2 = 1.$$

In order to integrate (2.4), a matrix differential equation with variable coefficients, we eliminate there σ_r and get

$$(2.10) \qquad \frac{d^2u^*}{dr^2} + \frac{1}{r}\frac{du^*}{dr} - \frac{c}{r^2}u^* = 0,$$

with

$$(2.11) \qquad c = ML + N^2.$$

Taking into account (2.6), (2.7), (2.6)′ and (2.7)′ there follows

$$(2.12) \qquad c = \begin{cases} \dfrac{1}{1-\nu^2}\dfrac{E}{E'}\left(1-\nu'^2\dfrac{E}{E'}\right) > 0 & \text{for case (a)}, \\ \dfrac{E}{E'} > 0 & \text{for case (c)}, \\ 1 & \text{for cases (b) and (d)}, \end{cases}$$

where the positiveness of the strain energy density has been used. The change of variable r according to $r = e^t$ yields finally

$$\frac{d^2u^*}{dt^2} - cu^* = 0,$$

with the solution – again in terms of r –

$$(2.13) \qquad u^* = C_1 r^{\lambda_1} + C_2 r^{\lambda_2},$$

where C_1 and C_2 mean integration constants, and $\lambda_{1,2}$ – the roots of the characteristic equation:

$$(2.14) \qquad \lambda_{1,2} = \pm\lambda; \qquad \lambda = \sqrt{c}.$$

Due to relations (2.12), these roots are real.

With (2.3) and (2.1), there follows from (2.2) (plane strain) and (2.2)′ (plane stress) the radial stress component

$$\sigma_r = \frac{1}{M}\left(\frac{du^*}{dr} + \frac{N}{r}u^*\right), \tag{2.15}$$

where M and N are given in (2.6), (2.7) and (2.6)′, (2.7)′, respectively. Taking further into account (2.13), we obtain the solution of (2.4)

$$\begin{bmatrix}\sigma_r(r)\\ u^*(r)\end{bmatrix} \doteq \begin{bmatrix}\dfrac{N+\lambda}{M}r^{\lambda-1} & \dfrac{N-\lambda}{M}r^{-\lambda-1}\\ r^{\lambda} & r^{\lambda}\end{bmatrix}\begin{bmatrix}C_1\\ C_2\end{bmatrix}. \tag{2.16}$$

Inverting this equation for $r = r_0$, one obtains for the integration constants C_1 and C_2

$$\begin{bmatrix}C_1\\ C_2\end{bmatrix} = \frac{1}{2\lambda}\begin{bmatrix}Mr_0^{1-\lambda} & -(N-\lambda)r_0^{-\lambda}\\ -Mr_0^{1+\lambda} & (N+\lambda)r_0^{\lambda}\end{bmatrix}\begin{bmatrix}\sigma_{r_0}\\ u_0^*\end{bmatrix}. \tag{2.17}$$

Their elimination in (2.16) leads finally to

$$\mathbf{a}(r) = \mathbf{T}(r)\mathbf{a}(r_0), \tag{2.18}$$

where the *field-transfer-matrix* $\mathbf{T}(r)$ (from radius r_0 to radius r) is

$$\mathbf{T}(r) = \frac{1}{2\lambda}\begin{bmatrix}(N+\lambda)\left(\dfrac{r}{r_0}\right)^{\lambda-1} - (N-\lambda)\left(\dfrac{r}{r_0}\right)^{-\lambda-1} & \dfrac{(N+\lambda)(N-\lambda)}{Mr_0}\left[-\left(\dfrac{r}{r_0}\right)^{\lambda-1} + \left(\dfrac{r}{r_0}\right)^{-\lambda-1}\right]\\ Mr_0\left[\left(\dfrac{r}{r_0}\right)^{\lambda} - \left(\dfrac{r}{r_0}\right)^{-\lambda}\right] & -(N-\lambda)\left(\dfrac{r}{r_0}\right)^{\lambda} + (N+\lambda)\left(\dfrac{r}{r_0}\right)^{-\lambda}\end{bmatrix}. \tag{2.19}$$

For case (d) – plane stress and isotropic material – this formula corresponds to that given by SAUMWEBER [3].

REMARK 1. The fundamental matrix $\mathbf{A}$(r) in (2.5) and the field-transfer-matrix $\mathbf{T}$(r) are related to each other according to

$$\mathbf{A}(r_0) = \frac{d\mathbf{T}(r)}{dr}\Big|_{r=r_0}. \tag{2.20}$$

■

Let us consider, as an example, a thick-walled tube with radii r_0 and r_1 under internal pressure p. From the boundary conditions $\sigma_r(r_0) = -p$ and $\sigma_r(r_1) = 0$, one obtains from (2.12) and (2.13) at first

$$u_0^* = \frac{-(N+\lambda)\left(\frac{r_1}{r_0}\right)^{\lambda-1} + (N-\lambda)\left(\frac{r_1}{r_0}\right)^{-\lambda-1}}{(N+\lambda)(N-\lambda)\left[\left(\frac{r_1}{r_0}\right)^{\lambda-1} - \left(\frac{r_1}{r_0}\right)^{-\lambda-1}\right]} M r_0 p, \tag{2.21}$$

and then

$$u = \frac{pb}{\left(\dfrac{a}{b}\right)^{\lambda-1} - \left(\dfrac{a}{b}\right)^{-\lambda-1}} \frac{h^*}{E^*} M \left[\frac{\left(\dfrac{r}{b}\right)^{-\lambda}}{N-\lambda} - \frac{\left(\dfrac{r}{b}\right)^{\lambda}}{N+\lambda}\right],$$

$$\sigma_r = -\frac{p}{\left(\dfrac{a}{b}\right)^{\lambda-1} - \left(\dfrac{a}{b}\right)^{-\lambda-1}} \left[\left(\frac{r}{b}\right)^{\lambda-1} - \left(\frac{r}{b}\right)^{-\lambda-1}\right], \tag{2.22}$$

$$\sigma_\varphi = -\frac{p\lambda}{\left(\dfrac{a}{b}\right)^{\lambda-1} - \left(\dfrac{a}{b}\right)^{-\lambda-1}} \left[\left(\frac{r}{b}\right)^{\lambda-1} + \left(\frac{r}{b}\right)^{-\lambda-1}\right],$$

where $a = r_0$ and $b = r_1$. The last formula had been derived from (2.3). For an *isotropic material* – cases (b) and (d) – the well-known result is (see e.g. TIMOSHENKO and GOODYEAR [5]):

$$u = \frac{pb}{1 - \left(\frac{a}{b}\right)^{-2}} \frac{h^*}{E^*} M \left[\frac{\left(\frac{r}{b}\right)^{-1}}{N-1} - \frac{\frac{r}{b}}{N+1}\right],$$

$$\sigma_r = -\frac{p}{1 - \left(\frac{a}{b}\right)^{-2}} \left[1 - \left(\frac{r}{b}\right)^{-2}\right], \tag{2.23}$$

$$\sigma_\varphi = -\frac{p}{1 - \left(\frac{a}{b}\right)^{-2}} \left[1 + \left(\frac{r}{b}\right)^{-2}\right],$$

with M and N given by (2.7) and (2.7)′, respectively. Note that the state of stress for transversely isotropic material – cases (a) and (c) – depends on the material via λ, while it is independent of the material in the case of isotropy.

3. Arbitrarily layered thick tube

In Fig. 1 is sketched an n-layer system consisting of different transversely isotropic or isotropic layers which are considered to be either in a state of plane strain or in a state of plane stress. Layer (k) is characterized by the corresponding material parameters (marked by index k) and the radii r_{k-1} and r_k, or radius r_{k-1} and thickness h_k. The coordinate z and the dimensionless coordinate ρ for layer (k) are defined as follows:

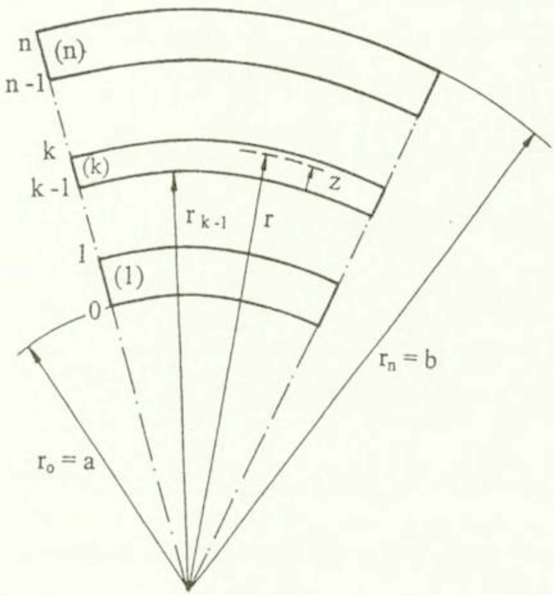

FIG. 1. Arbitrarily laminated thick tube consisting of n finite layers.

(3.1)
$$z = r - r_{k-1}; \qquad \rho = \frac{z}{r_{k-1}};$$
$$\rho_k = \frac{h_k}{r_{k-1}}; \qquad \frac{r}{r_{k-1}} = 1 + \rho;$$
$$0 \leqslant z \leqslant h_k, \qquad r_{k-1} \leqslant r \leqslant r_k, \qquad 0 \leqslant \rho \leqslant \rho_k.$$

Then with $r_0 = r_{k-1}$ in (2.19), the transfer Eq. (2.18) reads

$$\mathbf{a}^{(k)}(\rho) = \mathbf{T}^{(k)}(\rho)\mathbf{a}^{(k)}(0), \tag{3.2}$$

where

$$\mathbf{T}^{(k)}(\rho) = \begin{bmatrix} t_k^{11}(\rho) & t_k^{12}(\rho) \\ t_k^{21}(\rho) & t_k^{22}(\rho) \end{bmatrix}, \tag{3.3}$$

with

$$2\lambda_k t_k^{11}(\rho) = (N_k+\lambda_k)(1+\rho)^{\lambda_k-1} - (N_k-\lambda_k)(1+\rho)^{-\lambda_k-1},$$

$$2\lambda_k t_k^{12}(\rho) = \frac{(N_k+\lambda_k)(N_k-\lambda_k)}{M_k r_{k-1}}\left[-(1+\rho)^{\lambda_k-1} + (1+\rho)^{-\lambda_k-1}\right],$$

$$2\lambda_k t_k^{21}(\rho) = M_k r_{k-1}\left[(1+\rho)^{\lambda_k} - (1+\rho)^{-\lambda_k}\right],$$

$$2\lambda_k t_k^{22}(\rho) = -(N_k-\lambda_k)(1+\rho)^{\lambda_k} + (N_k+\lambda_k)(1+\rho)^{-\lambda_k},$$

is the *field-transfer matrix* of the layer k. Further there holds

$$\mathbf{a}^{(k)}(\rho_k) = \mathbf{T}^{(k)}(\rho_k)\mathbf{a}^{(k)}(0), \tag{3.4}$$

and the interface continuity condition is

$$\mathbf{a}^{(k)}(0) = \mathbf{a}^{(k-1)}(\rho_{k-1}) \equiv \mathbf{a}_{k-1} \qquad (k = 2, 3\ldots, n). \tag{3.5}$$

Consequently, (3.4) can be written in the form

$$\mathbf{a}_k = \mathbf{T}_k \mathbf{a}_{k-1}, \tag{3.6}$$

with

$$\mathbf{T}_k = \mathbf{T}^{(k)}(\rho_k), \tag{3.7}$$

representing the *layer-transfer-matrix*. Applying (3.6) from $k = 1$ to $k = n$, one obtains

$$\mathbf{a}_n = \mathbf{S}\mathbf{a}_0; \qquad \mathbf{S} = \mathbf{T}_n\mathbf{T}_{n-1}\ldots\mathbf{T}_1, \tag{3.8}$$

where $\mathbf{S}$ is called *system-transfer-matrix.*

Due to the boundary conditions, one component of $\mathbf{a}_0$ and one component of $\mathbf{a}_n$ are prescribed, and the remaining ones can be calculated from(3.8). Further, (3.6) yields all *initial state vectors* of the individual layers $\mathbf{a}_1\ldots\mathbf{a}_{n-1}$, and the exact state of radial stress and displacement follows from (3.2) with (3.3). Finally, the circumferential stress can be evaluated from (2.3). Concerning the transfer matrix method, the reader is referred to PESTEL and LECKIE [2].

REMARK 2. The layer-transfer-matrix (3.7) has the following properties:

(1) For two succesive layers (k) and $(k+1)$ with identical materials ($\lambda_k = \lambda$, $M_k = M$, $N_k = N$) we may write $\mathbf{T}^{(k)}(\rho_k) = \mathbf{T}(\rho_k)$ and $\mathbf{T}^{(k+1)}(\rho_{k+1}) = \mathbf{T}(\rho_{k+1})$, respectively in (3.3), and there holds

$$\mathbf{T}\left(\frac{h_{k+1}}{r_k}\right)\mathbf{T}\left(\frac{h_k}{r_{k-1}}\right) = \mathbf{T}\left(\frac{h_k+h_{k+1}}{r_{k-1}}\right). \tag{3.9}$$

(2) The inverse layer-transfer-matrix $\mathbf{T}^{-1}$ is obtained from the original one $\mathbf{T}(\rho_k) = \mathbf{T}\left(\frac{r_k}{r_{k-1}} - 1\right)$ by replacing r_k/r_{k-1} with its reciprocal r_{k-1}/r_k:

$$\left[\mathbf{T}\left(\frac{r_k}{r_{k-1}} - 1\right)\right]^{-1} = \mathbf{T}\left(\frac{r_{k-1}}{r_k} - 1\right). \tag{3.10}$$

These statements are plausible physically and can be proved directly using (3.3). Note that, for simplicity, $T(\cdot)$ is used in Sec. 2 (Eqs. (2.18) – (2.20)) and Sec. 3 (Eqs. (3.9) – (3.10)) simultaneously as a symbol for the transfer matrix and a symbol for a function. Of course, $\mathbf{T}(r)$ and $\mathbf{T}(\rho)$ mean the same transfer matrix, but different functions, since the corresponding arguments are different.

■

4. Periodically layered thick tube and its homogenization

In what follows we consider a tube which consists of many thin *layer groups* with equal thickness h. A layer group (Fig. 2) is composed of m (generally different) transversely isotropic or purely isotropic *basic layers* which are characterized (in addition to their radii) by their material parameters $E_k, E'_k, \nu_k, \nu'_k (k = 1, 2 \ldots m)$, and their thickness h_k. Denoting by κ_k the thickness ratio of basic layer (k), there holds

$$h_k = \kappa_k h; \qquad h = \sum_{k=1}^{m} h_k; \qquad \sum_{k=1}^{m} \kappa_k = 1 \tag{4.1}$$

$$(0 < \kappa_k < 1; \; k = 1, 2, \ldots m).$$

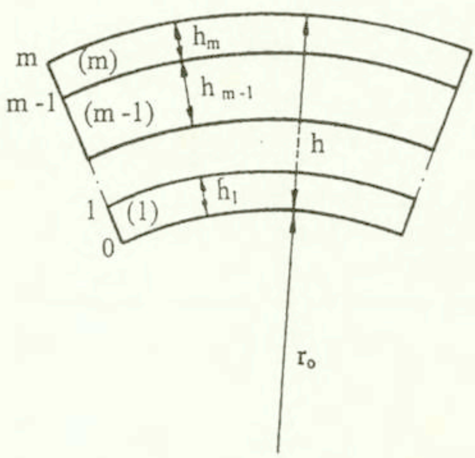

FIG. 2. Layer group consisting of m thin layers.

Let the layer group be bounded by the inner radius r_0 and the outer radius r_m, and let us assume $h \ll r_0$. Then after (3.1) there is

$$\rho_1 = \frac{h_1}{r_0} = \kappa_1 \frac{h}{r_0}; \qquad \rho_k = \frac{h_k}{r_{k-1}} = \frac{\kappa_k h}{r_0 + h\sum_{i=1}^{k-1} \kappa_i} = \frac{\kappa_k h}{r_0} + O((h/r_0)^2) \qquad (k = 2, 3, \dots m), \tag{4.2}$$

and a series expansion of (3.3) in (3.7) with respect to h/r_0 results in the *thin-layer transfer-matrix* of basic layer (k)

$$\overset{\circ}{\mathbf{T}}_k = \begin{bmatrix} 1 - \kappa_k K_k h/r_0 & \kappa_k L_k h/r_0^2 \\ \kappa_k M_k h & 1 - \kappa_k N_k h/r_0 \end{bmatrix} + \mathbf{O}((h/r_0)^2). \tag{4.3}$$

Consequently for the thin layer group, according to (3.8), there follows

$$\mathbf{a}_m = \overset{\circ}{\mathbf{S}}\, \mathbf{a}_0; \qquad \overset{\circ}{\mathbf{S}} = \overset{\circ}{\mathbf{T}}_m \overset{\circ}{\mathbf{T}}_{m-1} \dots \overset{\circ}{\mathbf{T}}_1, \tag{4.4}$$

particularly

$$\overset{\circ}{\mathbf{S}} = \begin{bmatrix} 1 - \bar{K}h/r_0 & \bar{L}h/r_0^2 \\ \bar{M}h & 1 - \bar{N}h/r_0 \end{bmatrix} + \mathbf{O}((h/r_0)^2), \tag{4.5}$$

with

$$\bar{K} = \sum \kappa_k K_k; \qquad \bar{L} = \sum \kappa_k L_k; \qquad \bar{M} = \sum \kappa_k M_k; \qquad \bar{N} = \sum \kappa_k N_k; \tag{4.6}$$

(The sum extends from $k = 1$ to $k = \mathrm{m}$; $\sum \kappa_k = 1$) .

The quantities K_k, L_k, M_k and N_k have to be taken from (2.6) or (2.7) or (2.6)′ or (2.7)′, after attaching index k at E, E', ν, ν', K, L, M and N. As can be easily verified , the following compatibility equation exists:

$$\bar{K} + \bar{N} = 1. \tag{4.7}$$

The *homogenization* is performed by replacing the finite relation (4.4), written in the form

$$\frac{\mathbf{a}_m - \mathbf{a}_0}{h} = \frac{\overset{\circ}{\mathbf{S}} - \mathbf{I}}{h} \mathbf{a}_0,$$

and to be interpreted as a difference equation, by the differential equation

$$\frac{d\mathbf{a}}{dr} = \lim_{h\to 0} \frac{\mathbf{a}_m - \mathbf{a}_0}{h} = \lim_{h\to 0} \frac{\overset{\circ}{\mathbf{S}} - \mathbf{I}}{h} \mathbf{a}_0 = \bar{\mathbf{A}}(r_0)\mathbf{a}_0,$$

or, taking $r_0 = r$ and $\mathbf{a}_0 = \mathbf{a}$, by

$$\frac{d\mathbf{a}}{dr} = \bar{\mathbf{A}}(r)\mathbf{a}, \tag{4.8}$$

with the *fundamental matrix of the periodical layering*

$$\bar{\mathbf{A}}(r) = \begin{bmatrix} -\bar{K}/r & \bar{L}/r^2 \\ \bar{M} & -\bar{N}/r \end{bmatrix}. \tag{4.9}$$

Since the fundamental matrix of the *basic layer k* is

$$\mathbf{A}_k(r) = \begin{bmatrix} -K_k/r & L_k/r^2 \\ M_k & -N_k/r \end{bmatrix},$$

taking into account (4.6), one obtains the relation

$$\bar{\mathbf{A}}(r) = \sum_{k=1}^{m} \kappa_k \mathbf{A}_k. \tag{4.10}$$

REMARK 3. We have derived the thin-layer transfer-matrix $\overset{\circ}{\mathbf{T}}_k$ in (4.3) via the layer-transfer-matrix $\mathbf{T}_k$ (given in (3.7) with (3.3)). Alternatively one can obtain this directly by integrating (2.4) in the neighbourhood of $r = r_0$, i.e. for a constant fundamental matrix $\mathbf{A}_k(r_0)$:

$$\mathbf{a}_k = e^{\mathbf{A}_k(r_0)h_k}\mathbf{a}_{k-1} = \left[\mathbf{I} + \mathbf{A}_k(r_0)\kappa_k h + \mathbf{O}(h^2)\right]\mathbf{a}_{k-1} = \overset{\circ}{\mathbf{T}}_k\, \mathbf{a}_{k-1}.$$

Consequently there is

$$\overset{\circ}{\mathbf{T}}_k = \mathbf{I} + \kappa_k \mathbf{A}_k(r_0)h + \mathbf{O}((h/r_0)^2) \tag{4.11}$$

in agreement with (4.3). ■

Since the governing differential equation of the homogenized periodically layered tube, (4.8) with (4.9), has the same structure as that for the homogeneous transversely isotropic tube, (2.4) with (2.5), the corresponding solutions of Sec. 3 can be taken over. One has only to substitute K, L, M, N by $\bar{K}, \bar{L}, \bar{M}, \bar{N}$ and as a consequence, c by $\bar{c}$ and λ by $\bar{\lambda}$, where

$$\bar{c} = \bar{M}\bar{L} + \bar{N}^2; \qquad \bar{\lambda} = \sqrt{\bar{c}}. \tag{4.12}$$

In this way one obtains from (2.22), for a thick tube with outer radius b and inner radius a and loaded by internal pressure p,

$$u = \frac{pb}{\left(\frac{a}{b}\right)^{\bar{\lambda}-1} - \left(\frac{a}{b}\right)^{-\bar{\lambda}-1}} \frac{h^*}{E^*} \bar{M} \left[\frac{\left(\frac{r}{b}\right)^{-\bar{\lambda}}}{\bar{N} - \bar{\lambda}} - \frac{\left(\frac{r}{b}\right)^{\bar{\lambda}}}{\bar{N} + \bar{\lambda}} \right];$$

(4.13)
$$\sigma_r = -\frac{p}{\left(\frac{a}{b}\right)^{\bar{\lambda}-1} - \left(\frac{a}{b}\right)^{-\bar{\lambda}-1}} \left[\left(\frac{r}{b}\right)^{\bar{\lambda}-1} - \left(\frac{r}{b}\right)^{-\bar{\lambda}-1} \right];$$

$$\sigma_\varphi = -\frac{p\bar{\lambda}}{\left(\frac{a}{b}\right)^{\bar{\lambda}-1} - \left(\frac{a}{b}\right)^{-\bar{\lambda}-1}} \left[\left(\frac{r}{b}\right)^{\bar{\lambda}-1} + \left(\frac{r}{b}\right)^{-\bar{\lambda}-1} \right].$$

Here σ_φ is to be considered as an average value; it follows from (2.3) with $\bar{N}$ and $\bar{L}$ instead of N and L, or equivalently from the equilibrium equation in (2.1) using σ_r from (4.13). Note, however, that for a finite layering, $u(r)$ and $\sigma_r(r)$ are continuous while $\sigma_\varphi(r)$ is discontinuous at the interfaces. Therefore the knowledge of σ_φ in the individual layers of a thin, but finite layer group is of interest. Assuming that for this case u and σ_r from (4.13) are a sufficient approximation, one obtains with (2.3) for layer (k) of a layer group at first

(4.14)
$$\sigma_{\varphi k}(r) = N_k \sigma_r(r) + \frac{L_k}{r} u^*(r) \qquad (k = 1, 2 \ldots m),$$

and finally

(4.15)
$$\sigma_{\varphi k} = -\frac{p}{\left(\frac{a}{b}\right)^{\bar{\lambda}-1} - \left(\frac{a}{b}\right)^{-\bar{\lambda}-1}} \left[\left(\frac{\bar{M} L_k}{\bar{N} + \bar{\lambda}} + N_k \right) \left(\frac{r}{b}\right)^{\bar{\lambda}-1} - \left(\frac{\bar{M} L_k}{\bar{N} - \bar{\lambda}} + N_k \right) \left(\frac{r}{b}\right)^{-\bar{\lambda}-1} \right].$$

It is further interesting to note that the individual circumferential stresses $\sigma_{\varphi k}$ (4.15) and the smeared stress σ_φ in (4.13) are related to each other according to

(4.16)
$$\sigma_\varphi = \sum_{k=1}^{m} \kappa_k \sigma_{\varphi k}.$$

For the proof, the relations (4.15), (4.6) and (4.12) are needed. It can easily be checked that in the case of identical materials of all basic layers, the difference between formulae (4.13) and formulae (2.22) disappears.

According to (4.9) (periodical layering) and (2.5) (homogeneous tube under plane stress or plane strain, respectively), both problems are formally equivalent. Especially a periodically layered tube consisting of isotropic basic layers can be substituted uniquely by a homogeneous, transversely isotropic tube in the case of plane stress. Indeed, equating (2.6)′ and (4.6), one obtains, taking into account (2.8), three independent equations for the three material parameters being involved, namely E, E' and ν'. The result is

$$E = \frac{E^*}{h^*}\bar{L} = \frac{E^*}{h^*}\sum \kappa_k L_k = \sum \kappa_k E_k;$$

$$\frac{1}{E'} = \frac{h^*}{E^*}\left(\bar{M} + \frac{\bar{N}^2}{\bar{L}}\right) = \frac{h^*}{E^*}\left(\sum \kappa_k M_k + \frac{(\sum \kappa_k N_k)^2}{\sum \kappa_k L_k}\right)$$

$$= \sum \kappa_k \frac{1-\nu_k^2}{E_k} + \frac{(\sum \kappa_k \nu_k)^2}{\sum \kappa_k E_k};$$

$$\frac{\nu'}{E'} = \frac{h^*}{E^*}\frac{\bar{N}}{\bar{L}} = \frac{h^*}{E^*}\frac{\sum \kappa_k N_k}{\sum \kappa_k L_k} = \frac{\sum \kappa_k \nu_k}{\sum \kappa_k E_k},$$

and with (4.1)

$$Eh = \sum E_k h_k; \qquad \frac{h}{E'} = \sum (1-\nu_k^2)\frac{h_k}{E_k} + \frac{(\sum \nu_k h_k)^2}{\sum E_k h_k}; \tag{4.17}$$

$$\frac{\nu'}{E'} = \frac{\sum \nu_k h_k}{\sum E_k h_k},$$

(all sums from $k = 1$ to $k = \mathrm{m}$).

As it can be seen from (4.18), the resulting stiffness of the layered tube in tangential direction is the sum of the stiffnesses of the individual layers. In radial direction the statement, that the resulting compliance of the layered tube equals the sum of the compliances of the individual layers holds exactly only if $\nu_k = 0$, otherwise approximatively.

5. Numerical Example

The numerical example concerns a tube under internal pressure consisting of five double layers, see Fig. 3. This periodically layered tube is analysed first by means of the exact (inhomogeneous) model. All basic layers have the same

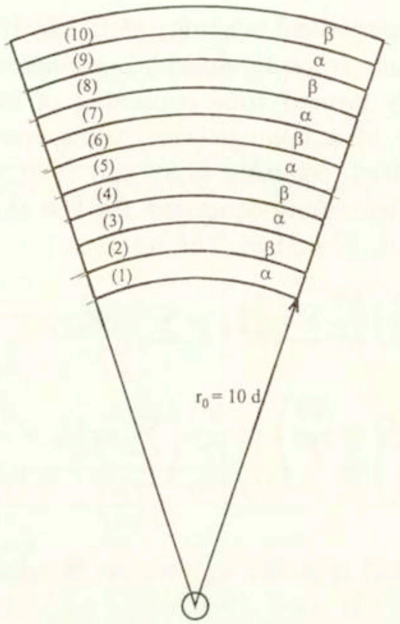

FIG. 3. Periodically layered tube consisting of 5 double layers.

thickness d and are assumed to be isotropic and under plane strain. For the layers of type α and the layers of type β (Fig. 3), we take

$$h_k = d \quad (k = 1, 2, \dots 10); \qquad \nu_\alpha = \nu_\beta = \frac{1}{3}; \quad E_\alpha = E; \quad E_\beta = 3E$$

and we choose $h^* = d$ and $E^* = E$. Then from (2.7), (2.12) and (2.14) there follows

$$K_\alpha = \frac{1}{2}; \qquad L_\alpha = \frac{9}{8}d; \qquad M_\alpha = \frac{2}{3}\frac{1}{d}; \qquad N_\alpha = \frac{1}{2}; \qquad \lambda_\alpha = 1;$$

$$K_\beta = \frac{1}{2}; \qquad L_\beta = \frac{27}{8}d; \qquad M_\beta = \frac{2}{9}\frac{1}{d}; \qquad N_\beta = \frac{1}{2}; \qquad \lambda_\beta = 1;$$

and after (3.1) $\rho_k = \dfrac{1}{9+k}$. Next the elements of the *layer transfer-matrices* $\mathbf{T}_k$ can be evaluated from (3.7) and (3.3). For instance, one obtains

$$t_k^{12}(\rho_k) = \left\{ \begin{array}{c} 1 \\ 3 \end{array} \right\} \frac{9}{16}\rho_k \left[1 - (1+\rho_k)^{-2}\right] \quad \text{for} \quad k = \left\{ \begin{array}{l} 1, 3, 5, 7, 9 \\ 2, 4, 6, 8, 10 \end{array} \right. .$$

Further from (3.8) and from the boundary conditions $\sigma_{r_0} = -p$ and $\sigma_{r_{10}} = 0$ one gets $u_0^* = (s_{11}/s_{12})p$ and hence the *initial state vector* $\mathbf{a}_0$ and subsequently – from (3.6) – $\mathbf{a}_1, \mathbf{a}_2, \dots \mathbf{a}_{10}$ i.e. the *radial stresses* σ_{rk} and the *radial displacements* u_k.

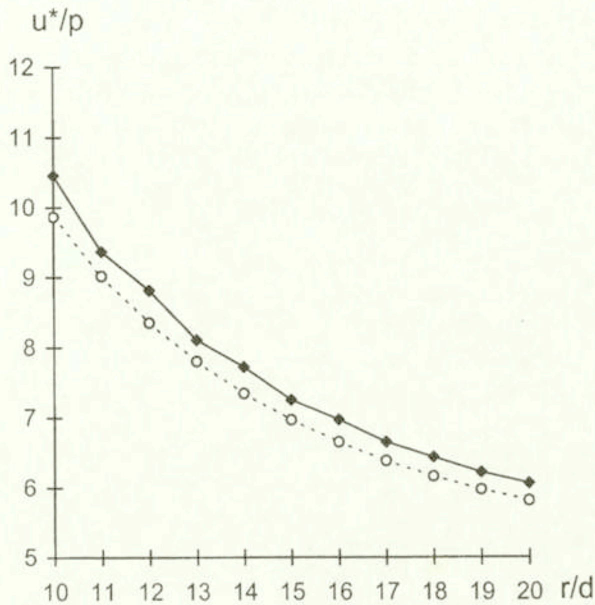

FIG. 4. Radial displacement distribution for the tube in Fig. 3, —— inhomogeneous model, - - - - homogenized model.

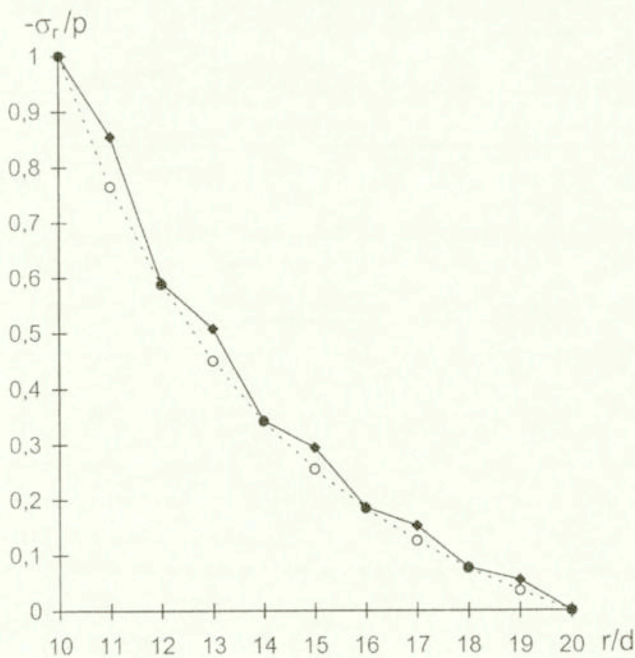

FIG. 5. Radial stress distribution for the tube in Fig. 3, —— inhomogeneous model, - - - - homogenized model.

These ones are sketched as solid lines in Figs. 4 and 5, respectively, where $u^* = (E/d)u$. The solid line in Fig. 6 represents the circumferential stress distribution which is discontinous due to the discontinuity of the stiffness parameters. They have been calculated from (2.3) for $r = r_k = (10+k)d$ and $k = 0, 1, \dots 10$

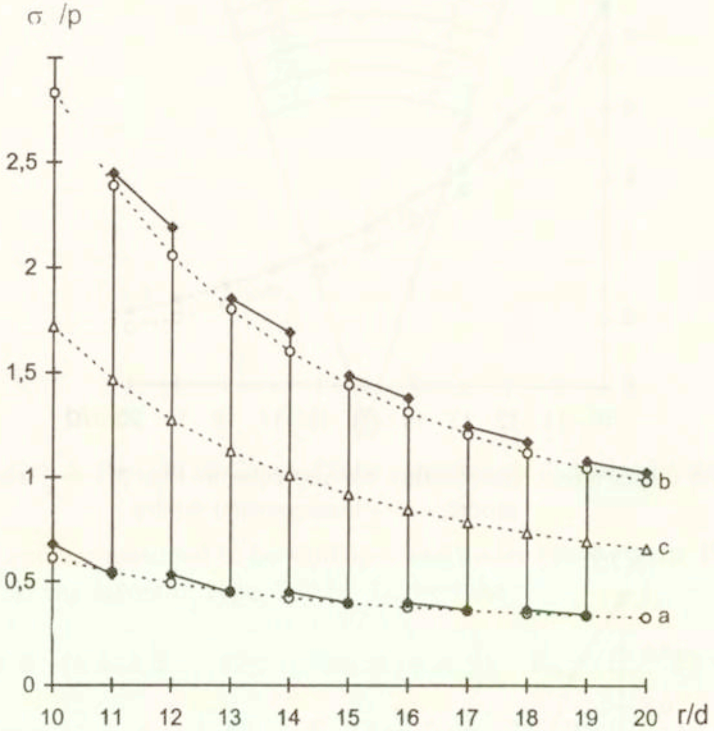

FIG. 6. Circumferential stress distribution for the tube in Fig. 3, —— inhomogeneous model, - - - - homogenized model, (a) for layers of type α, (b) for layers of type β; (c) mean values).

$$\left.\begin{matrix} \sigma_{k|\alpha} \\ \sigma_{k|\beta} \end{matrix}\right\} = \frac{1}{2}\sigma_{rk} + \left\{\begin{matrix} 1 \\ 3 \end{matrix}\right\} \frac{9}{8(10+k)}\frac{E}{d}u \quad \text{for layers} \left\{\begin{matrix} \alpha \\ \beta \end{matrix}\right. .$$

The corresponding *homogenized model* (being exact only in the case of infinitely many double layers with vanishing thickness) yields an approximate solution for the layered tube in Fig. 3. For this one there is

$$h = 2d; \quad a = 10d; \quad b = 20d; \quad \kappa_1 = \kappa_2 = \frac{1}{2};$$

$$K_1 = K_\alpha; \quad L_1 = L_\alpha; \quad M_1 = M_\alpha; \quad N_1 = N_\alpha;$$

$$K_2 = K_\beta; \quad L_2 = L_\beta; \quad M_2 = M_\beta; \quad N_2 = N_\beta$$

and according to (4.6) and (4.12)

$$\bar{K} = \frac{1}{2}; \quad \bar{L} = \frac{9}{4}d; \quad \bar{M} = \frac{4}{9}\frac{1}{d}; \quad \bar{N} = \frac{1}{2}; \quad \bar{c} = \frac{5}{4}; \quad \bar{\lambda} = \frac{\sqrt{5}}{2}.$$

The closed-form solution is given in (4.13). For instance, the radial displacements as a function of the radial coordinate r reads

$$u = \frac{d}{E}p\frac{160}{9}\frac{1}{2^{(\sqrt{5}+2)/2} - 2^{-(\sqrt{5}-2)/2}}\left[\frac{1}{\sqrt{5}+1}\left(\frac{r}{20d}\right)^{\sqrt{5}/2} + \frac{1}{\sqrt{5}-1}\left(\frac{r}{20d}\right)^{-\sqrt{5}/2}\right].$$

The corresponding graphs for u, σ_r and σ_φ are given in Fig. 4, 5 and 6 (dotted lines). The circumferential stress σ_φ $(4.13)_3$ is the mean value $(\sigma_{\varphi\alpha} + \sigma_{\varphi\beta})/2$, while the formulae (4.15), namely

$$\left.\begin{matrix}\sigma_{\varphi\alpha}\\ \sigma_{\varphi\beta}\end{matrix}\right\} = p\frac{1}{2^{(\sqrt{5}+2)/2} - 2^{-(\sqrt{5}-2)/2}}\left[\left(\left\{\begin{matrix}1\\3\end{matrix}\right\}\frac{1}{\sqrt{5}+1} + \frac{1}{2}\right)\left(\frac{r}{20d}\right)^{(\sqrt{5}-2)/2} + \left(\left\{\begin{matrix}1\\3\end{matrix}\right\}\frac{1}{\sqrt{5}-1} - \frac{1}{2}\right)\left(\frac{r}{20d}\right)^{-(\sqrt{5}+2)/2}\right] \quad \text{for layers} \quad \left\{\begin{matrix}\alpha\\ \beta\end{matrix}\right.$$

represent the values which are valid piecewise for the layers of type α (lower curve) and type β (upper curve), respectively.

The comparison of the results for the exact (inhomogeneous), but expensive and the approximate (homogenized) model shows that, even for a relatively low number of layer groups (double layers), one gets a rather good agreement.

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