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Macroscopic equations for nonstationary flow of Stokesian fluid through porous elastic medium

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The aim of this contribution is to apply homogenization methods in order to describe the nonstationary flow of a viscous fluid through a microperiodic porous elastic medium. By using the method of two-scale asymptotic expansions, the macroscopic phenomenological equations describing such a two-phase structure are derived and the formulae for the effective mechanical coefficients are given. The asymptotic approach is justified by the two-scale convergence. It is shown that Darcy's law is nonlocal in time.

1. Introduction

One can distinguish two approaches to modelling the mechanical behaviour of porous media. The first approach, more traditional, dates back to the papers by Biot [16 – 19], and may be called a phenomenological one. The nice book by Coussy [26] develops such an approach within the framework of modern continuum mechanics and applies it to elastic, thermoelastic and inelastic porous media, cf. also the paper by Cieszko and Kubik [23, 24] and the references cited therein. The second approach exploits the microstructure of the medium. The micro-macro passage is performed by using averaging techniques, the mixture theory or various homogenization methods. Allaire [6] and Mikelić [40] studied flows of Stokesian fluids through undeformed microperiodic porous media by using the homogenization theory. The first of these authors assumed the scaling of the viscosity while the second author scaled the liquid density. After homogenization they arrived at different Darcy's laws. More precisely, Darcy's law derived by Allaire is nonlocal in time while that obtained by Mikelić [40] coincides with the well known Darcy's law derived by many authors for stationary flow. The aim of the present contribution is to investigate nonstationary flows of Stokesian fluids through linear elastic porous medium, and in particular, to derive macroscopic equations of the Biot type. The scaling of viscosity is assumed. Thus our results extend those due to Allaire [6].

Our considerations differ from those performed in the papers [9 – 12, 22, 24, 25, 27, 37, 38]. In contrast to [10, 11, 12], we do not exploit the time transformation method. The approach employed by us is straightforward and exploits the two-scale asymptotic method which is justified by the two-scale convergence method developed by NGUETSENG [44] and ALLAIRE [7]. The comprehensive papers [2, 15] and the books [14, 20, 33, 46, 47] provide many applications of homogenization methods to modelling the flows through porous media. Stationary flow of electrolytes through such media was studied in [30, 49, 51, 52].

The plan of the paper is as follows. In Secs. 2 and 3 the microperiodic medium is introduced. Asymptotic homogenization is performed in Sec. 4. In Sec. 5 the formal results obtained in the previous section are justified by the two-scale convergence method. The passage to the stationary case is studied in Sec. 6. In Sec. 7 important earlier contributions are discussed. Two appendices complete the paper.

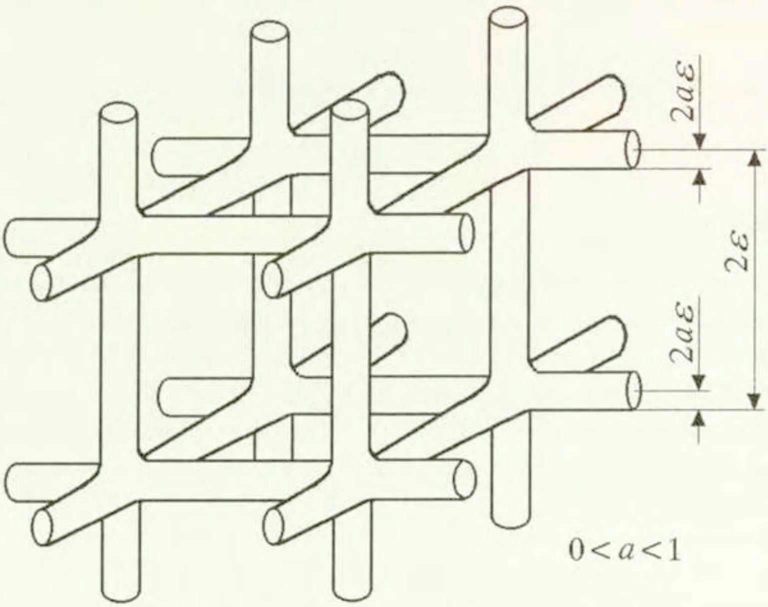

FIG. 1. Example of a skeleton, after ALLAIRE [4].

2. Notations and basic relations

Let $Y = (-1,1)^N \subset \mathbb{R}^N$ $(N \geqslant 2)$ be the basic cell and Y_S – a closed subset of $\overline{Y}$ (the bar denotes the closure of the set), cf. [4, 5]. Further, we set $Y_L = Y \backslash Y_S$; Y_L denotes that part of Y which is occupied by the fluid. Obviously, Y_S stands for the part of Y occupied by the solid. The closed set Y_S is repeated by Y-periodicity and fills the entire space $\mathbb{R}^N$, in order to obtain a closed set of $\mathbb{R}^N$, denoted E_S; further $E_L = \mathbb{R}^N \backslash E_S$ and $E_S = \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N \mid \exists(k_1, \ldots, k_N) \in \mathbb{Z}^N$ such that $(x_1 - 2k_1, \ldots, x_N - 2k_N) \in Y_S\}$. Here $\mathbb{Z}$ denotes the set of integers.

HYPOTHESES, cf. [4]

(i) Y_L and Y_S have strictly positive measures in $\overline{Y}$.

(ii) E_L and the interior of E_S are open sets with boundary of class C^1, and are locally situated on one side of their boundary. Moreover, E_L is connected.

(iii) Y_L is an open connected set with a locally Lipschitz boundary.

REMARK 2.1. The hypothesis (i) implies that the elementary cell Y contains fluid and solid together. Next, (ii) says that Y_L is Y-periodic (E_L has a boundary of class C^1) and $\overline{Y}$ has an intersection with each of its faces which has a strictly positive surface measure, cf. Figs. 2, 3.

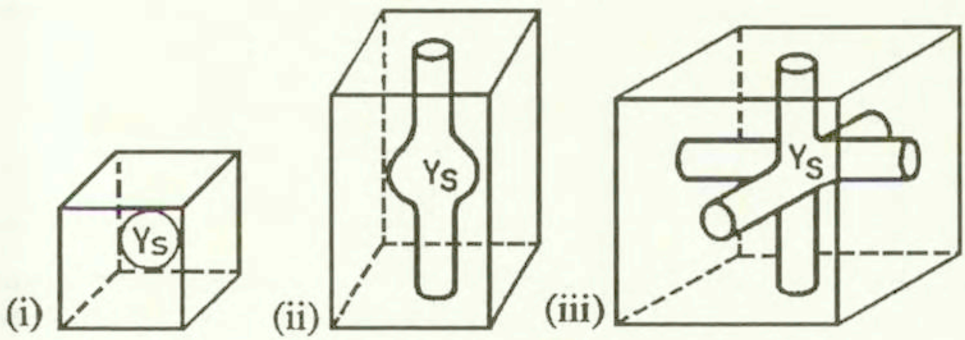

FIG. 2. Three typical situations, which agree with hypotheses (i)-(iii), after ALLAIRE [4].

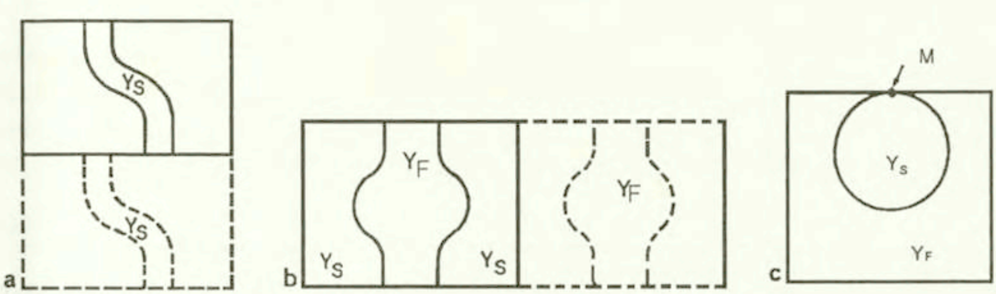

FIG. 3. Forbidden situations. (a) The boundary of E_S is not of class C^1 because Y_S is not Y-periodic. (b) No contact between the fluid parts of two adjacent cells implies that E_L is not connected. (c) Although E_L has a smooth boundary, ∂Y_L is not locally Lipschitz at point M, after ALLAIRE (1989).

Let Ω be an open bounded and connected set of $\mathbb{R}^N$ with the boundary $\partial\Omega$ of class C^1. In fact, it is sufficient to assume that $\partial\Omega$ and ∂E_L are locally Lipschitz. The domain Ω is assumed to have an εY-periodic structure, $\varepsilon = l/\mathsf{L}$, where l, L, are typical length scales associated with pores and the region and Ω, respectively. Next, we set

$$\Omega_L^\varepsilon = \Omega \setminus \bigcup_{i=1}^{N(\varepsilon)} Y_{S_i}^\varepsilon, \qquad \Omega_S^\varepsilon = \Omega \setminus \Omega_L^\varepsilon. \tag{2.1}$$

The set Ω is covered with a regular mesh of size ε, each cell being a cube Y_i^ε, $1 \leqslant i \leqslant N(\varepsilon)$, and

$$N(\varepsilon) = \frac{|\Omega|}{(2\varepsilon)^N}[1 + o(1)]. \tag{2.2}$$

The interface solid-liquid is denoted by Γ^ε; $\partial\Omega_S^\varepsilon$ ($\partial\Omega_L^\varepsilon$) is the boundary of Ω_S^ε (Ω_L^ε). For more details related to the description of the porous body the reader is referred to [4].
We set

$$\langle(\cdot)\rangle = \frac{1}{|Y|}\int_Y (\cdot)\,dy, \qquad \langle(\cdot)\rangle_\alpha = \frac{1}{|Y|}\int_{Y_\alpha} (\cdot)\,dy, \qquad \alpha = S, L \tag{2.3}$$

and $\partial Y_S = \Gamma_Y \cup P_S$, $\partial Y_L = \Gamma_Y \cup P_L$; Γ_Y is the local solid-liquid contact surface; and P_S and P_L are the parts of the surfaces of the solid and liquid, respectively, coinciding with the boundary of Y. The porosity f is defined as the volume fraction of the liquid in the considered solid-liquid composite medium

$$f = \frac{|Y_L|}{|Y|}, \qquad 1 - f = \frac{|Y_S|}{|Y|}. \tag{2.4}$$

By $\mathbf{n} = \mathbf{n}^L$ we denote the exterior unit vector normal to $\partial\Omega_L^\varepsilon$.

3. Equations of microperiodic porous media

For a fixed $\varepsilon > 0$ all the relevant quantities are denoted by the superscript ε. Let us denote by $\mathbf{u}^\varepsilon$ and $\mathbf{v}^\varepsilon$ the fields of displacements in the elastic skeleton and the velocity field in Ω_L^ε, respectively. By p^ε we denote the pressure in the liquid phase. These fields satisfy the following equations:

$$
(3.1)\quad \left|
\begin{aligned}
&\varrho^S \ddot{u}_i^\varepsilon(t,x) = \frac{\partial}{\partial x_j}\Big(a_{ijmn}e_{mn}(\mathbf{u}^\varepsilon(t,x))\Big) + F_i^S(t,x) \quad \text{in} \quad (0,T)\times\Omega_S^\varepsilon,\\
&\varrho^L \dot{v}_i(t,x) = \frac{\partial}{\partial x_j}\Big(-p^\varepsilon(t,x)\delta_{ij} + \varepsilon^2\eta_{ijmn}e_{mn}\left(\mathbf{v}^\varepsilon(t,x)\right)\Big) + F_i^L(t,x)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in} \quad (0,T)\times\Omega_L^\varepsilon,\\
&\operatorname{div}\mathbf{v}^\varepsilon(t,x) = 0 \quad \text{in} \quad (0,T)\times\Omega_L^\varepsilon.
\end{aligned}\right.
$$

The conditions imposed on the solid-liquid interface Γ^ε read

$$
(3.2)\qquad [\![\sigma_{ij}^\varepsilon]\!]n_j = 0 \quad \text{on} \quad (0,T)\times\Gamma^\varepsilon,
$$

$$
(3.3)\qquad \mathbf{v}^\varepsilon(t,x) = \dot{\mathbf{u}}^\varepsilon(t,x) \quad \text{on} \quad (0,T)\times\Gamma^\varepsilon.
$$

The stress tensor $\boldsymbol{\sigma}^\varepsilon$ is specified by

$$
(3.4)\qquad \sigma_{ij}^\varepsilon = \begin{cases} a_{ijmn}e_{mn}(\mathbf{u}^\varepsilon) & \text{in} \quad (0,T)\times\Omega_S^\varepsilon,\\ -p^\varepsilon\delta_{ij} + \varepsilon^2\eta_{ijmn}e_{mn}(\mathbf{v}^\varepsilon) & \text{in} \quad (0,T)\times\Omega_L^\varepsilon. \end{cases}
$$

Here $e_{ij}(\boldsymbol{z}) = z_{(i,j)} = \dfrac{1}{2}\left(\dfrac{\partial z_i}{\partial x_j} + \dfrac{\partial z_j}{\partial x_i}\right)$, $\mathbf{F}^S$ and $\mathbf{F}^L$ stand for the body forces. The jump $[\![\sigma_{ij}^\varepsilon]\!]$ on Γ^ε is given by

$$
[\![\sigma_{ij}^\varepsilon]\!]n_j = \sigma_{ij}^{L\varepsilon}n_j - \sigma_{ij}^{S\varepsilon}n_j.
$$

For an isotropic liquid the tensor $\boldsymbol{\eta} = (\eta_{ijkl})$ takes the following form:

$$
\eta_{ijmn} = \eta(\delta_{im}\delta_{jn} + \delta_{jm}\delta_{in} - \frac{2}{3}\delta_{ij}\delta_{mn}),
$$

where η denotes the viscosity. For the sake of simplicity, the nonlinear convective term in the equations of the fluid motion has been neglected. The moduli a_{ijkl} and η_{ijkl} satisfy the standard symmetry and coercivity conditions. Note that in Eq. $(3.1)_2$ and $(3.4)_2$ the following rescaling is introduced, cf. [4, 6, 22]

$$
(3.5)\qquad \eta_{ijmn} \rightsquigarrow \varepsilon^2\eta_{ijmn}.
$$

According to the method of two-scale asymptotic expansions we make the following *Ansatz*, cf. [47]:

$$
(3.6)\qquad \begin{aligned}
p^\varepsilon(t,x) &= p^{(0)}(t,x,y) + \varepsilon p^{(1)}(t,x,y) + \varepsilon^2 p^{(2)}(t,x,y) + \ldots, \qquad y = x/\varepsilon,\\
\mathbf{u}^\varepsilon(t,x) &= \mathbf{u}^{(0)}(t,x,y) + \varepsilon\mathbf{u}^{(1)}(t,x,y) + \varepsilon^2\mathbf{u}^{(2)}(t,x,y) + \ldots, \qquad y = x/\varepsilon,
\end{aligned}
$$

and similarly for $\mathbf{v}^\varepsilon(t,x)$. The functions $p^{(0)}(t,x,y)$, $p^{(1)}(t,x,y)$, etc., and $\mathbf{u}^{(0)}(t,x,y)$, $\mathbf{u}^{(1)}(t,x,y)$ etc., are Y-periodic in y.
The parabolic-hyperbolic system of Eqs. (3.1) – (3.4) is to be supplemented by the initial conditions, here assumed to be homogeneous

$$\mathbf{v}^\varepsilon(0,x) = \mathbf{0} \qquad \text{in} \quad \Omega_L^\varepsilon,$$

$$\mathbf{u}^\varepsilon(0,x) = \mathbf{0}; \qquad \dot{\mathbf{u}}^\varepsilon(0,x) = \mathbf{0} \quad \text{in} \quad \Omega_S^\varepsilon.$$

Next, taking into account the formula for the total derivative

$$\frac{d}{dx_i} f(x,y) = \left(\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\right) f(x,y), \qquad y = \frac{x}{\varepsilon},$$

and comparing terms with the same power of ε, we arrive at the homogenized set of equations.

We observe that the quasi-stationary flow of a viscous fluid through a linear elastic microperiodic media was examined in [9].

4. Homogenization

The interface condition (3.2) can be rewritten as follows:

$$\text{(4.1)} \qquad a_{ijmn} e_{mn}(\mathbf{u}^\varepsilon) n_j = \left(-p^\varepsilon \delta_{ij} + \varepsilon^2 \eta_{ijmn} \frac{\partial v_m^\varepsilon}{\partial x_n}\right) n_j.$$

After substitution of expansions (3.6) we get

$$\text{(4.2)} \qquad \left[a_{ijmn}\left(\frac{\partial}{\partial x_n} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_n}\right)\left(u_m^{(0)} + \varepsilon u_m^{(1)} + \varepsilon^2 u_m^{(2)} + \ldots\right)\right] n_j = -\left(p^{(0)} + \varepsilon p^{(1)}\right.$$

$$\left. + \varepsilon^2 p^{(2)} + \ldots\right)\delta_{ij} n_j + \varepsilon^2 \eta_{ijmn}\left(\frac{\partial}{\partial x_n} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_n}\right)(v_m^{(0)} + \varepsilon v_m^{(1)} + \ldots) n_j$$

$$\text{on} \quad (0,T) \times \Gamma^\varepsilon.$$

Comparing the terms associated with ε^{-1} we obtain

$$\text{(4.3)} \qquad a_{ijmn} e_{mn}^y(\mathbf{u}^{(0)}) N_j = 0 \qquad \text{on} \quad (0,T) \times \Omega \times \Gamma_Y,$$

where $e_{ij}^y(\boldsymbol{\xi}) = \left(\dfrac{\partial \xi_i}{\partial y_j} + \dfrac{\partial \xi_j}{\partial y_i}\right)\Big/ 2$ and $\mathbf{N}$ stands for the exterior unit vector normal to ∂Y_L. Further, the terms linked with ε^0 in Eq. (4.2) yield

$$\text{(4.4)} \qquad \left[a_{ijmn}\left(\frac{\partial u_m^{(0)}}{\partial x_n} + \frac{\partial u_m^{(1)}}{\partial y_n}\right)\right] N_j = -p^{(0)} N_i.$$

It can be shown that, cf. (A.3),

$$\mathbf{u}^{(0)} = \mathbf{u}^{(0)}(t,x). \tag{4.5}$$

Equations (A.4) and (4.4) are satisfied provided that

$$u_m^{(1)}(t,x,y) = A_m^{(pq)}(t,x,y)\,\frac{\partial}{\partial x_q}u_p^{(0)}(t,x) + P^{(m)}(t,x,y)\,p^{(0)}(t,x), \tag{4.6}$$

and the functions $\mathbf{A}^{(pq)}$ and $P^{(m)}$ are Y-periodic solutions to the following local equations on $(0,T)\times Y_S$:

$$\begin{aligned} &\frac{\partial}{\partial y_j}\Big(a_{ijpq} + a_{ijmn}e^y_{mn}(\mathbf{A}^{(pq)})\Big) = 0,\\ &\frac{\partial}{\partial y_j}\Big(a_{ijmn}\frac{\partial}{\partial y_n}P^{(m)} + \delta_{ij}\Big) = 0. \end{aligned} \tag{4.7}$$

Hence we get

$$\begin{aligned} &\Big(a_{ijpq} + a_{ijmn}e^y_{mn}(\mathbf{A}^{(pq)})\Big)N_j = 0, \quad \text{on} \quad (0,T)\times \varGamma_Y,\\ &\Big(a_{ijmn}\frac{\partial}{\partial y_n}P^{(m)} + \delta_{ij}\Big)N_j = 0, \quad \text{on} \quad (0,T)\times \varGamma_Y. \end{aligned} \tag{4.8}$$

In (4.7) and (4.8) the macroscopic variable x is treated as a parameter.

The interface condition (3.3) assumes the form

$$\begin{aligned} &\dot{\mathbf{u}}^{(0)}(t,x) + \varepsilon\dot{\mathbf{u}}^{(1)}(t,x,y) + \varepsilon^2\dot{\mathbf{u}}^{(2)}(t,x,y) + \ldots\\ &\qquad = \mathbf{v}^{(0)}(t,x,y) + \varepsilon\mathbf{v}^{(1)}(t,x,y) + \varepsilon^2\mathbf{v}^{(2)}(t,x,y) + \ldots \end{aligned} \tag{4.9}$$

Hence we obtain

$$\dot{\mathbf{u}}^{(\ell)} = \mathbf{v}^{(\ell)}, \qquad \ell = 0,1,2,\ldots \tag{4.10}$$

Applying asymptotic expansions to the constitutive relation (3.4) and comparing the terms linked with ε^0 we get

$$\sigma_{ij}^{(0)} = \begin{cases} a_{ijmn}\Big(e_{mn}(\mathbf{u}^{(0)}) + e^y_{mn}(\mathbf{u}^{(1)})\Big) & \text{in} \quad (0,T)\times\varOmega\times Y_S,\\ -p^{(0)}\delta_{ij} & \text{in} \quad (0,T)\times\varOmega\times Y_L. \end{cases} \tag{4.11}$$

We observe that

$$\langle\sigma_{ij}^{(0)}\rangle = \langle\sigma_{ij}^{(0)}\rangle_S + \langle\sigma_{ij}^{(0)}\rangle_L. \tag{4.12}$$

Using Eqs. (4.6), (4.11) and (4.12) we get

$$\langle\sigma_{ij}^{(0)}\rangle = a_{ijpq}^{h} e_{pq}(\mathbf{u}^{(0)}) + \Big(\langle a_{ijmn}\frac{\partial}{\partial y_n}P^{(m)}\rangle_S - f\delta_{ij}\Big)p^{(0)}, \tag{4.13}$$

where

$$a_{ijpq}^{h} = \langle a_{ijpq} + a_{ijmn}e_{mn}(\mathbf{A}^{(pq)})\rangle_S. \tag{4.14}$$

The coefficients $\mathbf{A}^{(pq)}$ and $P^{(m)}$ are to be determined from the local equations (4.7) jointly with Eq. (4.8).

From Eqs. $(3.1)_1$ and $(3.1)_2$, by comparing the terms linked with ε^0 we get

$$\begin{aligned} \varrho^S \ddot{u}_i^{(0)} = F_i^S &+ \frac{\partial}{\partial x_j}\Big[a_{ijmn}\Big(e_{mn}(\mathbf{u}^{(0)}) + e_{mn}^{y}(\mathbf{u}^{(1)})\Big)\Big] \\ &+ \frac{\partial}{\partial y_j}\Big[a_{ijmn}\Big(e_{mn}(\mathbf{u}^{(1)}) + e_{mn}^{y}(\mathbf{u}^{(2)})\Big)\Big] \quad \text{in} \quad (0,T)\times\Omega\times Y_S, \end{aligned} \tag{4.15}$$

$$\varrho^L \dot{v}_i^{(0)} = F_i^L - \frac{\partial}{\partial x_i}p^{(0)} - \frac{\partial}{\partial y_i}p^{(1)} + \frac{\partial}{\partial y_j}\Big(\eta_{ijmn}e_{mn}^{y}(\boldsymbol{v}^{(0)})\Big) \quad \text{in} \quad (0,T)\times\Omega\times Y_L. \tag{4.16}$$

On the other hand, the terms linked with ε in the interface condition (4.2) lead to the relation

$$\Big[a_{ijmn}\Big(e_{mn}(\mathbf{u}^{(1)}) + e_{mn}^{y}(\mathbf{u}^{(2)})\Big)\Big]N_j = \Big(-p^{(1)}\delta_{ij} - \eta_{ijmn}e_{mn}(\mathbf{v}^{(0)})\Big)N_j \quad \text{on} \quad (0,T)\times\Omega\times\Gamma_Y. \tag{4.17}$$

Integration of Eq. (4.15) over Y_S and (4.16) over Y_L yields

$$\begin{aligned} (1-f)\varrho^S\ddot{u}_i^{(0)} &= (1-f)F_i^S + \frac{\partial}{\partial x_j}\langle a_{ijmn}\Big(e_{mn}(\mathbf{u}^{(0)}) + e_{mn}^{y}(\mathbf{u}^{(1)})\Big)\rangle_S \\ &- \frac{1}{|Y|}\int_{\partial Y_S}\Big[a_{ijmn}\Big(e_{mn}(\mathbf{u}^{(1)}) + e_{mn}^{y}(\mathbf{u}^{(2)})\Big)\Big]N_j\,dA \quad \text{in} \quad (0,T)\times\Omega, \\ \langle\varrho^L\dot{v}_i^{(0)}\rangle_L &= fF_i^L - f\frac{\partial}{\partial x_i}p^{(0)} \\ &+ \frac{1}{|Y|}\int_{\partial Y_L}\Big[-p^{(1)}\delta_{ij} + \eta_{ijmn}e_{mn}^{y}(\mathbf{v}^{(0)})\Big]N_j\,dA, \quad \text{in} \quad (0,T)\times\Omega. \end{aligned} \tag{4.18}$$

Adding Eqs. (4.18), using the interface relation (4.17) and taking into account (4.11) and (4.12), we arrive at

$$(1-f)\varrho^S \ddot{u}_i^{(0)} + \varrho^L \langle \dot{v}_i^{(0)} \rangle_L = \frac{\partial}{\partial x_j} \langle \sigma_{ij}^{(0)} \rangle + (1-f)F_i^S + fF_i^L \qquad \text{in} \quad (0,T)\times\Omega. \tag{4.19}$$

This is the macroscopic equation of motion of the porous medium filled with liquid.

Since $\mathbf{u}^{(0)} = \mathbf{u}^{(0)}(t,x)$, therefore (4.16) furnishes

$$\varrho^L \dot{\overline{v}}_i^{(0)} = F_i^L - \varrho^L \ddot{u}_i^{(0)} - \frac{\partial}{\partial x_i} p^{(0)} - \frac{\partial}{\partial y_i} p^{(1)} + \frac{\partial}{\partial y_j}\Big(\eta_{ijmn} e_{mn}^y(\overline{\mathbf{v}}^{(0)})\Big), \tag{4.20}$$

where

$$\overline{v}_m^{(0)}(t,x,y) = v_m^{(0)}(t,x,y) - \dot{u}_m^{(0)}(t,x) \qquad \text{on} \quad (0,T)\times\Omega\times\Gamma_Y. \tag{4.21}$$

In virtue of (4.10), for $\ell = 0$ we get

$$\overline{\mathbf{v}}^{(0)}(t,x,y) = \mathbf{0} \qquad \text{on} \qquad (0,T)\times\Omega\times\Gamma_Y. \tag{4.22}$$

Since the problem considered is linear, therefore $p^{(1)}$ is of the form

$$p^{(1)}(t,x,y) = \gamma^{(m)}(t,y)\Big(F_m^L(t,x) - \varrho^L \ddot{u}_m^{(0)}(t,x) - \frac{\partial}{\partial x_m} p^{(0)}(t,x)\Big), \tag{4.23}$$

where $\gamma^{(m)}$ is Y-periodic in y. Then Eq. (4.20) is written as follows:

$$\varrho^L \dot{\overline{v}}_i^{(0)} = \left(F_m^L - \varrho^L \ddot{u}_m^{(0)} - \frac{\partial}{\partial x_m} p^{(0)}\right)\left(\delta_{im} - \frac{\partial}{\partial y_i}\gamma^{(m)}\right) + \frac{\partial}{\partial y_j}\left(\eta_{ijmn}\frac{\partial}{\partial y_n}\overline{v}_m^{(0)}\right). \tag{4.24}$$

The last equation is satisfied provided that $\overline{\mathbf{v}}^{(0)}$ is given in the form of time-convolution

$$\overline{v}_m^{(0)}(t,x,y) = \frac{1}{\varrho^L}\int_0^t \Big(F_s^L(\tau,x) - \varrho^L \ddot{u}_s^{(0)}(\tau,x) - \frac{\partial}{\partial x_s} p^{(0)}(\tau,x)\Big)\, \chi_m^{(s)}(t-\tau,y)\, d\tau \qquad \text{in } (0,T)\times\Omega\times Y_L. \tag{4.25}$$

The functions $\gamma^{(m)}(t,y)$ and $\chi_m^{(s)}(t,y)$ are Y-periodic solutions to the following local problem:

$$\frac{\partial}{\partial y_j}\Big(\eta_{ijmn} e_{mn}^y(\chi^{(k)}(t,y))\Big) = \left[\varrho^L \frac{\partial \chi_m^{(k)}}{\partial t}(t,y) - \left(\delta_{km} - \frac{\partial}{\partial y_m}\gamma^{(k)}(t,y)\right)\delta(t)\right]\delta_{im} \qquad \text{in} \quad (0,T)\times Y_L, \tag{4.26}$$

$$\text{(4.27)}\qquad \operatorname{div}_y \boldsymbol{\chi}^{(i)} = \frac{\partial \chi_k^{(i)}}{\partial y_k} = 0,$$

$$\boldsymbol{\chi}^{(k)}(t,y) = \mathbf{0} \quad \text{on} \quad (0,T)\times \Gamma_Y, \qquad \boldsymbol{\chi}^{(k)}(0,y) = \mathbf{0} \quad \text{in } Y_L,$$

where $\delta(t)$ stands for the Dirac delta. The relation (4.26) is a consequence of (A.9), and relation (4.27) is a consequence of Eqs. (4.21), (4.5) and the local incompressibility condition $\operatorname{div}_y \mathbf{v}^{(0)} = 0$, cf. (A.7).

After averaging of (4.25) we get

$$\text{(4.28)}\qquad \langle \overline{v}_m^{(0)} \rangle_L = \frac{1}{\varrho^L} \int_0^t \langle \chi_m^{(s)}(t-\tau,y)\rangle_L \left(F_s^L(\tau,x) - \varrho^L \ddot{u}_s^{(0)}(\tau,x) - \frac{\partial}{\partial x_s} p^{(0)}(\tau,x) \right) d\tau.$$

The last relation is the *nonstationary Darcy's law*.

From Eq. $(3.1)_3$, comparing the terms associated with ε^0 we conclude

$$\text{(4.29)}\qquad \operatorname{div}_x \mathbf{v}^{(0)}(t,x,y) + \operatorname{div}_y \mathbf{v}^{(1)}(t,x,y) = 0 \qquad \text{in} \quad (0,T)\times \Omega \times Y_L.$$

Hence, after averaging we get

$$\operatorname{div}_x \langle \mathbf{v}^{(0)} \rangle_L = -\langle \operatorname{div}_y \mathbf{v}^{(1)} \rangle_L, \qquad \text{in} \quad (0,T)\times\Omega,$$

or

$$\text{(4.30)}\qquad \operatorname{div}_x \langle \mathbf{v}^{(0)}(t,x,y)\rangle_L = -\frac{1}{|Y|}\int_{\partial Y_L} v_i^{(1)}(t,x,y) N_i \, dA.$$

Moreover, in virtue of (4.21), (4.22), (4.6) and periodicity of $\dot{\mathbf{u}}^{(1)}$, $\mathbf{v}^{(1)}$ with respect to y, we also have

$$\text{(4.31)}\qquad \operatorname{div}_x \langle \mathbf{v}^{(0)}(t,x,y)\rangle_L = \frac{1}{|Y|}\int_{\partial Y_S} \dot{u}_i^{(1)}(t,x,y) N_i \, dA$$

$$= \frac{1}{|Y|}\int_{\partial Y_S} \frac{\partial}{\partial t}\left[A_m^{(pq)}(t,x,y)\, \frac{\partial}{\partial x_q} u_p^{(0)}(t,x) + P^{(m)}(t,x)\, p^{(0)}(t,x) \right] N_m \, dA,$$

or

$$\text{(4.32)}\qquad \operatorname{div}_x \langle \mathbf{v}^{(0)} \rangle_L = \frac{\partial}{\partial t}\left[\langle e_{mm}^y(\mathbf{A}^{(pq)}(t,x,y))\rangle_S \, e_{pq}(\mathbf{u}^{(0)}(t,x)) + \langle \frac{\partial}{\partial y_m} P^{(m)}(t,x)\rangle_S \, p^{(0)}(t,x) \right].$$

Equations (4.28) and (4.32) are Biot's type equations modelling the nonstationary flow of viscous fluid through the microperiodic porous media. Jointly with

Eq. (4.19) they constitute a system of 7 equations for the determination of 7 macroscopic fields: $\mathbf{u}^{(0)}$, $\langle \mathbf{v}^{(0)} \rangle_L$ and $p^{(0)}$.

REMARK 4.1. Let us set $\chi^{(k)} = \dot{\mathbf{w}}^{(k)}$ and $\gamma^{(k)} = \dot{q}^{(k)}$ in (4.26) and (4.27), and perform integration over the interval $(0, t)$. Then we obtain:

$$
\begin{aligned}
&\varrho^L \dot{\mathbf{w}}^{(k)}(t,y) = \operatorname{div}_y (\boldsymbol{\eta} \mathbf{e}^y(\mathbf{w}^{(k)}(t,y))) - \nabla_y q^{(k)}(t,y) + \mathbf{e}_k,\\
&\qquad \text{in} \quad (0,T) \times Y_L,\\
&\operatorname{div}_y \mathbf{w}^{(k)}(t,y) = 0, \qquad \text{in} \quad (0,T) \times Y_L,\\
&\mathbf{w}^{(k)}(t,y) = 0 \qquad \text{in} \quad (0,T) \times \Gamma_Y, \qquad \mathbf{w}^{(k)}(0,y) = 0 \quad \text{on} \quad Y_L,
\end{aligned} \tag{4.33}
$$

where $\mathbf{e}_k$ $(k = 1, \ldots N)$ are the unit base vectors in $\mathbb{R}^N$. The local problem (4.33) coincides with the corresponding local problem derived by ALLAIRE [6] for the flow through rigid skeleton. The Darcy law (4.28), however, involves the motion of the elastic skeleton. The permeability matrix $\mathbf{A}(t)$ is given by

$$
\begin{aligned}
A_{ij}(t) = \frac{1}{\varrho^L |Y|} \int_{Y_L} \frac{\partial \mathbf{w}^{(i)}(t,y)}{\partial t} \cdot \mathbf{e}_j \, dy = \frac{1}{|Y|} \int\limits_{Y_L} \dot{\mathbf{w}}^{(i)} \cdot \dot{\mathbf{w}}^{(j)} \, dy\\
+ \frac{1}{\varrho^L |Y|} \int\limits_{Y_L} \eta_{mnpq} e^y_{pq}(\mathbf{w}^{(i)}) e^y_{mn}(\dot{\mathbf{w}}^{(j)}) \, dy.
\end{aligned} \tag{4.34}
$$

The last formula is obtained from $(4.33)_1$ by multiplying it by $\dot{\mathbf{w}}^{(j)}$ and dividing by ϱ^L. Performing then averaging over Y, integrating by parts and exploiting the periodicity conditions, we arrive at (4.34).

REMARK 4.2. The system (4.19), (4.28) and (4.32) involves the macroscopic displacement field $\mathbf{u}^{(0)}$, macroscopic velocity $\langle \mathbf{v}^{(0)} \rangle_L$ and macroscopic pressure $p^{(0)}$. Various boundary conditions can be imposed to solve the initial-boundary value problem for such system, cf. [9, 35]. For instance, let $\partial \Omega = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$. On Γ_0 one can assume the homogeneous Dirichlet conditions whilst on the complementary part Γ_1 the homogeneous Neumann conditions are plausible.

REMARK 4.3. It would be interesting to weaken the assumption of periodicity and exploit the ideas proposed in [3, 21].

5. Justification of the asymptotic analysis by the two-scale convergence

The aim of this section is to justify rigorously the results obtained by the formal method of two-scale asymptotic expansions. To this end, we exploit the notion of the two-scale convergence, cf. [7] and Appendix B.

For the sake of simplicity we assume that the liquid is isotropic with the viscosity $\eta = 1$.

Consequently, the following system of equations is investigated:

$$\varrho^S \ddot{\mathbf{u}}^\varepsilon(t,x) = \operatorname{div}[\mathbf{a}\,\mathbf{e}(\mathbf{u}^\varepsilon(t,x))] + \mathbf{F}^S(t,x) \quad \text{in} \quad (0,T)\times\Omega^\varepsilon_S, \tag{5.1}$$

$$\varrho^L \dot{\mathbf{v}}^\varepsilon(t,x) = \varepsilon^2 \Delta \mathbf{v}^\varepsilon(t,x) - \nabla p^\varepsilon(t,x) + \mathbf{F}^L(t,x) \quad \text{in} \quad (0,T)\times\Omega^\varepsilon_L, \tag{5.2}$$

$$\operatorname{div}\mathbf{v}^\varepsilon(t,x) = 0 \quad \text{in} \quad (0,T)\times\Omega^\varepsilon_L, \tag{5.3}$$

$$(\mathbf{a}\,\mathbf{e}(\mathbf{u}^\varepsilon))\,\mathbf{n} = (-p^\varepsilon \mathbf{I} + \varepsilon^2 \mathbf{e}(\mathbf{v}^\varepsilon))\,\mathbf{n} \quad \text{on} \quad (0,T)\times\Gamma^\varepsilon, \tag{5.4}$$

$$\mathbf{v}^\varepsilon(t,x) = \dot{\mathbf{u}}(t,x) \quad \text{on} \quad (0,T)\times\Gamma^\varepsilon, \tag{5.5}$$

$$\mathbf{u}^\varepsilon(0,x) = \dot{\mathbf{u}}(0,x) = 0 \quad \text{in} \quad \Omega^\varepsilon_S, \qquad \dot{\mathbf{v}}^\varepsilon(0,x) = 0 \quad \text{in} \quad \Omega^\varepsilon_L. \tag{5.6}$$

For definitions of Lebesgue and Sobolev spaces the reader is referred to the book by ADAMS [1]. The main result of this section is formulated as

THEOREM 5.1. *The sequence* $\{\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon, \mathbf{p}^\varepsilon\}_{\varepsilon>0}$ *of solutions of the system (5.1) – (5.6) two-scale converges to the solution* $(\mathbf{u}^{(0)}(t,x), \mathbf{v}^{(0)}(t,x,y), p^{(0)}(t,x))$ *of the two-scale homogenized problem:*

$$(1-f)\varrho^S \ddot{\mathbf{u}}^\varepsilon(t,x) + \varrho^L \langle \dot{\mathbf{v}}^{(0)} \rangle_{Y_L} = \operatorname{div}_x \int_{Y_S} [\mathbf{a} : (\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)})]\, dy$$
$$- \int_{Y_L} \nabla_y p^{(1)}(t,x,y)\, dy - f\nabla_x p^{(0)}(t,x) + (1-f)\mathbf{F}^S(t,x) + f\mathbf{F}^L(t,x)$$
$$\text{in} \quad (0,T)\times\Omega, \tag{5.7}$$

$$\operatorname{div}_y[\mathbf{a} : (\nabla_x \mathbf{u}^{(0)}(t,x) + \nabla_y \mathbf{u}^{(1)}(t,x,y))] = 0 \quad \text{in} \quad (0,T)\times\Omega\times Y_S, \tag{5.8}$$

$$\varrho^L \dot{\mathbf{v}}^{(0)}(t,x,y) = \Delta_y \mathbf{v}^{(0)}(t,x,y) - \nabla_x p^{(0)}(t,x) - \nabla_y p^{(1)}(t,x,y)$$
$$+ \mathbf{F}^L(t,x) \quad \text{in} \quad (0,T)\times\Omega\times Y_L. \tag{5.9}$$

(5.10) $$\mathrm{div}_y \mathbf{v}^{(0)}(t,x,y) = 0 \qquad \text{in} \quad (0,T)\times\Omega\times Y_L,$$

(5.11) $$\mathrm{div}_x \int\limits_{Y_L} \mathbf{v}^{(0)}(t,x,y)dy = \frac{1}{|Y|}\int\limits_{\Gamma_Y} \dot{\mathbf{u}}^{(1)}(t,x,y)\cdot\mathbf{N}\,ds \qquad \text{in} \quad (0,T)\times\Omega,$$

(5.12) $$\mathbf{a} : (\nabla_x \mathbf{u}^{(0)}(t,x) + \nabla_y \mathbf{u}^{(1)}(t,x,y))\mathbf{N} = -p^{(0)}(t,x)\mathbf{N} \qquad \text{in} \qquad (0,T)\times\Omega\times\Gamma_Y,$$

(5.13) $$\mathbf{u}^{(0)}(0,x) = \dot{\mathbf{u}}^{(0)}(0,x) = \mathbf{0} \qquad \text{in} \quad \Omega,$$

(5.14) $$\mathbf{v}^{(0)}(0,x,y) = \mathbf{0} \qquad \text{in} \quad \Omega\times Y_F.$$

P r o o f. It is not difficult to show that there exists a constant $c > 0$ independent of ε such that

$$\|\mathbf{u}^\varepsilon\|_{H^1(\Omega)^3} \leqslant c, \ \|\mathbf{v}^\varepsilon\|_{L^2(\Omega)^3} \leqslant c, \ \varepsilon\|\nabla\mathbf{v}^\varepsilon\|_{L^2(\Omega,\mathbb{E}^N)} \leqslant c,$$

where $\mathbb{E}^N$ stands for the space of $N\times N$ matrices. The two-scale limits of these sequences satisfy the following properties, cf. Appendix B:

$$\mathrm{div}_y \mathbf{v}^{(0)}(t,x,y) = 0 \qquad \text{in} \qquad (0,T)\times\Omega\times Y_L,$$

$$\mathrm{div}_x \mathbf{v}^{(0)}(t,x,y) + \mathrm{div}_y \mathbf{v}^{(1)}(t,x,y) = 0 \qquad \text{in} \qquad (0,T)\times\Omega\times Y_L,$$

$$\nabla_y \mathbf{u}^{(0)}(t,x,y) = 0, \qquad \text{or} \qquad \mathbf{u}^{(0)} = \mathbf{u}^{(0)}(t,x), \ t\in(0,T), \ x\in\Omega.$$

Moreover from (5.2) one can conclude that, cf. [6],

$$\nabla_y p^{(0)}(t,x,y) = 0, \ \text{or} \ p^{(0)} = p^{(0)}(t,x) \ t\in(0,T), \ x\in\Omega.$$

Let $\chi_S^\varepsilon(x)$ and $1-\chi_S^\varepsilon(x) = \chi_L^\varepsilon(x)$ be the characteristic functions of the domains Ω_S^ε and Ω_L^ε, respectively. Let $\phi = \phi(t) \in C^\infty(0,T)$ be such that $\phi(0) = \dot{\phi}(T) = 0$. Let $\mathbf{\Phi}^\varepsilon(x)$ be a test function such that

$$\mathbf{\Phi}^\varepsilon(x) = \boldsymbol{\eta}(x) + \varepsilon\boldsymbol{\psi}\left(x, \frac{x}{\varepsilon}\right),$$

where $\boldsymbol{\eta}\in\mathcal{D}(\Omega)^3$ and $\boldsymbol{\psi}(x,y)\in\mathcal{D}[\Omega; C^\infty_{per}(Y)]^3$. Multiplying (5.1) by χ_S^ε and (5.2) by χ_L^ε and by the test functions $\phi(t)\mathbf{\Phi}^\varepsilon(x)$ and next integrating over the $[0,T]\times\Omega$ and finally, integrating by parts with respect to time t, we obtain

$$(5.15)\quad \int_0^T\int_\Omega \chi_S^\varepsilon(x)\varrho^S(x)\mathbf{u}^\varepsilon(t,x)\cdot\left[\boldsymbol{\eta}(x)+\varepsilon\boldsymbol{\psi}\left(x,\frac{x}{\varepsilon}\right)\right]\ddot{\phi}(t)\,dx\,dt$$

$$-\int_0^T\int_\Omega \chi_L^\varepsilon(x)\varrho^L\mathbf{v}^\varepsilon(t,x)\cdot\left[\boldsymbol{\eta}(x)+\varepsilon\boldsymbol{\psi}\left(x,\frac{x}{\varepsilon}\right)\right]\dot{\phi}(t)\,dx\,dt$$

$$=\int_0^T\int_\Omega \chi_S^\varepsilon(x)\,\mathrm{div}[\mathbf{a}\nabla_x\mathbf{u}^\varepsilon(t,x)]\cdot\left[\boldsymbol{\eta}(x)+\varepsilon\boldsymbol{\psi}\left(x,\frac{x}{\varepsilon}\right)\right]\phi(t)\,dx\,dt$$

$$+\int_0^T\int_\Omega \chi_L^\varepsilon(x)\varepsilon^2\Delta\mathbf{v}^\varepsilon(t,x)\cdot\left[\boldsymbol{\eta}(x)+\varepsilon\boldsymbol{\psi}\left(x,\frac{x}{\varepsilon}\right)\right]\phi(t)\,dx\,dt$$

$$+\int_0^T\int_\Omega \chi_L^\varepsilon(x)[-\nabla p^\varepsilon(t,x)]\cdot\left[\boldsymbol{\eta}(x)+\varepsilon\boldsymbol{\psi}\left(x,\frac{x}{\varepsilon}\right)\right]\phi(t)\,dx\,dt$$

$$+\int_0^T\int_\Omega \chi_S^\varepsilon(x)\mathbf{F}^S(t,x)\cdot\left[\boldsymbol{\eta}(x)+\varepsilon\boldsymbol{\psi}\left(x,\frac{x}{\varepsilon}\right)\right]\phi(t)\,dx\,dt$$

$$+\int_0^T\int_\Omega \chi_L^\varepsilon(x)\mathbf{F}^L(t,x)\cdot\left[\boldsymbol{\eta}(x)+\varepsilon\boldsymbol{\psi}\left(x,\frac{x}{\varepsilon}\right)\right]\phi(t)\,dx\,dt.$$

Passing to the two-scale limit when $\varepsilon\to 0$ and applying the Theorems B.2, B.3 for the vector case and eventually making integration by parts with respect to time t, we get

$$(5.16)\quad \int_0^T\int_\Omega\int_Y \chi_S(y)\varrho^S\ddot{\mathbf{u}}^{(0)}(t,x)\cdot[\phi(t)\boldsymbol{\eta}(x)]\,dx\,dy\,dt$$

$$+\int_0^T\int_\Omega\int_Y \chi_L(y)\varrho^L\dot{\mathbf{v}}^{(0)}(t,x,y)\cdot[\phi(t)\boldsymbol{\eta}(x)]\,dx\,dy\,dt$$

$$=-\int_0^T\int_\Omega\int_Y \chi_S(y)\mathbf{a}\left[\nabla_x\mathbf{u}^{(0)}(t,x)+\ \nabla_y\mathbf{u}^{(1)}(t,x,y)\right]:[\nabla_x\boldsymbol{\eta}(x)$$

$$+\nabla_y\boldsymbol{\psi}(x,y)]\phi(t)\,dx\,dy\,dt+0$$

$$+\int_0^T\int_\Omega\int_Y \chi_L(y)p^{(0)}(t,x)[\mathrm{div}_x\boldsymbol{\eta}(x)+\mathrm{div}_y\boldsymbol{\psi}(x,y)]\phi(t)\,dx\,dy\,dt$$

$$+\int_0^T\int_\Omega\int_Y \left[\chi_S(y)\mathbf{F}^S(t,x)+\ \chi_L(y)\mathbf{F}_L(t,x)\right]\cdot\boldsymbol{\eta}(x)\phi(t)\,dx\,dy\,dt.$$

We recall that

$$\chi_S^\varepsilon \stackrel{*}{\rightharpoonup} \frac{1}{|Y|}\int\limits_{Y_S} dy = \frac{|Y_S|}{|Y|} = \frac{1}{|Y|}\int\limits_{Y} \chi_S(y)\,dy \qquad \text{in} \quad L^\infty(Y) \text{ weak}-*,$$

and similarly for χ_L^ε. Let $\boldsymbol{\eta} \equiv \mathbf{0}$ in Eq. (5.16). Then we get

$$(5.17) \qquad \int\limits_0^T\int\limits_\Omega\int\limits_{Y_S} \operatorname{div}_y \left[\mathbf{a}(\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)})\right] \cdot \boldsymbol{\psi}(x,y)\phi(t)\,dx\,dy\,dt$$
$$- \int\limits_0^T\int\limits_\Omega\int\limits_{Y_F} p^{(0)}(t,x)\operatorname{div}_y \boldsymbol{\psi}(x,y)\phi(t)\,dx\,dy\,dt = 0.$$

After some calculations we arrive at

$$(5.18) \qquad \int\limits_0^T\int\limits_\Omega\int\limits_{Y_S} \operatorname{div}_y \left(\mathbf{a}(\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)}\right) \boldsymbol{\psi}(x,y))\phi(t)\,dx\,dy\,dt$$
$$- \int\limits_0^T\int\limits_\Omega\int_{Y_S} \operatorname{div}_y \left[\mathbf{a}\left(\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)}\right)\right] \cdot \boldsymbol{\psi}(x,y)\phi(t)\,dx\,dy\,dt$$
$$- \int\limits_0^T\int\limits_\Omega\int\limits_{Y_L} \operatorname{div}_y \left[p^{(0)}(t,x)\boldsymbol{\psi}(x,y)\right]\phi(t)\,dx\,dy\,dt = 0.$$

Hence we conclude that

$$\operatorname{div}_y \left[\mathbf{a}\left(\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)}\right)\right] = 0 \qquad \text{in} \quad (0,T)\times\Omega\times Y_S,$$

and

$$\left[\mathbf{a}\left(\nabla_x \mathbf{u}^{(0)} + \nabla_y \mathbf{u}^{(1)}\right)\right]\mathbf{N} - p^{(0)}\mathbf{N} = 0 \qquad \text{on} \quad (0,T)\times\Omega\times\Gamma_Y.$$

Taking now in (5.16) $\boldsymbol{\psi} \equiv 0$ we obtain

$$(1-f)\varrho^S \ddot{\mathbf{u}}^{(0)}(t,x) + \frac{1}{|Y|}\int\limits_{Y_L} \varrho^L \dot{\mathbf{v}}^{(0)}(t,x,y)\,dy$$
$$= \frac{1}{|Y|}\operatorname{div}_x \int\limits_{Y_S} \mathbf{a}(\nabla_x \mathbf{u}^{(0)} + \nabla \mathbf{u}^{(1)})\,dy$$
$$- f\nabla_x p^{(0)}(t,x) + (1-f)\mathbf{F}^S(t,x) + f\mathbf{F}^L(t,x) \qquad \text{in} \quad (0,T)\times\Omega.$$

Multiplying Eq. (5.2) by a test function $\psi\left(x, \frac{x}{\varepsilon}\right)$ with support in Ω_L^ε and next integrating over Ω, we arrive at

$$\text{(5.19)}\qquad \int_\Omega \varrho^L \mathbf{v}^\varepsilon(t,x)\cdot\psi\left(x,\frac{x}{\varepsilon}\right)dx = \int_\Omega \varepsilon^2\Delta\mathbf{v}^\varepsilon(t,x)\cdot\psi\left(x,\frac{x}{\varepsilon}\right)dx - \int_\Omega \nabla p^\varepsilon(t,x)\cdot\psi\left(x,\frac{x}{\varepsilon}\right)dx + \int_\Omega \mathbf{F}_L(t,x)\cdot\psi\left(x,\frac{x}{\varepsilon}\right)dx.$$

In the last relation $\mathbf{v}^\varepsilon$ is to be viewed as any extension to Ω such that $\mathbf{v}^\varepsilon = \dot{\mathbf{u}}^\varepsilon$ on Γ^ε, cf. [45]. We have

$$\text{(5.20)}\qquad \int_\Omega \varepsilon^2\Delta\mathbf{v}^\varepsilon\cdot\psi\left(x,\frac{x}{\varepsilon}\right)dx = \int_\Omega \varepsilon^2\operatorname{div}\left(\nabla\mathbf{v}^\varepsilon\,\psi\left(x,\frac{x}{\varepsilon}\right)\right)dx - \int_\Omega \varepsilon^2\nabla\mathbf{v}^\varepsilon : \left[\nabla_x\psi\left(x,\frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon}\nabla_y\psi\left(x,\frac{x}{\varepsilon}\right)\right]dx.$$

Taking into account (5.20) in (5.19) we find the following two-scale limit as $\varepsilon \to 0$:

$$\text{(5.21)}\qquad \int_\Omega\int_{Y_L} \varrho^L\dot{\mathbf{v}}^{(0)}(t,x,y)\cdot\psi(x,y)\,dx\,dy = -\int_\Omega\int_{Y_L}\nabla_y\mathbf{v}^{(0)}(t,x,y):\nabla_y\psi(x,y)\,dx\,dy - \int_\Omega\int_{Y_L}(\nabla_x p^{(0)}(t,x) + \nabla_y p^{(1)}(t,x,y))\cdot\psi(x,y)\,dx\,dy + \int_\Omega\int_{Y_L}\mathbf{F}_L(t,x)\cdot\psi(x,y)\,dx\,dy.$$

From (5.21) we get, cf. Eq. (5.9),

$$\varrho^L\dot{\mathbf{v}}^{(0)}(t,x,y) = \Delta_y\mathbf{v}^{(0)}(t,x,y) - \nabla_x p^{(0)}(t,x) - \nabla_y p^{(1)}(t,x,y) + \mathbf{F}^L(t,x) \quad \text{in } (0,T)\times\Omega\times Y_S.$$

This completes the proof. □

6. Passage to the stationary case

Allaire [8] suggested us how to pass to the stationary flow. Assume now that $\eta = 1$, cf. Sec. 5. We observe that the stationary case is not obtained by simply putting $\dfrac{\partial \mathbf{w}^{(i)}}{\partial t} = 0$, cf. (4.33). In fact we have to pass with time to infinity. We find, cf. (4.34)

$$\int_0^t A_{ij}(t-s)\,ds = \frac{1}{\varrho^L |Y|} \int_{Y_L} \left(\int_0^t \frac{\partial \mathbf{w}^{(i)}(t-\sigma, y)}{\partial \sigma}\, d\sigma \right) \cdot \mathbf{e}_j\, dy$$

$$= \frac{1}{\varrho^L |Y|} \int_{Y_L} \mathbf{w}^{(i)}(t,y) \cdot \mathbf{e}_j\, dy.$$

The forcing $\left(F_i^L - \varrho^L \ddot{u}_i^{(0)} - \dfrac{\partial p^{(0)}}{\partial x_i} \right)$ is now time-independent. Hence

$$\text{(6.1)} \qquad \langle \bar{\mathbf{v}}_i^{(0)} \rangle_L = \left(\int_0^t A_{ij}(t-s)\,ds \right) \left[F_j^L(x) - \frac{\partial p^{(0)}(x)}{\partial x_j} \right],$$

because $\ddot{\mathbf{u}}^{(0)} = 0$. From (6.1) and (6.2), taking into account (4.28) we get

$$\text{(6.2)} \qquad \langle \bar{\mathbf{v}}_i^{(0)} \rangle_L = \frac{1}{\varrho^L |Y|} \left(\int_{Y_L} \mathbf{w}^{(i)}(t,y) \cdot \mathbf{e}_j\, dy \right) \left[F_j^L(x) - \frac{\partial p^{(0)}(x)}{\partial x_j} \right].$$

Letting t tend to infinity we find

$$\text{(6.3)} \qquad \langle \bar{\mathbf{v}}_i^{(0)} \rangle_L = \frac{1}{\varrho^L |Y|} \left(\int_{Y_L} \mathbf{w}_\infty^{(i)}(y) \cdot \mathbf{e}_j\, dy \right) \left[F_j^L(x) - \frac{\partial p^{(0)}(x)}{\partial x_j} \right]$$

$$= K_{ij} \left[F_j^L(x) - \frac{\partial p^{(0)}(x)}{\partial x_j} \right],$$

where $\langle \bar{\mathbf{w}}^{(0)} \rangle_L = \langle \mathbf{v}^{(0)} \rangle_L$ and $\mathbf{K} = (K_{ij})$ coincides with the well-known permeability matrix for the stationary flow, cf. [4, 15, 47]. Indeed, the local equation (4.33) yields

$$\text{(6.4)} \qquad \varrho^L \lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{\partial \mathbf{w}^{(i)}(s,y)}{\partial s}\, ds + \lim_{t\to\infty} \frac{1}{t} \int_0^t \nabla_y q^{(i)}(s,y)\, ds$$

$$- \lim_{t\to\infty} \frac{1}{t} \Delta_y \mathbf{w}^{(i)}(s,y)\, ds = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbf{e}_i\, ds = \mathbf{e}_i.$$

We have

$$\text{(6.5)} \qquad \lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{\partial \mathbf{w}^{(i)}}{\partial s}(s,y)\, ds = \lim_{t\to\infty} \frac{1}{t} \mathbf{w}^{(i)}(t,y) = 0,$$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \nabla_y q^{(i)}(s,y)\,ds = \nabla_y \lim_{t\to\infty}\frac{1}{t}\int_0^t q^{(i)}(s,y)\,ds = \nabla_y q^{(i)}_\infty(y). \tag{6.6}$$

Thus

$$\begin{aligned} \nabla_y q^{(i)}_\infty(y) - \Delta_y \mathbf{w}^{(i)}_\infty(y) &= \mathbf{e}_i \qquad \text{in} \quad Y_L,\\ \operatorname{div}_y \mathbf{w}^{(i)}_\infty &= 0 \qquad \text{in} \quad Y_L. \end{aligned} \tag{6.7}$$

The components of the permeability matrix $\mathbf{K}$ are eventually given by

$$K_{ij} = \frac{1}{\varrho^L|Y|}\int_{Y_L}\left(\nabla_y \mathbf{w}^{(i)}_\infty\right) : \nabla_y \mathbf{w}^{(j)}_\infty\,dy. \tag{6.8}$$

We recall that it was assumed that $\eta = 1$. In the general case we have

$$K_{ij} = \frac{1}{\varrho^L|Y|}\int_{Y_L}\eta\left(\nabla_y \mathbf{w}^{(i)}_\infty\right) : \nabla_y \mathbf{w}^{(j)}_\infty\,dy. \tag{6.9}$$

7. Comments on related papers

7.1. The works by BIOT [16 – 19] are now classical. In [16] an elastic skeleton with a statistical distribution of interconnected pores is considered. The porosity is denoted by

$$f = \frac{V_p}{V_b}, \tag{7.1}$$

where V_p is the volume of pores contained in a sample of bulk volume V_b, cf. the definition $(2.4)_1$. It is assumed, however, that f represents also a ratio of areas

$$f = \frac{S_p}{S_b}, \tag{7.2}$$

i.e. the fraction S_p occupied by the pores in any cross-sectional area S_b. This assumption is not needed in our approach. In the next two papers BIOT [17, 18] proposed to consider frequency-dependent characteristics of materials of skeleton and liquid and derived a time convolution-type law for nonstationary flow through a porous medium.

The resulting stress tensor S^Z_{ij} in porous material is assumed in the form

$$S^Z_{ij} = s_{ij} + \sigma\delta_{ij}, \qquad \sigma = -fp \tag{7.3}$$

where p denotes the hydrostatic pressure of the liquid. If a cube of unit size of the bulk material is considered, σ represents the total normal tension applied to the liquid part of the faces of the cube, while s_{ij} represent the forces applied to the portion of the cube faces occupied by the solid. This assumption clearly corresponds to (4.12); nevertheless, the assumption (7.2) was not needed in our consideration.

This liquid-solid system is regarded as an elastic system. The liquid may be compressible. The average displacement of the solid is denoted by $\mathbf{u}$ and that of the liquid by $\mathbf{U}$, whilst $e = \operatorname{div}\mathbf{U}$ stands for the dilatation of the liquid. At this point there is a difference with our theory. In our case the liquid is incompressible, cf. $(3.1)_3$; however the divergence of the average velocity does not vanish, cf. (4.30), and $e \neq 0$ also in our theory.

The potential energy W per unit volume of aggregate is given by

$$W = \frac{1}{2}\Big(s_{ij}\,e_{ij} + \sigma\,e\Big). \tag{7.4}$$

The stress-strain relations are expressed by

$$s_{ij} = \frac{\partial W}{\partial e_{ij}}, \qquad \sigma = \frac{\partial W}{\partial e}, \tag{7.5}$$

and reduce to the form

$$s_{ij} = c_{ijmn}e_{mn} + Q_{ij}e, \qquad \sigma = Q_{ij}e_{ij} + Re. \tag{7.6}$$

The coefficients c_{ijmn} are elastic moduli, whilst R is a measure of the pressure required to force a certain volume of the liquid into the aggregate provided that the total volume remains constant. The coefficients Q_{ij} describe the coupling effect. The equation (7.6) is of the type (4.13) thus confirming Biot's idea. Biot assumed the equation of motion in the form

$$\varrho_{11}\ddot{u}_i + \varrho_{12}\ddot{U}_i + b_{ij}\Big(\dot{u}_j - \dot{U}_j\Big) = s_{ij,j} + \check{F}_i^S, \tag{7.7}$$

$$\varrho_{12}\ddot{u}_i + \varrho_{22}\ddot{U}_i - b_{ij}\Big(\dot{u}_j - \dot{U}_j\Big) = \sigma_{,i} + \check{F}_i^L. \tag{7.8}$$

The coefficients ϱ_{11}, ϱ_{12}, ϱ_{22} are mass-like coefficients which take into account the fact that the relative liquid flow through pores is not uniform. The meaning of ϱ_{12} is rather obscure. One shows that

$$\varrho_{11} + 2\varrho_{12} + \varrho_{22} = \varrho, \tag{7.9}$$

represents the total mass of the solid-liquid aggregate per unit volume, and $b_{ij}K_{ij} = f$.

For frequency-dependent permeability matrix **K**, AURIAULT *et al.* [11] propose to treat the mass-like coefficients: ϱ_{11}, ϱ_{12}, ϱ_{22} as tensors

$$(7.10)\qquad \begin{aligned} (\varrho_{11})_{ij} &= \langle\varrho^S\rangle\delta_{ij} - (\varrho_{12})_{ij},\\ (\varrho_{22})_{ij} &= \frac{f^2}{\omega}H^{\mathcal{I}}_{ij},\\ (\varrho_{12})_{ij} &= (\varrho_{21})_{ij} = f\varrho^L\delta_{ij} - (\varrho_{22})_{ij},\\ \varrho &= (1-f)\varrho^S + f\varrho^L, \end{aligned}$$

where $<\varrho^S>=(1-f)\varrho^S$ and

$$\mathbf{H} = \mathbf{H}^{\mathcal{R}} + i\mathbf{H}^{\mathcal{I}} = \mathbf{K}^{-1}.$$

Hence, in the isotropic case one has $\varrho_{11} \geqslant \varrho$, $\varrho_{12} = \varrho_{21} \leqslant 0$, $\varrho_{22} \geqslant f\varrho$.

In Biot's theory, Darcy's law is given by

$$(7.11)\qquad (\dot{U}_i - \dot{u}_i) = K_{ij}\Big(\frac{\partial\sigma}{\partial x_j} + F^L_j\Big).$$

However, we observe the discrepancy between the Darcy law (7.11) and the equation of motion (7.8). Only for stationary flows both equations coincide. Adding Eqs. (7.7) and (7.8) we obtain

$$(7.12)\qquad (\varrho_{11}+\varrho_{12})\ddot{u}_i + (\varrho_{12}+\varrho_{22})\ddot{U}_i = \frac{\partial}{\partial x_j}(s_{ij}+\sigma\delta_{ij}) + \check{F}^S_i + \check{F}^L_i.$$

After comparison with (4.19) we conclude that

$$\check{F}^S_i = (1-f)Fi^S \qquad \check{F}^L_i = fF^L_i$$

$$(7.13)\qquad \varrho_{11}+\varrho_{12} = (1-f)\varrho^S,$$

and

$$(7.14)\qquad (\varrho_{12}+\varrho_{22})\ddot{U}_i = \varrho^L\langle\dot{v}^{(0)}_i\rangle_L,$$

or

$$(7.15)\qquad (\varrho_{12}+\varrho_{22})\ddot{U}_i = \varrho^L f\frac{1}{|Y_L|}\int\limits_{Y_L}\dot{v}^{(0)}_i\,dy.$$

Hence

$$(7.16)\qquad \varrho_{12}+\varrho_{22} = f\varrho^L,$$

and

$$(7.17)\qquad \varrho_{11}+2\varrho_{12}+\varrho_{22} = (1-f)\varrho^S + f\varrho^L = \varrho,$$

where ϱ is the same as in (7.9). Consider now the nonstationary Darcy's law (4.28). We have

$$\langle v_i^{(0)}\rangle_L - \dot{u}_i^{(0)} = K_{ij}(t) * \Big(F_j^L - \varrho^L \ddot{u}_j^{(0)} - \frac{\partial}{\partial x_j} p^{(0)}\Big)(t,x), \tag{7.18}$$

where

$$K_{ij} = K_{ij}(t) = \frac{1}{\varrho^L}\langle \chi_i^{(j)}(t,y)\rangle_L, \tag{7.19}$$

and $*$ denotes the convolution with respect to time t. After taking the Laplace transformation of (7.18) defined as

$$\tilde{f} = \int_0^\infty f(t)\, \mathrm{e}^{izt}\, dt, \quad z = \omega + i\zeta, \quad \omega = \Re z, \quad \zeta = \Im z > 0,$$

we get

$$\langle \tilde{v}_i^{(0)}\rangle_L - \tilde{\dot{u}}_i^{(0)} = \tilde{K_{ij}}(z)\Big(F_j^L - \varrho^L \ddot{u}_j^{(0)} - \frac{\partial}{\partial x_j} p^{(0)}\Big)^{\tilde{}}(z,x). \tag{7.20}$$

Thus, after the inversion

$$\Big(F_i^L - \varrho^L \ddot{u}_i^{(0)} - \frac{\partial}{\partial x_i} p^{(0)}\Big)^{\tilde{}} = \tilde{H}_{ij}\Big(\langle \tilde{v}_i^{(0)}\rangle_L - \tilde{\dot{u}}_i^{(0)}\Big), \tag{7.21}$$

where

$$\tilde{\mathbf{H}} = \tilde{\mathbf{K}}^{-1}.$$

The inverse transformation yields

$$F_i^L - \varrho^L \ddot{u}_i^{(0)} - \frac{\partial}{\partial x_i} p^{(0)} = H_{ij}(t) * \Big(\langle v_j^{(0)}\rangle_L - \dot{u}_j^{(0)}\Big)(t,x). \tag{7.22}$$

Therefore

$$F_i^L - \frac{\partial}{\partial x_i} p^{(0)} = H_{ij}(t) * \Big(\langle v_j^{(0)}\rangle_L - \dot{u}_j^{(0)}\Big) + \varrho^L \ddot{u}_i^{(0)}. \tag{7.23}$$

The matrix $\tilde{\mathbf{H}}$ can be written as follows:

$$\tilde{H}_{ij} = \tilde{H}_{ij}^{\Re} + i\tilde{H}_{ij}^{\Im}. \tag{7.24}$$

Then (7.21) becomes

$$\Bigg(F_j^L - \frac{\partial}{\partial x_j} p^{(0)}\Bigg)^{\tilde{}} = \tilde{H}_{ij}^{\Re}\Bigg(\langle \tilde{v}_i^{(0)}\rangle_L - \tilde{\dot{u}}_i^{(0)}\Bigg) + \Bigg(\varrho^L \delta_{ij} + \frac{1}{z}\tilde{H}_{ij}^{\Im}\Bigg)\tilde{\ddot{u}}_j^{(0)} - \frac{1}{z}\tilde{H}_{ij}^{\Im}\langle \tilde{\dot{v}}_i^{(0)}\rangle_L \tag{7.25}$$

and we find

$$(\tilde{\varrho}_{12})_{ij} = \varrho^L \delta_{ij} + \frac{1}{z}\tilde{H}^{\Im}_{ij} \qquad \text{and} \qquad (\tilde{\varrho}_{22})_{ij} = -\frac{1}{z}\tilde{H}^{\Im}_{ij}, \tag{7.26}$$

cf. also the paper by AURIAULT et al. [11].

7.2. BURRIDGE and KELLER [22] employed the homogenization theory, though not quite rigorously. Performing Fourier transformation in time, the basic system of equations is given by, cf. (2.3) - (2.8),

$$\begin{aligned}
-\omega^2\varrho^S\tilde{u}_i &= \tilde{S}_{ij,j} + F_i^S && \text{in } Y_S,\\
\tilde{S}_{ij} &= a_{ijmn}\tilde{e}_{mn} && \text{in } Y_S,\\
i\omega\varrho^L\tilde{v}_i &= \left(-\tilde{p}\delta_{ij} + \eta_{ijmn}\tilde{v}_{m,n}\right)_{,j} + F_i^L && \text{in } Y_L,\\
i\omega\tilde{p} &= -\kappa\,\mathrm{div}\,\tilde{\mathbf{v}} && \text{in } Y_L,\\
\tilde{S}_{ij}n_j &= \left(-\tilde{p}\delta_{ij} + \eta_{ijmn}e_{mn}(\tilde{\mathbf{v}})\right)n_j && \text{on } \partial Y_S \cap \partial Y_L,\\
i\omega\tilde{u}_i &= \tilde{v}_i && \text{on } \partial Y_S \cap \partial Y_L.
\end{aligned} \tag{7.27}$$

Here ω denotes the frequency. Now the incompressibility condition $(3.1)_3$ is replaced by compressibility equation $(7.27)_4$.

The two-scale asymptotic method was applied to the system (7.27). The macroscopic equations describe a liquid-filled porous elastic medium with isotropic stochastic distribution of pores. A comparison with Biot's result for an isotropic stochastic medium was performed. The averaging used is typical for stochastic media.

The use of time-Fourier transforms is also characteristic for other papers on the subject, cf. LÉVY [38], AURIAULT *et al.* [11]. Such an approach corresponds to steady vibration of the porous medium.

7.3. The question of the form of the nonlinear effects has been raised by FORCHHEIMER already in 1901, cf. [29], and was studied by LINDQUIST [39]. MUSKAT [43] distinguishes 3 zones of Reynolds number $\mathcal{R}$, namely: (i) Darcy zone where $\mathcal{R}$ is low, (ii) transition zone, and (iii) linear deviation zone for high values of $\mathcal{R}$.

WODIE and LÉVY [51] considered the influence of the convective term $\mathbf{v}\nabla\mathbf{v}$ on Darcy's law in the case of a stationary flow. If the nonlinear term is of the same order as the viscosity term, then

$$\eta \rightsquigarrow \epsilon^{3/2}\eta$$

instead of (3.5). We note the difference with the linear case where the scaling parameter is ϵ^2. These authors show that up to the second order approximation, Darcy's law is given by

$$(7.28) \qquad < v_i >= \frac{1}{\eta} K_{ij} \left(F_j - \frac{\partial p^{(0)}}{\partial x_j} \right) + \frac{\varrho}{\eta} T_{ijk} \left(F_j - \frac{\partial p^{(0)}}{\partial x_j} \right) \left(F_k - \frac{\partial p^{(0)}}{\partial x_k} \right),$$

where the tensor $\mathbf{T}$ satisfies the following relation:

$$T_{ijk} + T_{kij} + T_{jki} = 0.$$

In the case of macroscopic isotropy and one-dimensional flow, the correction term vanishes. However, one can calculate the third correction being the first nonvanishing nonlinear contribution in Darcy's law which can be written in the form

$$(7.29) \qquad \mathbf{F} - \nabla p^{(0)} = \frac{\eta}{K^\epsilon} < \mathbf{v} > + \frac{\varrho^2 c^\epsilon}{\eta (K^\epsilon)^4} | < \mathbf{v} > |^2 < \mathbf{v} >,$$

where K^ϵ and c^ϵ are positive constants of order ϵ^8 completely determined by the microstructure. Such law is not in agreement with the generally used empirical law of FORCHHEIMER [29] but agrees with the numerical calculation of BARRÈRE (1990).

The results of Wodie and Lévy on vanishing of the first nonlinear correction were confirmed by FIRDAOUSS *et al.* [28].

8. Concluding remarks

ALLAIRE'S [6] results were extended to the flow of a viscous fluid through a microperiodic, linear elastic anisotropic porous medium. To justify rigorously the formal asymptotic approach, the two-scale convergence method developed by ALLAIRE [7] and NGUETSENG [44] was successfully used. In contrast to several previous papers on similar flow problem [10, 22, 27, 45], our approach is straightforward and avoids using time-transformation. Also, the local problem was explicitly formulated and the formula for the permeability matrix was derived.

TORZILLI and MOW [50] employed the mixture theory to describe the interaction of the fluid and solid phases of articular cartilage. The equations of motion for each phase and the total mixture were derived from the extended Hamilton's principle, where the Rayleigh dissipative resistance is considered as a generalized body-force field. This procedure yields, as special cases, the classical Darcy's law for the liquid transport due to direct pressure gradients, as well as Biot's consolidation equation for the liquid transport due to dilatation of the solid phase. For further results on modelling of articular cartilage the reader is referred to [31, 32, 34, 36, 41, 42, 48]. We are convinced that, since the cartilage exhibits a hierarchical microstructure, the homogenization methods can be used to macroscopic modelling of its behaviour.

Appendix A

Analysis of terms of ε^{-2} order

Equation $(3.1)_1$ yields

$$\frac{\partial}{\partial y_j}\left[a_{ijmn}\frac{\partial}{\partial y_n}u_m^{(0)}\right]=0 \qquad \text{in} \quad Y_S. \tag{A.1}$$

Multiplying (A.1) by $u_i^{(0)}$ and integrating by parts we get

$$\int\limits_{\partial Y_S} a_{ijmn}e^y_{mn}(\mathbf{u}^{(0)})u_i^{(0)}N_j\,dA-\int\limits_{Y_S} a_{ijmn}e^y_{ij}(\mathbf{u}^{(0)})e^y_{mn}(\mathbf{u}^{(0)})\,dy=0. \tag{A.2}$$

By virtue of the interface condition (4.3), the surface integral in the last equation vanishes and consequently we get

$$\mathbf{u}^{(0)}=\mathbf{u}^{(0)}(t,x). \tag{A.3}$$

Analysis of terms of ε^{-1} order

Equation $(3.1)_1$ now yields

$$\frac{\partial}{\partial y_j}\left[a_{ijmn}\left(\frac{\partial}{\partial x_n}u_m^{(0)}+\frac{\partial}{\partial y_n}u_m^{(1)}\right)\right]=0, \qquad x\in\Omega, \quad y\in Y_S. \tag{A.4}$$

Similarly, Eq. $(3.1)_2$ gives

$$\nabla_y p^{(0)}=0, \quad x\in\Omega \qquad y\in Y_L. \tag{A.5}$$

Hence

$$p^{(0)}=p^{(0)}(t,x). \tag{A.6}$$

The incompressibility Eq. $(3.1)_3$ yields

$$\operatorname{div}_y\mathbf{v}^{(0)}=0. \tag{A.7}$$

Local function for the flow problem

By (4.25) we have

$$\dot{v}_m^{(0)}(t,x,y) = \frac{1}{\varrho^L}\left(F_s^L - \varrho^L \ddot{u}_s^{(0)} - \frac{\partial}{\partial x_s}p^{(0)}\right)(t,x)\chi_m^{(s)}(0,y)$$

$$+\frac{1}{\varrho^L}\int\limits_0^t \left(F_s^L - \varrho^L \ddot{u}_s^{(0)} - \frac{\partial}{\partial x_s}p^{(0)}\right)(\tau,x)\frac{d\chi_m^{(s)}(t-\tau,y)}{dt}\,d\tau.$$

Since $\chi_m^{(s)}(0,y) = 0$, we write

$$\text{(A.8)}\qquad \dot{v}_m^{(0)}(t,x,y) = \frac{1}{\varrho^L}\int\limits_0^t \left(F_s^L - \varrho^L \ddot{u}_s^{(0)} - \frac{\partial}{\partial x_s}p^{(0)}\right)(\tau,x)\frac{d\chi_m^{(s)}(t-\tau,y)}{dt}\,d\tau.$$

Therefore, substitution of (4.25) into (4.24) yields

$$\text{(A.9)}\qquad \int\limits_0^t \left(F_s^L(\tau,x) - \varrho^L \ddot{u}_s^{(0)}(\tau,x) - \frac{\partial}{\partial x_s}p^{(0)}(\tau,x)\right) \times \left[\varrho^L \frac{\partial \chi_i^{(s)}(t-\tau,y)}{\partial t}\right.$$

$$\left. -\left(\delta_{si} - \frac{\partial}{\partial y_i}\gamma_s(\tau,y)\right)\delta(t-\tau) - \frac{\partial}{\partial y_j}\left(\eta_{ijmn}\frac{\partial \chi_m^{(s)}(t-\tau,y)}{\partial y_n}\right)\right]d\tau = 0.$$

This equation implies the relation (4.26).

Appendix B. Two-scale convergence

The aim of this Appendix is to gather basic facts about two-scale convergence. For details the reader is referred to [7, 44]. We observe that this notion was introduced to justify the formal method of two-scale asymptotic expansions. We also note that the Γ-convergence method is confined to sequences of functionals.

Let Ω be an open set in $\mathbb{R}^N$ ($N \geqslant 1$) and Y – a closed cube. As usual, $L^2(\Omega)$ is the Lebesgue space of real-valued functions that are measurable and square integrable in Ω, cf. [1]. Let $C^\infty_{\text{per}}(Y)$ be the space of infinitely differentiable functions in $\mathbb{R}^N$ that are Y-periodic. Then $L^2_{per}(Y), (H^1_{\text{per}}(Y))$ is the completion for the norm of $L^2(Y)(H^1(Y))$ of $C^\infty_{\text{per}}(Y)$. We note that $L^2_{\text{per}}(Y)$ coincides with the space of functions in $L^2(Y)$ extended by Y-periodicity to the whole of $\mathbb{R}^N$. Consider a sequence of functions $\{u^\varepsilon\}_{\varepsilon>0}$ in $L^2(\Omega)$ ($\varepsilon > 0$ and $\varepsilon \to 0$).

DEFINITION B.1. *A sequence of functions $\{u^\varepsilon\}_{\varepsilon>0}$ in $L^2(\Omega)$ is said to two-scale converge to a limit $u^{(0)}(x,y) \in L^2(\Omega \times Y)$ if, for any function $\psi(x,y)$ in $\mathcal{D}[\Omega; C^\infty_{\text{per}}(Y)]$, we have*

$$\text{(B.1)} \qquad \lim_{\varepsilon \to 0} \int_{\Omega} u^{\varepsilon}(x)\psi(x, x/\varepsilon)\, dx = \frac{1}{|Y|} \int_{\Omega} \int_{Y} u^{(0)}(x,y)\psi(x,y)\, dx\, dy.$$

REMARK B.1. For evolution problems when $u^{\varepsilon} = u^{\varepsilon}(t,x)$, the variable t is treated as a parameter and instead of (B.1) we have [7]

$$\lim_{\varepsilon \to 0} \int_0^T \int_{\Omega} u^{\varepsilon}(x)\psi(x, x/\varepsilon)\phi(t)\, dx\, dt = \frac{1}{|Y|} \int_0^T \int_{\Omega} \int_{Y} u^{(0)}(x,y)\psi(x,y)\phi(t)\, dx\, dy\, dt,$$

where ϕ is a smooth test function.

The following compactness theorem establishes that the notion of two-scale convergence makes sense.

THEOREM B.1. *From each bounded sequence $\{u^{\varepsilon}\}_{\varepsilon>0}$ in $L^2(\Omega)$ we can extract a subsequence and there exists a limit $u^{(0)}(x,y) \in L^2(\Omega \times Y)$ such that this subsequence two-scale converges to $u^{(0)}$.* □

It is interesting to note that the two-scale limit furnishes more information than the weak limit in L^2. The relationship between the two-scale and weak L^2-convergence is established by

PROPOSITION B.1. Let $\{u^{\varepsilon}\}_{\varepsilon>0}$ be a sequence of functions in $L^2(\Omega)$, which two-scale converges to a limit $u^{(0)} \in L^2(\Omega \times Y)$. Then u^{ε} converges also to $u(x) = \frac{1}{|Y|} \int_Y u^{(0)}(x,y)\, dy$ in L^2-weakly. Furthermore, we have

$$\text{(B.2)} \qquad \lim_{\varepsilon \to 0} \|u^{\varepsilon}\|_{L^2(\Omega)} \geqslant \|u^{(0)}\|_{L^2(\Omega \times Y)} \geqslant \|u\|_{L^2(\Omega)}.$$

□

The next proposition gives two-scale limit of the product of two sequences.

PROPOSITION B.2. Let $\{u^{\varepsilon}\}_{\varepsilon>0}$ be a sequence of functions in $L^2(\Omega)$, which two-scale converges to $u^{(0)}(x,y) \in L^2(\Omega \times Y)$. Assume that

$$\text{(B.3)} \qquad \lim_{\varepsilon \to 0} \|u^{\varepsilon}\|_{L^2(\Omega)} = \|u^{(0)}\|_{L^2(\Omega \times Y)}.$$

Then for any sequence $\{v^{\varepsilon}\}_{\varepsilon>0}$ two-scale convergent to $v^{(0)} \in L^2(\Omega \times Y)$ one has

$$\text{(B.4)} \qquad u^{\varepsilon}(x)\, v^{\varepsilon}(x) \rightharpoonup \frac{1}{|Y|} \int_Y u^{(0)}(x,y)\, v^{(0)}(x,y)\, dy \qquad \text{in} \quad \mathcal{D}'(\Omega).$$

Moreover, if $u^{(0)} \in L^2(\Omega; C_{\mathrm{per}}(Y))$ then

$$\lim_{\varepsilon\to 0} \left\| u^\varepsilon(x) - u^{(0)}\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega)} = 0. \tag{B.5}$$

□

Of practical importance is the two-scale convergence of sequences with bounds on derivatives. The relevant results are summarized as follows.

THEOREM B.2. *Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a bounded sequence in $H^1(\Omega)$ that converges weakly to a limit u in $H^1(\Omega)$. Then u^ε two-scale converges to u and there exists a function $u^{(1)}(x,y)$ in $L^2[\Omega; H^1_{\mathrm{per}}(Y)/\mathbb{R}]$ such that, up to a subsequence, ∇u^ε two-scale converges to $\nabla_x u(x) + \nabla_y u^{(1)}(x,y)$.* □

THEOREM B.3. *Let $\{u^\varepsilon\}_{\varepsilon>0}$ and $\{\varepsilon\nabla u^\varepsilon\}_{\varepsilon>0}$ be two bounded sequences in $L^2(\Omega)$. Then there exists a function $u^{(0)}$ in $L^2(\Omega; H^1_{\mathrm{per}}(Y))$ such that, up to a subsequence, u^ε and $\varepsilon\nabla u^\varepsilon$ two-scale converge to $u^{(0)}(x,y)$ and $\nabla_y u^{(0)}(x,y)$, respectively.* □

THEOREM B.4. *Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a free-divergence bounded sequence in $L^2(\Omega)^N$, which two-scale converges to $u^{(0)}(x,y)$ in $[L^2(\Omega\times Y)]^N$. Then, the two-scale limit satisfies*

$$\operatorname{div}_y u^{(0)}(x,y) = 0 \tag{B.6}$$

and

$$\int_Y \operatorname{div}_x u^{(0)}(x,y)\, dy = 0. \tag{B.7}$$

□

EXAMPLE B.1. To illustrate the two-scale convergence method let us consider the problem of homogenization of linear second-order elliptic equations, cf. ALLAIRE [7].

Let Ω be a bounded open set of $\mathbb{R}^N$. Let f be a given function in $L^2(\Omega)$. Consider the following equation:

$$-\operatorname{div}\left(A\left(x, \frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right) = f \quad \text{in} \quad \Omega, \tag{B.8}$$

$$u^\varepsilon = 0 \quad \text{on} \quad \partial\Omega, \tag{B.9}$$

where $\mathbf{A}\left(x, \dfrac{x}{\varepsilon}\right)$ is a matrix defined on $\Omega \times Y$, Y-periodic in y, such that there exist two positive constants $0 < \alpha \leqslant \beta$ satisfying

$$\alpha|\xi|^2 \leqslant \sum_{i,j=1}^{N} A_{ij}(x,y)\xi_i\xi_j \leqslant \beta|\xi|^2 \quad \text{for any} \quad \xi \in \mathbb{R}^N. \tag{B.10}$$

Additionally we assume that $A_{ij}(x, x/\varepsilon)$ satisfies

$$\lim_{\varepsilon\to 0}\int_\Omega \left[A_{ij}\left(x, \frac{x}{\varepsilon}\right)\right]^2 dx = \int_\Omega\int_Y [A_{ij}(x,y)]^2\, dx\, dy. \tag{B.11}$$

Under the assumptions (B.10) and (B.11), the system of Eqs. (B.8) – (B.9) admits a unique solution u^ε in $H_0^1(\Omega)$, which satisfies the a priori estimate

$$\|u^\varepsilon\|_{H_0^1(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}, \tag{B.12}$$

where C is a positive constant that depends only on Ω and α, and not on ε. The homogenized problem is given by

$$-\operatorname{div}[\mathbf{A}^h(x)\nabla u(x)] = f \qquad \text{in} \quad \Omega, \tag{B.13}$$

$$u = 0 \qquad \text{on} \quad \partial\Omega, \tag{B.14}$$

where the matrix $\mathbf{A}^h$ has the following form

$$A_{ij}^h(x) = \frac{1}{|Y|}\int_Y \mathbf{A}(x,y)\left[\nabla_y w^{(i)}(x,y) + \mathbf{e}_i\right]\cdot\left[\nabla_y w^{(j)}(x,y) + \mathbf{e}_j\right] dy \tag{B.15}$$

and, for $1 \leqslant i \leqslant N$, $w^{(i)}$ is the solution of the so-called cell problem

$$-\operatorname{div}_y\left[\mathbf{A}(x,y)(\nabla_y w^{(i)}(x,y) + \mathbf{e}_i)\right] = 0 \qquad \text{in} \quad Y. \tag{B.16}$$

Here $w^{(i)}(x,y)$ is Y-periodic with respect to y. The result of two-scale convergence are summarized is the following form:

THEOREM B. 5. *The sequence $\{u^\varepsilon\}$ of solutions of the problem (B.8) converges weakly to $u(x)$ in $H_0^1(\Omega)$, and the sequence ∇u^ε two-scale converges to $\nabla u(x) + \nabla_y u^{(1)}(x,y)$, where $(u, u^{(1)})$ is the unique solution in $H_0^1(\Omega) \times L^2[\Omega; H_{\text{per}}^1(Y)/\mathbb{R}]$ of the following two-scale homogenized system:*

$$-\operatorname{div}_y\left(\mathbf{A}(x,y)\left[\nabla u(x) + \nabla_y u^{(1)}(x,y)\right]\right) = 0 \qquad \text{in} \quad \Omega\times Y, \tag{B.17}$$

(B.18)
$$-\operatorname{div}_x\left(\int_Y \mathbf{A}(x,y)\left[\nabla u(x)+\nabla_y u^{(1)}(x,y)\right]dy\right)=f \qquad \text{in} \quad \Omega,$$

(B.19)
$$u(x)=0 \qquad \text{on} \quad \partial\Omega.$$

Furthermore the system is equivalent to the usual homogenized and cell equations through the relation

(B.20)
$$u^{(1)}(x,y)=\sum_{i=1}^{N}\frac{\partial u}{\partial x_i}(x)w^{(i)}(x,y).$$

P r o o f. Due to the a priori estimates (B.12), there exists a limit u such that, up to a subsequence, u^{ε} converges weakly to u in $H_0^1(\Omega)$. We multiply now the Eq. (B.8) by a test function $\eta(x)+\varepsilon\psi(x,x/\varepsilon)$ with $\eta(x)\in\mathcal{D}(\Omega)$ and $\psi(x,y)\in\mathcal{D}[\Omega;C^{\infty}_{\text{per}}(Y)]$. This yields

(B.21)
$$\int_\Omega \mathbf{A}\left(x,\frac{x}{\varepsilon}\right)\nabla u^{\varepsilon}\left[\nabla\eta(x)+\nabla_y\psi\left(x,\frac{x}{\varepsilon}\right)+\varepsilon\nabla_x\psi\left(x,\frac{x}{\varepsilon}\right)\right]dx$$
$$=\int_\Omega f(x)\left[\eta(x)+\varepsilon\psi\left(x,\frac{x}{\varepsilon}\right)\right]dx.$$

By Theorem B.1 and B.2 we can pass to the two-scale limit in (B.21):

(B.22)
$$\int_\Omega\int_Y \mathbf{A}(x,y)\left[\nabla u(x)+\nabla_y u^{(1)}(x,y)\right]\cdot\left[\nabla\eta(x)+\nabla_y\psi(x,y)\right]dx\,dy$$
$$=\int_\Omega f(x)\eta(x)\,dx.$$

By density, (B.22) holds true for any (η,ψ) in $H_0^1(\Omega)\times L^2[\Omega;H^1_{\text{per}}(Y)/\mathbb{R}]$. The integration by parts show that (B.22) is a variational formulation associated to (B.13) – (B.15). To prove that (B.17) – (B.19) is equivalent to (B.13) – (B.16), it is sufficient to use (B.20). □

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