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Quasi-homoclinic solutions to a system of ODEs

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TIS PAPER considers the problem of existence of "quasi-homoclinic" solution to a system of three first order ODEs containing a small parameter. These equations describe travelling wave solutions to a one-temperature model of laser-sustained plasma with absorption. This solution is homoclinic with respect to the first two variables. We use the methods of geometric singular perturbation theory to prove the existence of strictly homoclinic trajectory and then prove that it implies the existence of the desired solution.

Key words: Homoclinic solution, singular perturbation, exchange lemma, laser-sustained plasma.

1. Introduction

PLASMA SUSTAINED by a laser beam occurs naturally in different processes in which laser radiation is used, e.g. in laser welding. The full systems of equations modelling such phenomena are very complicated and it is impossible to know the quantitative details of their solutions. In this work we are looking for solutions in a form of homoclinic travelling waves of temperature. Such waves, though one-dimensional, may serve as a local approximation of the real solution and may supply us with some qualitative relations between the basic magnitudes characterizing the considered system (e.g. the speed of the moving boundary of plasma region and the intensity of the laser radiation). In this paper we consider the same problem as in [1], however the proof is simpler. *The one-temperature model of laser-maintained plasma with absorption is sketched in Sec. 3.*

2. Formulation of the problem

In this paper we consider the following system of ODEs:

$$\widetilde{u}' = \widetilde{v}, \tag{2.1}$$

$$\widetilde{v}' = qc(\widetilde{u})\widetilde{v} - f(\widetilde{u}) - \kappa(\widetilde{u})\widetilde{I}, \tag{2.2}$$

$$(2.3) \qquad \widetilde{I}' = -\varepsilon\kappa(\widetilde{u})\widetilde{I} ,$$

where $' := \dfrac{d}{d\xi}$, $\xi \in (-\infty, \infty)$, $\widetilde{u}$, $\widetilde{v}$, $\widetilde{I}: R^1 \to R^1$ and $q \in R^1$, $\varepsilon \in R^1$ with $0 < \varepsilon \ll 1$.

Assumption 1. The functions c, f, $\kappa: R^1 \to R^1$ are sufficiently smooth.

Assumption 2. $c(u) > 0$ for all $u \in (-\infty, \infty)$. There exists $u_0 > 0$, such that $\kappa(u) \equiv 0$ for $u \leqslant u_0$, and $\kappa(u) > 0$ for $u > u_0$.

We assume that for I from some appropriate interval, the source function $(f(u) + \kappa(u)I)$ behaves qualitatively like a cubic polynomial, i.e. that:

Assumption 3. There exist numbers $I_c > 0$, and $I^c > I_c$ such that for $I \in (I_c, I^c)$ the equation $F(u, I) = f(u) + \kappa(u)I = 0$ has exactly three solutions: 0, $u_1(I) > u_0$, and $u_2(I) > u_1(I)$ such that $F_{,u}(0, I) < 0$, $F_{,u}(u_1(I), I) > 0$, $F_{,u}(u_2(I), I) < 0$, $F(u, I) > 0$ for $u < 0$, $F(u, I) < 0$ for $u \in (0, u_1(I))$, $F(u, I) > 0$ for $u \in (u_1(I), u_2(I))$ and $F(u, I)) < 0$ for $u > u_2(I)$.

Remark 1. Due to Assumption 3 and the implicit function theorem we note that $u_{1,I}(I) < 0$ and $u_{2,I}(I) > 0$ for $I \in (I_c, I^c)$ (see Fig. 1). □

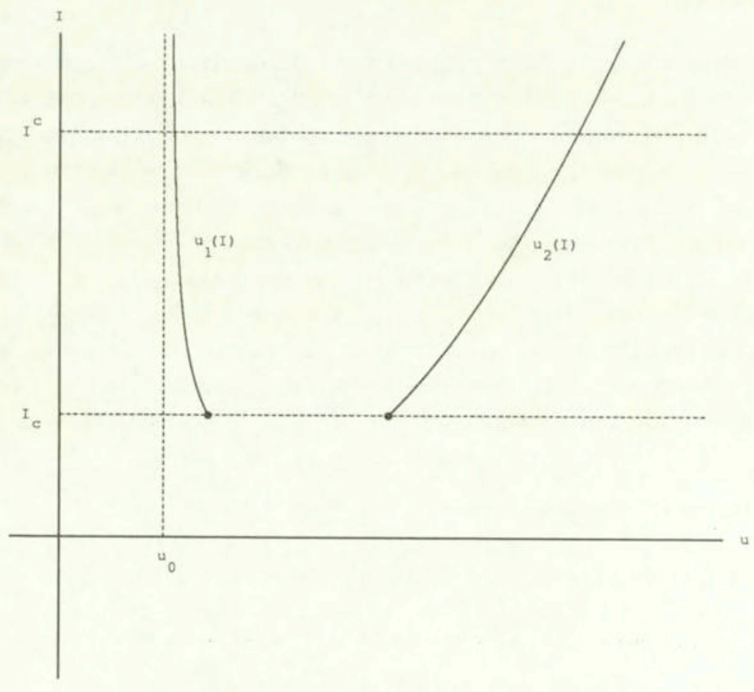

Fig. 1. The three branches of solutions to the equations $F(u, I)$.

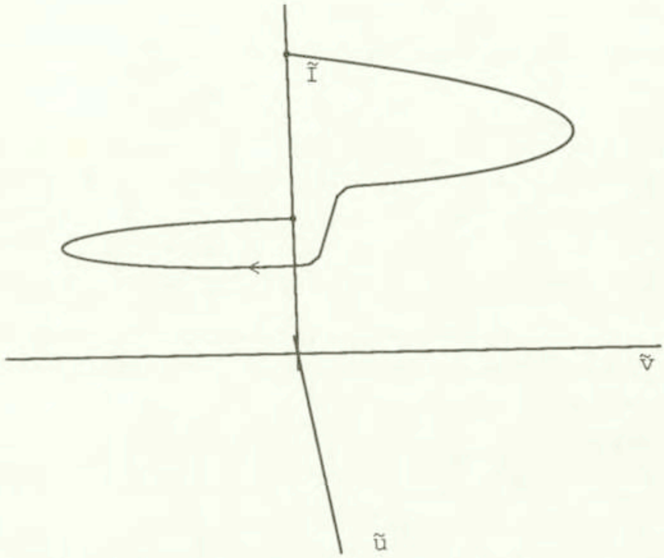

FIG. 2. A possible trajectory satisfying the conditions (2.4) and (2.5).

Our problem consists in showing that for all I_0 from some subinterval of (I_c, I^c) and for all sufficiently small $\varepsilon > 0$, there exists at least one q such that the system (2.1) – (2.3) has a solution defined for $\xi \in R^1$ and satisfying the conditions:

(2.4) $$\widetilde{I}(-\infty) = I_0, \qquad \widetilde{v}(\pm\infty) = 0,$$

(2.5) $$\widetilde{u}(\pm\infty) = 0\ , \qquad \max_{\xi \in R^1} \widetilde{u}(\xi) > u_2(I_0) + O(\varepsilon) \quad \text{as} \quad \varepsilon \searrow 0.$$

Such a problem occurs e.g. in a simplified one-dimensional analysis of plasma sustained by a laser beam. This example is shortly described in the next section.

On the one hand this problem is simpler than seeking a solution homoclinic to the point $(\widetilde{u}, \widetilde{v}, \widetilde{I}) = (0, 0, I_0)$, because we do not demand the condition $\widetilde{I}(-\infty) = \widetilde{I}(\infty)$. On the other hand it is, in a way, not standard as we have to do with the whole line of nonseparate singular points of the form $(\widetilde{u}, \widetilde{v}, \widetilde{I}) = (0, 0, \widetilde{I})$. We are looking for a solution starting from one point of this line and arriving at another one (with smaller $\widetilde{I}$; see Fig. 2). Our idea to solve the problem is to modify the considered system, so that the line of singular points shrinks to one singular point $(0, 0, I_0)$, and to apply the efficient methods of asymptotic analysis of existence of homoclinic orbits (see [2], [3], [4]). Then we prove that the obtained homoclinic solution can be modified in turn to become a solution to the system (2.1) – (2.3).

To construct the homoclinic orbit, we must make two crucial assumptions about the existence of forward and backward heteroclinic solutions for $\varepsilon = 0$ limit of the system (2.1) – (2.3).

Assumption 4. There exists exactly one $q_0 > 0$ such that for $q = q_0$ and $\widetilde{I} = I_0$ there exists a unique (modulo shifts in ξ) heteroclinic solution $h_f(\xi) = (u_f(\xi), v_f(\xi))$ to the system (2.1) – (2.2) connecting the points $(0,0) = h_f(-\infty)$ and $(u_2(I_0), 0) = h_f(\infty)$ such that $v_f(\xi) > 0$ for $|\xi| < \infty$.

Assumption 5. There exists $I^* \in (I_c, I_0)$ such that for $q = q_0$ and $\widetilde{I} = I^*$ there exists a unique (modulo shifts in ξ) heteroclinic solution $h_b(\xi) = (u_b(\xi), v_b(\xi))$ to the system (2.1) – (2.2) connecting the points $(u_2(I^*), 0) = h_b(-\infty)$ and $(0,0) = h_b(\infty)$ such that $v_b(\xi) < 0$ for $|\xi| < \infty$.

Remark 2. It is necessary to comment on the validity of the existence assumptions (Assumptions 4 and 5). First, the necessary conditions for Assumptions 4 and 5 to hold are:

$$\int_0^{u_2(I_c)} (f(s) + \kappa(s) I_c)\, ds < 0 \ , \qquad \int_0^{u_2(I^c)} (f(s) + \kappa(s) I^c)\, ds > 0 \ .$$

Now, let $I_l > I_c$ be such that

$$\int_0^{u_2(I_l)} (f(s) + \kappa(s) I_l)\, ds = 0$$

and

$$\int_0^{u_2(I)} (f(s) + \kappa(s) I)\, ds < 0$$

for all $I \in (I_c, I_l)$. A possible range of I_0 may be estimated in the following way. Let us consider the system (2.1), (2.2) for $\widetilde{I} = I_\vartheta \in (I_c, I_l)$. Let q_ϑ denote the corresponding value of q for which the (unique) heteroclinic solution $h_b(\xi)$ (such that $h_b(-\infty) = (u_2(I_\vartheta), 0)$, $h_b(\infty) = (0,0)$) exists. Since $(-f(u) - \kappa(u)\widetilde{I})$ increases for every $u > u_0$ as $\widetilde{I}$ decreases, then it is easy to note that q_ϑ increases. Thus, if I_ϑ tends to I_c from above, then $q_\vartheta(I_\vartheta) \nearrow q_c > 0$.

Now, we may estimate I^u – the upper bound for the value of I_0 . It is just the value of I, for which there exists a (unique) heteroclinic $h_f(\xi)$ (such that $h_f(-\infty) = (0,0)$, $h_f(\infty) = (u_2(I_\vartheta), 0)$) for $q = q_c$. (Again, one notes that if $h_f(\xi)$ exists for two pairs $(q_1, \widetilde{I}_1)$ and $(q_2, \widetilde{I}_2)$ and $q_2 > q_1$, then we also have $\widetilde{I}_2 > \widetilde{I}_1$.) If $I^u \leqslant I^c$ then Assumptions 4, 5 will be valid for $I_0 \in (I_l, I^u) \subseteq (I_c, I^c)$. □

3. Physical example

The system (2.1) – (2.3) arises naturally, when we look for travelling wave solutions to partial differential equations describing plasma sustained by a laser beam. Under the assumption of constant pressure, the temperature field in plasma maintained by a laser radiation may be modelled by means of a nonlinear heat conduction equation (see e.g. [6] or [7]):

$$\rho C(\partial_t T + \mathbf{v}_k \cdot \mathrm{grad}_x T) = \mathrm{div}_x(\sigma\, \mathrm{grad}_x T) + W(T, \mathbf{x}), \tag{3.1}$$

where $\mathbf{x} \in R^3$, ρ is the mass density, C is the heat capacity per unit mass at constant pressure, $\sigma(T)$ is the total heat conductivity coefficient and $W(T, \mathbf{x})$ is the nonlinear source function responsible for the energy balance: the gain of energy from the laser beam and its losses through radiation, and $\mathbf{v}_k$ is the local velocity of the gas stream. Eq. (3.1) was analyzed in many papers (see e.g. [7], [8]). The source term has the form:

$$W(T, \mathbf{x}) = -L(T) + \kappa(T)\widetilde{I}(\mathbf{x}). \tag{3.2}$$

$L(T)$ denotes the energetic losses of plasma. They are negligible below a certain temperature $T = \eta$ of the order of 10^3K. $\widetilde{I}(\mathbf{x})$ is the intensity of the laser beam at the point $\mathbf{x}$ and $\kappa(T)$ is the coefficient of energy absorption from the laser radiation. The magnitude of $\kappa(T)$ is negligible below a certain temperature $T = \eta_* \approx 10^4$K. In optically thin plasma the dependence of $\widetilde{I}$ on $\mathbf{x}$ is described by the equation:

$$\mathrm{div}(\mathbf{n}_p(t, \mathbf{x})\widetilde{I}) = -\kappa(T)\widetilde{I}, \tag{3.3}$$

where $\mathbf{n}_p(t, \mathbf{x})$ is a unit vector parallel to the Poynting vector of the electromagnetic wave at point $\mathbf{x}$. The equations (3.1), (3.3) may be written in an dimensionless form. Let us choose a typical $\widetilde{I} = J$ and let τ be the first root of the equation

$$-L(T) + \kappa(T)J = 0$$

such that $\tau > \eta_*$. Let

$$U = U(T) = (T - \eta)(\tau - \eta)^{-1}\,.$$

Then

$$T(U) = (\tau - \eta)U + \eta\,.$$

Let us set

$$\underline{\rho}(U) = \rho_0^{-1}\rho(T(U))\,, \quad \underline{C} = C_0^{-1}C(T(U)), \quad \underline{\sigma}(U) = \sigma(T(U))\sigma_0^{-1}\,,$$

$$\underline{L}(U) = L(T(U))(\,L(\tau))^{-1}\,, \quad \underline{\kappa}(U) = \kappa(T(U))\kappa_0^{-1}\,,$$

$$\rho_0 = \rho(\tau)\,, \quad C_0 = C(\tau), \quad \sigma_0 = \sigma(\tau)\,, \quad L_0 = L(\tau)(\tau - \eta)^{-1}\,,$$

$$\kappa_0 = \kappa(\tau)\,, \qquad \underline{\mathbf{x}} = \kappa_0 \mathbf{x}\,, \quad \underline{t} = \frac{L_0}{C_0\rho_0}\,t\,.$$

Then Eq. (3.1) takes the form:

$$\underline{\rho}\underline{C}(\partial_{\underline{t}}U + \varepsilon \mathbf{v} \cdot \mathrm{grad}_{\underline{x}}U) = \varepsilon^2 \mathrm{div}_{\underline{x}}(\underline{\sigma}\, \mathrm{grad}_{\underline{x}}U) + \underline{W}(U, \underline{\mathbf{x}}) \,, \tag{3.4}$$

where

$$\varepsilon^2 = \sigma_0 \frac{\kappa_0^2}{L_0} \,, \quad \mathbf{v} = \mathbf{v}_k\, \rho_0 C_0 (L_0\sigma_0)^{-\frac{1}{2}} \,,$$

$$\underline{W}(U, \underline{\mathbf{x}}) = -\underline{L}(U) + \underline{\kappa}(U)\underline{I}(\underline{\mathbf{x}}),$$

$$\underline{I}(\underline{\mathbf{x}}) = \kappa_0 I(\underline{\mathbf{x}}) \frac{1}{L(\tau)},$$

$$\underline{\rho}(U) = \rho_0^{-1}\rho(T(U)) \,, \quad \underline{C} = C_0^{-1}C(T(U)), \quad \underline{\sigma}(U) = \sigma(T(U))\sigma_0^{-1} \,.$$

In a similar way, Eq. (3.3) may be written in the form:

$$\mathrm{div}_{\underline{x}}(\mathbf{n}_p(\underline{\mathbf{x}})\widetilde{\underline{I}}(\underline{\mathbf{x}}) = -\underline{\kappa}(U))\widetilde{\underline{I}}(\underline{\mathbf{x}}). \tag{3.5}$$

In most experiments with laser-sustained plasma, the value of ε is relatively small. For example, for argon plasma (for pressure equal to 1 atm) it is of the order of 0.1 (see [9, 10]).

Under suitable conditions (sufficiently large laser radiation intensity and properly chosen speed of the gas stream), in a certain region of space we can cause the gas ionization. Here the gas may be in the state of plasma – its temperature is equal to T_2 ($\approx 2 \cdot 10^4$K for argon plasma and pressure equal to 1 atm). Out of this region the inflowing gas is cold, unionized and its temperature is equal to η ($\approx 10^3$K). (See [6, 7, 8] and references therein). We can examine the qualitative character of the temperature profile as well as the motion of the plasma boundaries by assuming that ε is (sufficiently) small. Moreover, we will confine ourselves to the axis of symmetry of the problem, which has the simplifying property: $n_p(\underline{\mathbf{x}})$ is parallel to it at every $\underline{\mathbf{x}}$, i.e. the laser beam propagates along this axis. Also $\underline{\mathbf{v}}(\underline{\mathbf{x}})$ can be assumed to be parallel to it. This 1-dimensional analysis may serve as an initial point to a more advanced consideration of the problem. So, let us look for solutions in the form of a travelling wave moving with the speed χ in the direction $\mathbf{n}$ parallel to the axis of symmetry. Let:

$$U(\underline{t}, \underline{\mathbf{x}}) = U(\xi),$$

where

$$\xi = \varepsilon^{-1}\, \mathbf{n} \cdot (\underline{\mathbf{x}} - \underline{\mathbf{x}}_0) + \chi \underline{t},$$

$\underline{\mathbf{x}}_0$ lies at the axis of symmetry and $\mathbf{n} \in R^3$ is a fixed unit vector parallel to it. Thus one arrives at the system:

$$C(U)qU' = (\sigma(U)U')' + W(U,I) \,, \tag{3.6}$$

$$\widetilde{I}' = -\varepsilon\kappa(U)\widetilde{I}, \tag{3.7}$$

where $q = \rho(U)(\chi + \mathbf{v}\cdot\mathbf{n})$, $' = \dfrac{d}{d\xi}$, $\xi \in (-\infty, \infty)$, where we have ommited bars under the coefficients symbols. One can easily check that, due to the continuity equation and the assumed symmetry of the problem, q = const, q can be interpreted as the mass speed of the wave (in the direction of decreasing ξ) in the system of coordinates moving with the gas.

Finally, after introducing the relative heat potential:

$$\widetilde{u}(U) := \int_0^U \sigma(y)dy$$

we arrive at the following system of equations:

$$\widetilde{u}' = \widetilde{v}, \tag{3.8}$$

$$\widetilde{v}' = qC(\widetilde{u})(\sigma(\widetilde{u}))^{-1}\widetilde{v} + L(\widetilde{u}) - \kappa(\widetilde{u})\widetilde{I}, \tag{3.9}$$

$$\widetilde{I}'(\xi) = -\varepsilon\kappa(\widetilde{u})\widetilde{I}(\xi), \tag{3.10}$$

where $C(\widetilde{u}) = C(U(\widetilde{u}))$, $L(\widetilde{u}) = L(U(\widetilde{u}))$ and $\kappa(\widetilde{u}) = \kappa(U(\widetilde{u}))$. Due to the second law of thermodynamics $\sigma(y) > 0$ for all $y > 0$, so the above transformation is invertible, i.e. there exists a smooth function $U = U(\widetilde{u})$.

When we denote:

$$c(\widetilde{u}) := C(\widetilde{u})(\sigma(\widetilde{u}))^{-1} \,,$$

$$f(\widetilde{u}) := -L(\widetilde{u}),$$

then we obtain the system (2.1) – (2.3). We can modify slightly the function $f(u)$ for $u \in (-\infty, \delta)$, δ small, and choose I_c and I^c in such a way that ASSUMPTIONS 2, 3, 4 and 5 are satisfied. The roots u_0 and u_2 of ASSUMPTION 3 correspond to the temperatures of cold incoming gas and T_2, respectively.

4. Modified system

First, we will analyze the system:

$$u' = v, \tag{4.1}$$

$$v' = qc(u)v - f(u) - \kappa(u)I, \tag{4.2}$$

$$I' = -\varepsilon(\kappa(u)I + k(u)(I - I_0)) \,, \tag{4.3}$$

where $k(u)$ is a function with its support disjoint with the support of $\kappa(u)$ and $I_0 = \tilde{I}(-\infty)$. To this system we add the auxilliary equation:

$$q' = 0. \tag{4.4}$$

As we have mentioned, our plan is to prove the existence of an orbit homoclinic to the point $(0, 0, I_0)$ for the system (4.1) – (4.3), and then to prove that it implies the existence of a solution to our initial system (2.1), (2.2), (2.3). We assume that the function $k(u)$ is sufficiently smooth and satisfies the condition:

ASSUMPTION 6. $k : R^1 \to R^1$, $k(u) \equiv 0$ for $u \geqslant \frac{1}{2}u_0$, and $k(u) > 0$ for $u < \frac{1}{2}u_0$.

5. Singular orbit

The system (4.1) – (4.4) has two different time scales. By changing the "time" scale from ξ to $\tau = \xi\varepsilon$ we obtain the so-called slow system:

$$\varepsilon\,\dot{u} = v\,, \tag{5.1}$$

$$\varepsilon\,\dot{v} = qc(u)v - f(u) - \kappa(u)I\,, \tag{5.2}$$

$$\dot{I} = -(\kappa(u)I + k(u)(I - I_0))\,, \tag{5.3}$$

$$\dot{q} = 0\,, \tag{5.4}$$

where $\dot{(\;)}$ denotes differentiation of $(\;)$ with respect to the variable τ. On the other hand, the system (4.1) – (4.4) is called the fast system.

In the limit as $\varepsilon \to 0$ the set of critical points of the fast system (4.1) – (4.4) is determined by the set of equations:

$$v = 0, \tag{5.5}$$

$$f(u) + \kappa(u)I = 0. \tag{5.6}$$

The manifold determined by the above system will be denoted below by S. Let us note that S is invariant with respect to q, so $S = \bigcup_{q\in R^1} S(q)$. Note that the limiting flow (as $\varepsilon \to 0$) for the slow system takes place on S. According to ASSUMPTION 3, in the strip $I \in (I_c, I^c)$ every $S(q)$ consists of three branches corresponding to the three solutions to the Eq. (5.6). These branches will be denoted respectively by $\mathcal{M}^0(q)$, $\mathcal{M}^1(q)$ and $\mathcal{M}^2(q)$. Thus $\mathcal{M}^0(q) = \{(u, v, I, q) : u = 0, v = 0, I \in (I_c, I^c)\}$, $\mathcal{M}^1(q) = \{(u, v, I, q) : u = u_1(I), v = 0, I \in (I_c, I^c)\}$ and $\mathcal{M}^2(q) = \{(u, v, I, q) : u = u_2(I), v = 0, I \in (I_c, I^c)\}$. Let us denote

$$\mathcal{M}^i = \bigcup_{q\in R^1} \mathcal{M}^i(q)\,, \qquad i = 0, 1, 2.$$

Every $\mathcal{M}^i(q)$, $i = 0,1,2$, is parametrized by $I \in (I_c, I^c)$. Consequently, the limiting (as $\varepsilon \to 0$) flow on S (for $I \in (I_c, I_0+\delta)$) is in fact determined by a single autonomous equation for I (with $v=0$ and q fixed) of the form:

$$\dot{I} = -k(0)(I - I_0) \tag{5.7}$$

on $\mathcal{M}^0$, and

$$\dot{I} = -\kappa(u_i(I))I \tag{5.8}$$

on $\mathcal{M}^i$, $i = 1,2$.

Under the assumption $\varepsilon \ll 1$ our aim is to show the existence of an orbit homoclinic to the point $(u, v, I, q) = (0, 0, I_0, q(\varepsilon, I_0))$, where I_0 is from some open interval and $q(\varepsilon, I_0)$ is to be chosen appropriately. This will be done by the use of the geometric singular perturbations theory developed e.g. in [3], [4], [2], [5]. The starting point of the proof is the construction of a singular homoclinic orbit, i.e. an orbit consisting of solutions to the $\varepsilon = 0$ limit of the fast system, with their "ends" joined by trajectories of the $\varepsilon = 0$ limit of the slow system. More precisely:

DEFINITION 1. *Let $\mathcal{H}_f$ denote the orbit in the (u, v, I, q)-space corresponding to the solution (h_f, I_0, q_0) of $\varepsilon = 0$ limit of the system (4.1) – (4.4). Let $\mathcal{H}_b$ denote the orbit in the (u, v, I, q)-space corresponding to the solution (h_b, I^*, q_0) of $\varepsilon = 0$ limit of the system (4.1) – (4.4).*

DEFINITION 2. *The singular homoclinic orbit consists of:*

(1) Heteroclinic orbits $\mathcal{H}_f$ and $\mathcal{H}_b$ joining the points $(0, 0, I_0, q_0)$ with $(u_2(I_0), 0, I_0, q_0)$ and $(u_2(I^*), 0, I^*, q_0)$ with $(0, 0, I^*, q_0)$.

(2) Singular trajectories of the $\varepsilon = 0$ limit of the slow system lying on the manifolds $\mathcal{M}^0(q_0)$, and $\mathcal{M}^2(q_0)$ joining the "ends" of the heteroclinics: the right orbit $\mathcal{J}_1$ (lying on $\mathcal{M}^2(q_0)$) joins the point $(u_2(I_0), 0, I_0, q_0)$ with $(u_2(I^*), 0, I^*, q_0)$, and the left orbit $\mathcal{J}_2$ (lying on $\mathcal{M}^0(q_0)$) joins the point $(0, 0, I^*, q_0)$ with $(0, 0, I_0, q_0)$.

6. Existence of a homoclinic solution

Let us recall that two manifolds $\mathcal{N}_1$ and $\mathcal{N}_2$, both in $R^m, m \geqslant 1$, are said to intersect transversely at point $p \in \mathcal{N}_1 \cap \mathcal{N}_2$ if

$$T_p\mathcal{N}_1 + T_p\mathcal{N}_2 = T_pR^m \equiv R^m.$$

The basic question undertaken by the geometric singular perturbation theory for ODEs consists in finding conditions sufficient for the existence of an orbit which stays near the singular one for sufficiently small $\varepsilon \neq 0$. One of the possible answers is given by the following theorem (Theorem of Section 4 in [2], Theorem 8 in [4]):

THEOREM 1. *Consider the system:*

$$x' = X(x, y, q, \varepsilon), \tag{6.1}$$

$$y' = \varepsilon Y(x, y, q, \varepsilon), \tag{6.2}$$

$$q' = 0, \tag{6.3}$$

where $x \in R^{k+l}, y \in R^n$. *Assume that for each* $q \in R^1$ *and* $\varepsilon > 0$, *there is a locally unique hyperbolic equilibrium point* $P(q)$. *Suppose that the linearization matrix at* $P(q)$ *possesses* k *positive and* $l+n$ *negative eigenvalues and that exactly* n *negative eigenvalues tend to* 0 *together with* ε . *Let* $\{S^i\}, i = 0, ..., N$ *denote a family of slow manifolds for* $\varepsilon = 0$ *equations (with the equlibrium point for* $\varepsilon \neq 0$ *in* S^0*) and assume that for each* i, S^i *is normally hyperbolic with splitting:* k *stable and* l *unstable. Assume further that there is a singular homoclinic orbit, with finitely many heteroclinic orbits* $\mathcal{H}_{i,i+1}$, *each from* S^i *to* S^{i+1} *for some* i, *and trajectories* $\mathcal{J}_{i+1}$ *of the* $\varepsilon = 0$ *limit flow of the slow system corresponding to (6.1) - (6.3) connecting the ends of heteroclinic orbits (lying on* S^{i+1}*). Let* $\hat{\mathcal{J}}_i$ *denote the trajectories* $\mathcal{J}_i$ *extended beyond these jump points. Finally, assume that the following transversality conditions hold for the* $\varepsilon = 0$ *system:*

Let $[P(q), q] \subset S^0$ *denote the graph as* q *is varied of the* $\varepsilon = 0$ *limit of the* $\varepsilon \neq 0$ *equlibrium point. Let* $S^{N+1} \equiv S^0$. *We require that:*

- $W^u([P(q), q])$ *transversely intersects* $W^s(S^1)$ *in* (x, y, q) *- space along* $\mathcal{H}_{0,1}$.
- $W^u(\hat{\mathcal{J}}_i)$ *transversely intersects* $W^s(S^{i+1})$ *in* (x, y, q) *- space along* $\mathcal{H}_{i,i+1}$.

Then for $\varepsilon > 0$ *sufficiently small, there is a locally unique homoclinic solution to the considered system near the singular orbit.*

REMARK 3.

1. This theorem was slightly changed in that we assumed that $\varepsilon \geqslant 0$. But, one may easily note that it does not affect its validity. (Formally, we can satisfy the conditions of the Theorem of Section 4 in [2] by substituting $\varepsilon = \varepsilon_*^2$ and considering ε_* as a small parameter)

2. Let $\mathcal{A}$ denote a subset of a slow manifold of the system (6.1) - (6.3). By the stable (unstable) manifold of $\mathcal{A} \subseteq S$ (at $\varepsilon = 0$) we mean the sum:

$$W_0^s(\mathcal{A}) = \bigcup_{p \in \mathcal{A}} W_0^s(p), \qquad (W_0^u(S) = \bigcup_{p \in \mathcal{A}} W_0^u(p)),$$

where $W_0^s(p)$ $(W_0^u(p))$ denotes the stable manifold of the point p with respect to the $\varepsilon = 0$ limit of the system (6.1) – (6.3).

3. In the context of the above system, a slow manifold S^i is said to be normally hyperbolic if the matrix $\mathbf{D_x}X(x_0, y_0, q_0, 0)$ has only eigenvalues with nonzero real parts for all $(x_0, y_0, q_0) \in S^i$ (see [2] p. 67, [4] p. 49).

4. "Near the singular orbit" means that the trajectory of the homoclinic solution lies within the $O(\varepsilon)$ neighbourhood of the singular orbit (see [4] Theorem 8 p. 102). □

According to Assumptions 2 and 3 we have:

LEMMA 1. For all $\varepsilon > 0$, $I_0 \in (I_c, I^c)$, the point $(u, v, I) = (0, 0, I_0)$ is an equilibrium point of the system (4.1) – (4.3) for all $q \in R^1$ with $k = 1$, $l = 1$, and $n = 1$.

To apply the theorem we will take:

$$N = 1, \quad S^0 = S^2 = \mathcal{M}^0, \quad S^1 = \mathcal{M}^2, \quad \mathcal{H}_{01} = \mathcal{H}_f, \quad \mathcal{H}_{12} = \mathcal{H}_b .$$

Thus to prove the existence of a locally unique homoclinic orbit, we have to verify the transversality condition. The proof of this transversality may be found in [4], [2] or [3], but for the reader's convenience we will sketch it below.

6.1. Transversality of $W^u(S^0)\mid_{[P(q),q]}$ and $W^s(S^1)$

In our case $W^u(S^0)\mid_{[P(q),q]}$ is just the set of points $\{(u, v, I, q) : u = v = 0, I = I_0, q \in R^1\}$. Of course, according to Assumption 4, the manifolds $W^u(S^0)\mid_{[P(q),q]}$ and $W^s(S^1)$ intersect in the plane $\{(u, v, I, q) : q = q_0, I = I_0\}$ along the heteroclinic orbit $\mathcal{H}_f$. Let $v^*(q, u)$ denote the unstable manifold of the singular point $(0, 0, I_0, q)$ with respect to the $\varepsilon = 0$ limit of the system (4.1) – (4.4) (for $I = I_0$ and q fixed). We have the following relations:

$$\tag{6.4} \frac{1}{2}v^{*2}(q, u) = \int_0^u \{qc(y)v^*(q, y) - f(y) - \kappa(y)I_0\}dy ,$$

$$\tag{6.5} \frac{1}{2}v_*^2(q, u) = -\int_u^{u_2(I_0)} \{qc(y)v_*(q, y) - f(y) - \kappa(y)I_0\}dy .$$

In fact, in Eq. (6.4), for q near q_0, we must confine ourselves to $u \in [0, u^*(q)]$, where $u^*(q) \to u_2(I_0)$ as $q \to q_0$. In the same way, in Eq. (6.5) $u \in [u_*(q), u_2(I_0)]$, $u_*(q) \to 0$ as $q \to q_0$.

By differentiating with respect to q at $q = q_0$ one obtains the relations:

$$\tag{6.6} v^*_{,q}(q_0, u)v^*(q_0, u) = \int_0^u c(y)\{q_0 v^*_{,q}(q_0, y) + v^*(q_0, y)\}dy ,$$

$$(6.7)\qquad v_{*,q}(q_0,u)v_*(q_0,u) = -\int\limits_u^{u_2(I_0)} c(y)\{q_0 v_{*,q}(q_0,y) + v_*(q_0,y)\}dy\ ,$$

which are valid for all $u \in (0, u_2(I_0))$. As every trajectory in $W^s(S^1)$ stays in a plane $I = \text{const}$, thus to prove the desired transversality, it suffices to show that the projection of $W^s(S^1)$ onto the space $I = I_0$ intersects $W^u(S^0)\,|_{[P(q),q]}$ along the curve $u \to (u, v^0(q_0,u), I_0, q_0)$, for $u \in (0, u_2(I_0))$ (corresponding to $\mathcal{H}_f$) transversely in the plane $\{(u,v,I,q) : I = I_0\}$ which is isomorphic to R^3. Here $v^0(q_0,u) := v^*(q_0,u) = v_*(q_0,u)$. At any point $(u_0, v, I_0, q_0), v = v^0(q_0,u_0)$, an arbitrary vector from the space tangent to the surface $(u,q) \to (u, v^*(q,u), I_0, q)$ has the form

$$[du, v^0_{,u}(q_0,u_0)du + v^*_{,q}(q_0,u_0)dq, 0, dq]$$

and a vector from the space tangent to the surface $(u,q) \to (u, v_*(q,u), I_0, q)$ has the form

$$[du, v^0_{,u}(q_0,u_0)du + v_{*,q}(q_0,u_0)dq, 0, dq].$$

One notes that all vectors from R^3 can be expressed in terms of these vectors iff $v^*_{,q}(q_0,u_0) \neq v_{*,q}(q_0,u_0)$.

LEMMA 2. For $u \in (0, u_2(I_0))$:

$$v^*_{,q}(q_0,u) > 0 \quad \text{and} \quad v_{*,q}(q_0,u)) < 0.$$

Moreover,

$$\lim_{u \nearrow u_2(I)} v^*_{,q}(q_0,u) \to \infty, \qquad \lim_{u \searrow 0} v_{*,q}(q_0,u) \to -\infty\ .$$

Proof. Let us note that $v^*_{,q}(q_0,0) = v_{*,q}(q_0,u_2(I_0)) = 0, v^*_{,qu}(q_0,0) > 0$ and $v_{*,qu}(q_0,u_2(I_0)) > 0$. Hence, $v^*_{,q}(q_0,u) > 0$ for $u \in (0,\delta)$ whereas $v_{*,q}(q_0,u)) < 0$ for $u \in (u_2(I_0) - \delta, u_2(I_0))$ for some $\delta > 0$ sufficiently small. Now, suppose that u_m is the largest u in $(0, u_2(I_0))$, such that $v_{*,q}(q_0,u_m) = 0$. As $v_{*,q}(q_0,u)$ satisfies the equation of the form:

$$v_{,qu}v + v_{,q}v_u = q_0 c(u) v_{,q} + cv\ \ ,$$

this would imply that $v_{*,qu}(q_0,u_m) = c(u_m) > 0$ – a contradiction. Thus $v_{*,q}(q_0,u) < 0$ for all $u \in (0, u_2(I_0))$. In the same way we prove that $v^*_{,q}(q_0,u) > 0$ for all $u \in (0, u_2(I_0))$. Also $m := \lim\limits_{u \searrow 0} v_{*,q}(q_0,u) < 0$ and $\lim\limits_{u \nearrow u_2(I)} v^*_{,q}(q_0,u) > 0$. Using (6.6) we obtain the third relation of this lemma. Now, we will prove that the last relation is true. Suppose to the contrary that $m \in (-\infty, 0)$. Then both sides of Eq. (6.) are (in the limit) equal to 0 for $u = 0$ and we obtain for $u \searrow 0$:

$$[\lambda_- + o(1)][m + o(1)]u = q_0[c(0) + o(1)][m + o(1)] + O(u^2),$$

$$[\lambda_- - q_0 c(0) + o(1)] = O(u^2),$$

where λ_- is the positive eigenvalue of linearization matrix at $(u, v) = (0, 0)$. As $\lambda_- > q_0 c(0)$, then passing to the limit $u \searrow 0$ we arrive at a contradiction. The lemma is proved. □

6.2. Transversality of $W^u(S^1)\,|_{sing.orbit}$ and $W^s(S^0)$

The intersection of these two manifolds takes place in the set $\{(u, v, I, q) : q = q_0, I = I^*\}$ along the orbit corresponding to $\mathcal{H}_b$. It is easy to note that this intersection is transversal, if it is transversal in the $\{(u, v, I, q) : q = q_0\}$ space. So, as above, for I from some neighbourhood of I^*, let us denote by $\sigma^*(I, u)$ the stable manifold of the singular point $(0, 0, I, q_0)$ with respect to the $\varepsilon = 0$ limit of the system (4.1) – (4.4) (for $q = q_0$ and I fixed) parametrized locally by u. Analogically, let $\sigma_*(q, u)$ denote the unstable manifold of the point $(u_2(I^*), 0, I, q_0)$. Then, for I from some neighbourhood of I^* we have:

$$\frac{1}{2}\sigma_*^2(I, u) = \int_{u_2(I)}^{u} \{q_0 c(y)\sigma_*(I, y) - f(y) - \kappa(y)I\}dy \quad , \tag{6.8}$$

$$\frac{1}{2}\sigma^{*2}(I, u) = \int_0^u \{q_0 c(y)\sigma^*(I, y) - f(y) - \kappa(y)I\}dy \, . \tag{6.9}$$

Equation (6.8) is valid for $u \in [u_*(I), u_2(I)]$, where $u_*(I) \to 0$ as $I \to I^*$. In the same way Eq. (6.9) is valid for $u \in [0, u^*(I)]$, where $u^*(I) \to u_2(I^*)$ as $I \to I^*$. For $I = I^*$, $\sigma^*(I, u) = \sigma_*(I, u)$. Of course $\sigma^*(I^*, u) = \sigma_*(I^*, u)$ is negative for $u \in (0, u_2(I^*))$. Differentiation with respect to I at $I = I^*$ gives us the relations:

$$\sigma_{*,I}(I^*, u)\sigma_*(I^*, u) = \int_u^{u_2(I^*)} \{-q_0 c(y)\sigma_{*,I}(I, y) + \kappa(y)\}dy \quad , \tag{6.10}$$

$$\sigma^*_{,I}(I^*, u)\sigma^*(I^*, u) = -\int_0^u \{-q_0 c(y)\sigma^*_{,I}(I, y) + \kappa(y)\}dy \, . \tag{6.11}$$

Now, it easy to see that the following lemma holds:

LEMMA 3. $\sigma_{*,I}(I^*, u)$, and $\sigma_{*,I}(I^*, u)$ satisfy the relations:

$$\sigma_{*,I}(I^*, u) < 0 \qquad \text{for} \qquad u \in (0, u_2(I^*)],$$

$$\sigma^*_{,I}(I^*,u) = 0 \qquad \text{for} \qquad u \in [0,u_0] \ ,$$

$$\sigma^*_{,I}(I^*,u) > 0 \qquad \text{for} \qquad u \in (u_0, u_2(I^*)) \ ,$$

$$\lim_{u\searrow 0} \sigma_{*,I}(I^*,u) = -\infty \ , \qquad \lim_{u\nearrow u_2(I^*)} \sigma^*_{,I}(I^*,u) = \infty \ .$$

P r o o f. Due to Assumption 3 we note that $\sigma_{*,I}(I^*, u_2(I^*)) = 0$. Thus, by (6.10), we infer that $\sigma_{*,I}(I^*,u) < 0$ for $u \in (0, u_2(I^*)]$. Moreover, $\sigma_{*,I}(I^*,u) \searrow -\infty$ as $u \searrow 0$. It follows from (6.11) that $\sigma^*_{,I}(I^*,u) = 0$ for $u \in [0,u_0]$. Suppose that $\sigma^*_{,I}(I^*,\eta) \leqslant 0$ for some $\eta > u_0$. Then, by (6.11), $\sigma^*_{,I}(I^*,u) < 0$ for $u < \eta$, sufficiently close to η, so it cannot become 0 for $u = u_0$. Hence $\sigma^*_{,I}(I^*,u) > 0$ for $u \in (u_0, u_2(I^*))$. Finally, as $\kappa(u_2(I^*)) > 0$, then $\lim_{u\nearrow u_2(I^*)} \sigma^*_{,I}(I^*,u) \neq 0$, as otherwise the right-hand side of Eq. (6.11) cannot tend to 0 as $u \nearrow u_2(I^*)$. Now, as in the proof of Lemma 2, one can show that $\lim_{u\nearrow u_2(I^*)} \sigma^*_{,I}(I^*,u) = \infty$. The lemma is proved. □

This lemma guarantees the transversality of the considered manifolds.

7. Existence of a solution to the initial system

Below, we will prove that from the homoclinic solution obtained by means of Theorem 1 we can construct a solution to the system (2.1) – (2.3) satisfying the conditions (2.4) – (2.5). First, we will state some properties of the solution homoclinic to the point $(0,0,I_0)$ obtained by means of Theorem 1.

LEMMA 4. Let $(u(\xi), v(\xi), I(\xi))$ be a solution to the system (4.1) – (4.3) homoclinic to the point $(0,0,I_0)$. Then:

1. $u(\xi) > 0$ for $|\xi| \neq \infty$.

2. Let $(1, w_-)$ and $(1, w_+)$ denote eigenvectors corresponding to the eigenvalues $\lambda_- > 0$ and $\lambda_+ < 0$ of the linearization matrix of Eqs. (4.1), (4.2) for $q = q(\varepsilon, I_0)$ at $(0,0)$. Then $w_- = \lambda_-$, $w_+ = \lambda_+$ and $(u(\xi), v(\xi)) = u(\xi)(1, w_\pm + o(1))$ as $\xi \to \pm\infty$.

P r o o f. Suppose to the contrary that the first part of the lemma is not true. Then there must exist such ξ that $u(\xi)$ is a negative minimum. So, $-f(u(\xi)) - \kappa(u(\xi))I(\xi) \geqslant 0$. This is, however, impossible according to Assumption 3. The second part is obvious as, due to Assumption 2, $\kappa(u) \equiv 0$ for $u \in [0,u_0]$. □

LEMMA 5. For $\varepsilon > 0$ sufficiently small, the u-component of the homoclinic solution to Eqs. (4.1) – (4.3) may have only one maximum and no minimum.

P r o o f. As $\varepsilon \searrow 0$, then homoclinic solution to Eqs.(4.1) – (4.4) obtained by Theorem 1 tends to the singular solution defined in Section 5. The projection of the singular orbit onto the (u, v) space touches the $v = 0$ axis for $u = 0$ and $u \in [u_2(I^*), u_2(I_0)]$. Suppose that the function u attains a maximum different from the global maximum (or from a fixed global maximum if there are more than one) at some point ζ_M . According to Lemma 4, for ε small, $u(\zeta_M)$ must be close to the set $[u_2(I^*), u_2(I_0)]$. As there is another maximum, the function u must attain a minimum at some point ζ_m , where $\zeta_m < u^*$ and $u^* \to u_1(I^*)$ for $\varepsilon \to 0$. However, due to Assumption 3, $u_2(I^*) > u_1(I^*) + d$, $d > 0$, for $I \in (I_c, I^c)$. This would imply that the distance between the considered solution and the singular solution would not tend to 0 as $\varepsilon \searrow 0$. Hence we arrive at a contradiction with Theorem 1. The lemma is proved. □

LEMMA 6. Let $u(\xi) < u_0$ for $\xi < \xi_-$. Then $I(\xi) \equiv I_0$ for $\xi \in (-\infty, \xi_-)$.

P r o o f. Suppose that this is not true and that for some $\zeta < \xi_-$ we have $I(\zeta) \neq I_0$. Then $I(\xi) = I_0 + (I(\zeta) - I_0) exp(-\varepsilon \int_\zeta^\xi k(u(s))ds)$ and consequently, $I(\xi)$ would diverge as $\xi \to -\infty$ unless $(I(\zeta) - I_0) = 0$. □

In view of Lemma 4, 5 and 6, the following theorem of existence is valid:

THEOREM 2. *Let Assumptions 1-5 be fulfilled. Then for each $\varepsilon > 0$ sufficiently small there exists $q(\varepsilon, I_0)$ such that for $q = q(\varepsilon)$ there exists a (locally unique modulo shifts in ξ) solution to the system* (2.1) – (2.3) *satisfying the conditions* (2.4), (2.5).

P r o o f. Fix $\varepsilon > 0$ sufficiently small. Let ξ_- be the same as in Lemma 6 and let ξ_+ be such that $u(\xi) < u_0$ for $\xi > \xi_+$. (Of course ξ_- and ξ_+ may depend on ε.) Then a solution to our problem can be determined by the equations:

$$\widetilde{u}(\xi) = u(\xi), \quad \widetilde{v}(\xi) = v(\xi) \quad \text{for} \quad \xi \in (-\infty, \infty) \ ,$$

$$\widetilde{I}(\xi) = I(\xi) = I_0 \quad \text{for} \quad \xi \in (-\infty, \xi_-) \ ,$$

$$\widetilde{I}(\xi) = I(\xi) \quad \text{for} \quad \xi \in [\xi_-, \xi_+) \ ,$$

$$\widetilde{I}(\xi) = I(\xi_+) \quad \text{for} \quad \xi \in [\xi_+, \infty) \ .$$

The theorem is proved. □

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References

1. B. Każmierczak, *Travelling waves in a system modelling laser-sustained plasma*, to be published in J. Appl. Math. Phys.
2. C.K.R.T. Jones, N. Kopell, *Tracking invariant manifolds with differential forms in singularly perturbed systems*, J. Differential Equations, **108**, 64–88, 1994.
3. P. Szmolyan, Heteroclinic and homoclinic orbits in singular perturbation problems, *J.Differential Equations*, **92**, 252–281, 1991.
4. C.K.R.T. Jones, *Geometric singular perturbation theory*, in *Dynamical Systems*, Montecacini Terme, 1994, Lecture Notes in Mathematics, 1609, 45–118, 1994.
5. B. Deng, The existence of infinitely many traveling front and back waves in the FitzHugh-Nagumo Equations *SIAM J. Math. Anal.*, **22** (1991) (1631–1650).
6. Iu.P. Raizer, *Fizika gazovogo razrjada* (in Russian), Moskva, Nauka, 1987.
7. Z. Peradzyński, Continuous Optical Discharge, Properties and Modelling, III, ICPIG XXI, Ruhr-Universität Bochum (1993), 227–236.
8. W. Eckhaus, A. van Harten, Z. Peradzyński, A Singularly Perturbed Free Boundary Problem Describing a Laser Sustained Plasma, *SIAM J. Appl. Math*, **45** (1985), 1–31.
9. J. Kurzyna, *Numerical Modelling of Laser Sustained Plasma and Its Comparison with Experiments* (in Polish)–Ph.D. thesis, IFTR-PAS, Warsaw, 1992.
10. R.S. Devoto, Transport coefficients of partially Ionized argon, *Physics of Fluids*, **10** (1967), 354–364.

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