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Dynamic solutions of non-uniform extensional motions as applied to instability of fibre-forming processes

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THE CONCEPT of non-uniform extensional motion of materially non-uniform simple, locally isotropic materials is applied to the case of dynamic solutions of fibre-forming processes. The approach presented enables discussion of instability problems taking into account inertia as well as shearing effects. It is shown that small stationary disturbances may lead to unstable situations while small oscillatory disturbances grow limitlessly only for certain discrete frequencies. Some comparisons with previous results are made.

1. Introduction

THIS PAPER PRESENTS further application of the theory of non-uniform extensional motions (NUEM) of materially non-uniform (inhomogenous) media developed in [1, 2] and applied to kinematic description of steady fibre-forming processes in [3]. Under the assumption that the filament shape slightly differs from the cylindrical one, we used a consequent linearization procedure leading to the relatively simple equations describing also shearing effects.

In the present paper, in a way similar to that developed in [3], we consider the corresponding approximations of the unsteady governing equations taking into account shearing as well as inertial effects. Such an approach leads to the dynamic solutions enabling effective discussion of instability problems of the stationary and oscillatory types. Certain comparisons of our results with those obtained so far for the instability problems and draw-resonance phenomena in non-isothermal fibre-spinning processes (cf. [4, 5, 6]) are discussed in grater detail at the end.

It is worth noting that certain particular instability problems were also considered in the case of flows with dominating extension (FDE) in our previous papers [7, 8]. Apart from formal similarities to the present results, the solutions obtained for FDE refer rather to the sensitivity problems caused by disturbances superposed on the corresponding stresses, velocity gradients, etc.

In Sec. 2 we remind the concept of steady NUEM of materially non-uniform media and introduce the additional motions superposed on the quasi-elongational ones and formulate the relevant constitutive equations. This section is based in whole on our previous considerations [3]. Section 3 briefly presents the equations of dynamic equilibrium and the boundary conditions appropriate for fibre-forming processes. The subsequent approximations of the governing equations are

considered in Sec. 4. The dynamic solutions and the instability problems of the stationary and oscillatory types are discussed in greater detail in Sec. 5. The last Sec. 6 summarizes certain conclusions and compares our results on instability with those obtained previously under different assumptions.

2. Non-uniform extensional motions (NUEM) of materially non-uniform media

Consider an isochoric, quasi-elongational motion in cylindrical coordinates for which the deformation gradient at the current time t, relative to a configuration at time 0, is of the following diagonal form:

$$[\mathbf{F}(\mathbf{X},t) = \begin{bmatrix} \lambda^{-1/2} & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \qquad \det \mathbf{F} = 1, \tag{2.1}$$

where the non-uniform stretch ratio $\lambda(\mathbf{X},t)$, depending on the position $\mathbf{X}$ of a particle in the reference configuration at time 0, is defined as

$$\lambda = V/V_0, \qquad \dot{\lambda} = V'\lambda, \tag{2.2}$$

where V and V_0 denote the variable axial velocity and the velocity at the exit, respectively, and the prime denotes derivative with respect to the axial coordinate z.

It has been shown in [3] that the quasi-elongational motion described by Eq. (2.1) is consistent with the definition of NUEM and enables determination of such quantities as the velocity gradient $\mathbf{L}(\mathbf{X},t)$, the left Cauchy-Green deformation tensor $\mathbf{B}(\mathbf{X},t)$ and the first Rivlin-Ericksen kinematic tensor $\mathbf{A}_1(\mathbf{X},t)$ (cf. [9]). For steady NUEM these quantities are independent of time and the constitutive equations of materially non-uniform, simple, locally isotropic media can be written in the following spatial form:

$$\mathbf{T}(z) = \mathbf{k}(V'(z),\ V(z),\ \varrho(z);\ z), \tag{2.3}$$

where $\mathbf{k}$ is a tensor function of scalar arguments: the velocity V, the velocity gradient V', the density ϱ, and the axial coordinate z. An explicit dependence of the stress tensor $\mathbf{T}$ on the coordinate z takes into account the material properties variable along the axis and caused by the corresponding temperature, crystallization, orientation, solidification, etc. effects (cf. [6, 3]).

For axisymmetric, quasi-elongational motions Eq. (2.3) leads to

$$\begin{aligned} T^{11} &= T^{22} = \sigma_1(V', V, \varrho; z), \\ T^{33} &= \sigma_3(V', V, \varrho; z), \\ T^{13} &= 0. \end{aligned} \tag{2.4}$$

Since for steady flows the pair of variables V, V' is equivalent to the pair ε, $\dot{\varepsilon}$ (ε denotes the Hencky strain), we may quote PETRIE'S [10] statement that "... the use of rate of strain and strain as independent variables, was the most plausible choice. The fact that this choice is appropriate for both extremes of material behaviour, the purely viscous and the purely elastic, lend weight to the argument."

If the inclination of filament surface is a small quantity, i.e. $R' = 0(\varepsilon)$, $\varepsilon = R_0/L \ll 1$, where R, R_0 and L denote the outer current radius $R(z)$, the radius at the exit and the total length, respectively, we consider the following small additional velocity field superposed on the fundamental motion characterized by the axial velocity V:

$$[w(r,z,t)] = \begin{bmatrix} u \\ 0 \\ w \end{bmatrix} = [0(\varepsilon)]. \tag{2.5}$$

The above additional velocity field depens on both coordinates r and z, on time t, and enables determination of all the linear increments necessary for further considerations (cf. [3]). We have, in particular,

$$\Delta\lambda = \frac{w}{V_0}, \qquad \Delta\lambda' = \frac{1}{V_0}\frac{\partial w}{\partial z}. \tag{2.6}$$

Under the assumption that, in the perturbed expressions for stresses, viz.

$$T^*(z) = T(z) + \Delta T(r,z,t), \tag{2.7}$$

only the increments ΔT may depend on time, we finally arrive at the following equations:

$$\begin{aligned}
T^{*11} &= \sigma_1 + \frac{\partial\sigma_1}{\partial V}w + \frac{\partial\sigma_1}{\partial V'}\frac{\partial w}{\partial z} + \frac{\partial\sigma}{\partial\varrho}\Delta\varrho + \alpha\frac{\partial u}{\partial r}, \\
T^{*22} &= \sigma_1 + \frac{\partial\sigma_1}{\partial V}w + \frac{\partial\sigma_1}{\partial V'}\frac{\partial w}{\partial z} + \frac{\partial\sigma}{\partial\varrho}\Delta\varrho + \beta\frac{u}{r}, \\
T^{*33} &= \sigma_3 + \frac{\partial\sigma_3}{\partial V}w + \frac{\partial\sigma_3}{\partial V'}\frac{\partial w}{\partial z} + \frac{\partial\sigma}{\partial\varrho}\Delta\varrho, \\
T^{*13} &= \eta\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) + \gamma u,
\end{aligned} \tag{2.8}$$

where α, β, γ and η are new material functions depending on the same arguments as σ_1 and σ_3.

It has been proved in [3] that for steady fundamental motions for which, moreover, div.$V = 0$, the density ϱ is constant and $\Delta\varrho$ denotes the corresponding

increment depending on r and z. Similarly, from the global continuity condition, viz.

$$W = \varrho \pi R^2 V = \text{const}, \tag{2.9}$$

where W denotes the mass output, it results that for density increments

$$\int_0^R \Delta\varrho\, r\, dr = 0, \tag{2.10}$$

if the additional velocity distribution over the fibre cross-section is such that (see Sec. 4)

$$\int_0^R w\, r\, dr = 0. \tag{2.11}$$

In what follows, we assume validity of the so-called thin-thread approximation (cf. [3]) introducing the following dimensionless quantities marked with broken dashes

$$u = \varepsilon U \hat{u}, \qquad w = U\hat{w}, \qquad r = R_0 \hat{r}, \qquad z = L\hat{z}, \tag{2.12}$$

where U is some characteristic velocity. Therefore, it is a consequence of the assumption (2.5) that $u = 0(\varepsilon^2)$, if $w = 0(\varepsilon)$.

3. Equations of dynamic equilibrium and boundary conditions

The axisymmetric equations of dynamic equilibrium can be written in the form:

$$\begin{aligned} &\frac{\partial T^{*11}}{\partial r} + \frac{1}{r}\left(T^{*11} - T^{*22}\right) + \frac{\partial T^{*13}}{\partial z} = \varrho\left(\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial z}\right), \\ &\frac{\partial T^{*13}}{\partial r} + \frac{1}{r}T^{*13} + \frac{\partial T^{*33}}{\partial z} = \varrho\left(\frac{\partial w}{\partial t} + VV' + V\frac{\partial w}{\partial z} + V'w\right). \end{aligned} \tag{3.1}$$

Introducing the stresses defined by Eq. (2.8), differentiating with respect to z and r, and subtracting the first Eq. (3.1) from the second one, we arrive, after integration with respect to r, at

$$\begin{aligned} \eta\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + r\frac{d}{dz}\left(\sigma + \frac{\partial\sigma}{\partial V}w + \frac{\partial\sigma}{\partial V'}\frac{\partial w}{\partial z} + \frac{\partial\sigma}{\partial\varrho}\Delta\varrho\right) \\ = C + \varrho\left(\frac{\partial w}{\partial t} + VV' + V\frac{\partial w}{\partial z} + V'w\right) - \frac{\gamma}{r}\frac{\partial}{\partial r}(ru), \end{aligned} \tag{3.2}$$

where C is an integration constant,

$$\sigma = T^{33} - T^{11} = \sigma_3 - \sigma_1 , \tag{3.3}$$

and only terms up to ε^2 have been retained. Here, the symbol d/dz denotes the total derivative with respect to z.

For the fibre-forming processes considered in the present paper it is usually assumed that at the exit (the feeding velocity V_0) and at the end (the take-up velocity V_L), the following boundary conditions are satisfied:

$$V(0) = V_0 \qquad \text{and} \qquad V(L) = V_L , \tag{3.4}$$

respectively. Since by assumption $R'(z) = 0(\varepsilon)$, it is reasonable to consider that the additional velocity field only modifies the uniform velocity profile of the fundamental motion. Thus, if the mass output W is constant along the fibre, we should apply the additional condition in the integral form (2.11).

On the free surface of the filament all the forces acting have to be mutually balanced. For small R' this leads to (cf. [3])

$$R' \left(T^{*33} - T^{*11}\right)_{r=R} = T^{*13}\Big|_{r=R} , \tag{3.5}$$

or alternatively to

$$R' \left[\sigma + \frac{\partial \sigma}{\partial V} w + \frac{\partial \sigma}{\partial V'} \frac{\partial w}{\partial z} + \frac{\partial \sigma}{\partial \varrho} \Delta \varrho\right]_{r=R} = \eta \left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right)_{r=R} + \gamma u\Big|_{r=R} , \tag{3.6}$$

where only terms of order ε^2 have been retained.

4. Approximations of governing equations

Assuming on the basis of Eq. (2.5) that the additional velocity field can be written as

$$w = \varepsilon w_1 + \varepsilon^2 w_2 \ldots , \qquad u = \varepsilon^2 u_1 + \varepsilon^3 u_2 \ldots , \tag{4.1}$$

Eq. (3.2) amounts to

$$\eta \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial w_1}{\partial r}\right) + \frac{d}{dz} \sigma = C_1 + \varrho \frac{\partial w_1}{\partial t} + \varrho V V' , \tag{4.2}$$

for terms of order ε, and to

$$\eta \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial w_2}{\partial r}\right) + \frac{d}{dz} \left(\frac{\partial \sigma}{\partial V} w_1 + \frac{\partial \sigma}{\partial V'} \frac{\partial w_1}{\partial z} + \frac{\partial \sigma}{\partial \varrho} \Delta \varrho\right) = C_2 + \varrho \left(\frac{\partial w_2}{\partial t} + V' w_1 + V \frac{\partial w_1}{\partial z}\right) - \gamma \frac{\partial}{\partial r} (r u_1), \tag{4.3}$$

for terms of order ε^2. C_1 and C_2 are the corresponding integration constants for the first and second approximation, respectively, and u_1 results from the continuity equation.

Being interested in dynamic (harmonic) solutions of the problems considered, we postulate that the total additional velocity field consists of steady part $\overline{w}$ and dynamic part $\widetilde{w}$, viz.

$$w_1 = \overline{w}_1 + \widetilde{w}_1 \exp \omega(t - t_0), \tag{4.4}$$

where the quantity ω may be complex, real or purely imaginary. Equation (4.4) implies that also

$$\begin{aligned} C_1 &= \overline{C}_1 + \widetilde{C}_1 \exp \omega(t - t_0), \\ R' &= \overline{R}' + \widetilde{R}' \exp \omega(t - t_0), \\ \Delta\varrho &= \Delta\overline{\varrho} + \Delta\widetilde{\varrho} \exp \omega(t - t_0), \end{aligned} \tag{4.5}$$

where $\overline{R}'$ denotes the surface inclination in a steady-state motion, and $\widetilde{R}$ is the amplitude of dynamic increment. Similar notations are used for the density increments $\Delta\overline{\varrho}$ and $\Delta\widetilde{\varrho}$.

Under the above assumptions the first approximation of the governing equations (4.2) leads to

$$\eta \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \overline{w}_1}{\partial r} \right) + \frac{d\sigma}{dz} + \varrho V V' = \overline{\overline{C}}_1, \tag{4.6}$$

for steady-state motions, and to

$$\eta \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \widetilde{w}_1}{\partial r} \right) - \varrho \omega \widetilde{w}_1 = \widetilde{C}_1 \tag{4.7}$$

for dynamic (harmonic) motions.

Of course, the condition (2.11) takes the forms:

$$\int_0^{\overline{R}} \overline{w}\, r\, dr = 0, \qquad \int_0^{\overline{R}} \widetilde{w}\, r\, dr = 0. \tag{4.8}$$

Similarly, the boundary condition (4.2), after taking into account Eqs. (4.4), (4.5), gives

$$\overline{R}' \sigma = \eta \left. \frac{\partial \overline{w}_1}{\partial r} \right|_{r=\overline{R}}, \tag{4.9}$$

for steady-state motions, and

$$\widetilde{R}' \sigma = \eta \left. \frac{\partial \widetilde{w}_1}{\partial r} \right|_{r=\overline{R}}, \tag{4.10}$$

for dynamic (harmonic) motions. It is worthwhile to remember that Eqs. (4.7) and (4.10) are valid only if the first harmonics are analysed.

5. Dynamic solutions and instability problems

The governing differential equation (4.6) together with the conditions $(4.8)_1$ and (4.9) can be integrated in a straightforward manner leading to the solution

$$\overline{w}_1 = \frac{\sigma}{2\eta}\frac{\overline{R}'}{R}\left(r^2 - \frac{\overline{R}^2}{2}\right), \tag{5.1}$$

describing steady-state motions. This is exactly the same result as that obtained previously in [3].

The governing differential equation (4.7) can easily be solved, leading to the expression

$$\widetilde{w}_1 = \widetilde{D}_1 I_0\left(\sqrt{\frac{\varrho\omega}{\eta}}r\right) - \frac{\widetilde{C}_1}{\varrho\omega}, \tag{5.2}$$

where I_0 denotes the modified Bessel function of the first kind and order 0, and $\widetilde{D}_1$ is a new integration constant. Taking into account the conditions $(4.8)_2$ and (4.10), we arrive at

$$\widetilde{C}_1 = 2\sigma\frac{\widetilde{R}'}{\overline{R}}, \qquad \widetilde{D}_1 = \frac{\sigma\widetilde{R}'}{\eta\sqrt{\frac{\varrho\omega}{\lambda}}I_1\left(\sqrt{\frac{\varrho\omega}{\eta}}\right)}, \tag{5.3}$$

and finally at

$$w_1 = 2\widetilde{R}'\overline{R}\frac{\sigma}{\eta}\left[\frac{1}{2x}\frac{I_0\left(x\frac{r}{\overline{R}}\right)}{I_1(x)} - \frac{1}{x^2}\right], \qquad x = \sqrt{\frac{\varrho\omega}{\eta}}\overline{R}, \tag{5.4}$$

where I_1 denotes the modified Bessel function of the first kind and order 1.

Since, in general, the quantity ω is complex, viz.

$$\omega = \omega_1 - i\omega_2, \tag{5.5}$$

where both ω_1 and ω_1 are real (the minus sign is a matter of convention), the result (5.2) can also be presented as

$$\widetilde{w}_1 = \widetilde{D}_2 J_0(M)e^{i\varphi} - \frac{\widetilde{C}_1}{\varrho(\omega_1 - i\omega_2)}, \tag{5.6}$$

where J_0 denotes the ordinary Bessel function of the first kind and order 0. The corresponding modulus M and the argument φ are defined as follows:

$$M^2 = \sqrt{\omega_1^2 + \omega_2^2}\frac{\varrho r^2}{\eta}, \tag{5.7}$$

$$\operatorname{tg}^2\varphi = \left(\pm\sqrt{\omega_1^2 + \omega_2^2} + \omega_1\right) / \left(\pm\sqrt{\omega_1^2 + \omega_2^2} - \omega_1\right). \tag{5.8}$$

In the particular case of $\omega_1 \equiv 0$, the solution (5.4) can easily be presented by the real and imaginary parts, viz.

$$\tilde{w}_1 = \mathrm{Re}\,\tilde{w}_1 + i\mathrm{Im}\,\tilde{w}_1, \tag{5.9}$$

where

$$\mathrm{Re}\,\tilde{w}_1 = \frac{\sigma}{\varrho\omega_2}\frac{\tilde{R}'}{\overline{R}}\frac{J_0(M)}{J_1(M(\overline{R}))}, \tag{5.10}$$

$$\mathrm{Im}\,\tilde{w}_1 = -2\frac{\sigma}{\varrho\omega_2}\frac{\tilde{R}'}{\overline{R}}\left[\frac{1}{2}\frac{J_0(M)}{J_1(M(\overline{R}))}\sqrt{\frac{\varrho\omega_2}{2\eta}}\overline{R} - 1\right], \tag{5.11}$$

and according to Eqs. (5.7), (5.8),

$$M^2(\overline{R}) = \frac{\varrho\omega_2}{\eta}\overline{R}^2, \qquad \mathrm{tg}^2\,\varphi = 1. \tag{5.12}$$

Although the above solutions resemble those widely discussed in [3], it should be emphasized, however, that they have different meanings and are based on different model assumptions.

In the case of harmonic disturbances in the form (4.4) and (4.5), the motion is said to be unstable, if there exists at least one solution for which the amplitudes limitlessly increase in time. If we assume that ω is written in the form (5.5), the sufficient condition of instability can be expressed as

$$\mathrm{Re}\,\omega = \omega_1 > 0. \tag{5.13}$$

We have the case of "neutral" stability, if $\mathrm{Re}\omega = 0$, passing from negative to positive values. Depending on what value is taken by $\mathrm{Im}\,\omega$, two types of the stability loss may be observed: the stationary type when $\omega_2 = 0$, and the oscillatory type when $\omega_2 \neq 0$.

In the motions discussed all the types of instability may occur independently, and "the principle of exchange of stabilities" (cf. [11]), valid for purely viscous fluids, cannot be proved in general. Thus, we discuss separately the following cases.

1. *The case of stationary instability*

If $\omega_2 = 0$ and $\omega_1 \geq 0$, the particular dynamic solution takes the form:

$$\tilde{w} = \tilde{w}_1 \exp\omega(t - t_0), \tag{5.14}$$

where the additional velocity amplitude $\tilde{w}_1$ is determined by Eq. (5.4) with ω replaced by ω_1. Since for finite x the quantity ω_1 is also finite, the result (5.14) implies that any small disturbances imposed on the additional velocity field either will be preserved ($\omega_1 = 0$) or will increase limitlessly in time (for $\omega_1 > 0$). In

other words, this means that various types of spots, neckings and corrugations occuring along the filament either will be preserved or will grow limitlessly in time.

2. *The case of oscillatory instability*

If $\omega_1 = 0$ and $\omega_2 \neq 0$, the solution (5.14) is valid with ω replaced by $-i\omega_2$, where the real and imaginary parts of the additional velocity amplitude $\tilde{w}_1$ are given by Eqs. (5.10), (5.11).

Assuming that only the real part of Eq. (5.14) has physical meaning, we obtain the implication:

$$\text{Im}\,\tilde{w} = 0 \quad \Rightarrow \quad \sin\omega_2(t-t_0) = \frac{\text{Im}\,\tilde{w}_1}{\text{Re}\,\tilde{w}_1}\cos\omega_2(t-t_0) \tag{5.15}$$

and finally

$$\text{Re}\,\tilde{w} = \left(\text{Re}\,\tilde{w}_1 + \frac{(\text{Im}\,\tilde{w}_1)^2}{\text{Re}\,\tilde{w}_1}\right)\cos\omega_2(t-t_0) = 2\frac{\sigma}{\varrho\omega_2}\frac{\tilde{R}'}{\overline{R}}\left(p + \frac{(p-1)^2}{p}\right)\cos\omega_2(t-t_0), \tag{5.16}$$

where

$$p = \frac{1}{2}\frac{J_0(M)}{J_1(M(\overline{R}))}\sqrt{\frac{\varrho\omega_2}{2\eta}}\,\overline{R}^2, \tag{5.17}$$

and M, $M(\overline{R})$ have been defined through Eqs. (5.7), $(5.12)_1$, respectively.

It is seen by inspection from Eq. (5.16) that the real part of the additional velocity $\text{Re}\,\tilde{w}$ tends to infinity, if the quantity p, defined in Eq. (5.17), tends to infinity. This is the case, if

$$J_1(M(\overline{R})) = 0, \tag{5.18}$$

i.e. for $M(\overline{R}) = 3.83$, 7.02, 10.17, etc. The above result means that any small oscillatory disturbances imposed on the additional velocity field will increase limitlessly only for the frequencies determined from

$$\omega_2 = \frac{M^2(\overline{R})\eta\lambda}{\varrho R_0^2} = \frac{\pi M^2(\overline{R})V_0}{W}\eta(z)\lambda(z), \tag{5.19}$$

where W denotes the constant mass ouput and $\lambda(z) = V(z)/V_0$ is the stretch ratio at the position z along the filament. In other words, this means that various oscillatory disturbances may grow limitlessly only for particular frequencies. For given values of the stretch ratio λ, the viscosity η etc., the least value of the frequency ω_2 results from Eq. (5.19) for $M(\overline{R}) = 3.83$.

Similar results can be obtained from the analysis of the radial components of additional velocities $\tilde{u}$. To this end, we introduce Eq. (5.2) into the local continuity condition expressed by the corresponding increments (cf. [3]):

$$\operatorname{div} \tilde{w}_1 = \frac{1}{r}\frac{\partial}{\partial r}(r\tilde{u}_1) + \frac{\partial \tilde{w}_1}{\partial z} = -\frac{V}{\varrho}\frac{\partial \Delta\tilde{\varrho}}{\partial z}. \tag{5.20}$$

Such a procedure leads to the equation for $\tilde{u}_1$, the solution of which, after integration with respect to r, and taking into account the conditions $(4.8)_2$ and (4.10), amounts to

$$\tilde{u}_1 = -\frac{d}{dz}\left\{\frac{\sigma\tilde{R}'}{\varrho\omega}\left[\frac{I_1\left(\sqrt{\frac{\varrho\omega}{\eta}}\,r\right)}{I_1\left(\sqrt{\frac{\varrho\omega}{\eta}}\,\overline{R}\right)} - \frac{r}{\overline{R}}\right]\right\} - \frac{1}{r}\frac{V}{\varrho}\frac{\partial}{\partial z}\int_0^r \Delta\tilde{\varrho} r\, dr. \tag{5.21}$$

We obtain, moreover, the following results:

$$\tilde{u}_1\Big|_{r=0} = 0, \qquad \tilde{u}_1\Big|_{r=\overline{R}} = 0, \tag{5.22}$$

if Eq. (2.10) is satified for $R = \overline{R}$ and $\Delta\varrho = \Delta\tilde{\varrho}$. Presentation of the above solution in forms similar to (5.9), (5.10) and (5.11) immediately leads to the conditions (5.18) or (5.19).

6. Conclusions and comparisons

The results obtained in the paper enable formulation of the following conclusions.

1. The dynamic solutions discussed in the paper essentially depend on the material properties of the non-uniform medium: the normal stress difference σ and the viscosity function η. For purely viscous fluids this dependence can be expressed by the kinematic quantities: the axial velocity V and its axial gradient V'. A change of material behaviour from a fluid-like to a solid-like (freezing process) can also be taken into account.

2. As a consequence of the first conclusion, the amplitudes of disturbances depend on the material properties, while their limitless growth does not. In particular, the stationary type of stability loss (neckings, corrugations etc.) is always possible for arbitrarily small disturbances distributed in any way along the filament axis.

3. Contrary to the previous well-known studies (cf.[4, 5, 6]), it results from our considerations that the hydrodynamic instability or the draw-resonance are not the only reasons for the appearance of irregular fibres, or for the breakage of the spinning line. This point of view is consistent with that expressed by ZIABICKI [6].

4. The oscillatory type of instability is possible only for certain frequencies of disturbances, depending on the viscosity $\eta(z)$ and velocity $V(z)$ distributions along the filament. The least value of frequency amounts to

$$\omega_{\min} = \frac{\pi \cdot 14.67}{W} \eta(z) V(z), \tag{6.1}$$

where W is a constant mass output.

5. The above result is similar to that discussed in the previous references [4, 5], in the sense that the oscillatory type of instability arise when the draw ratio $\mathrm{Dr} = V_L/V_0$ is greater than some critical value (e.g. $\mathrm{Dr} = 20.2$ for isothermal Newtonian flow) determined on the basis of the neutral stability curves depending on certain dimensionless groups. In our case the role of various dimensionless groups (e.g. Stanton number, Reynolds number, etc.) is replaced by the axial variability of $\eta(z)$ and $V(z)$ for any material described by the assumed pretty general constitutive equation (2.3).

When making various comparisons of the present results with those from other references, one has to bear in mind the fundamental differences between the compared approaches to instability problems. In our considerations we assumed, in particular, that:

- the additional disturbances as well as steady-state solutions depend on the radius r (or R), not only on the axial coordinate z;
- the boundary conditions at both ends of the fibre are quasi-homogeneous, in the sense that for additional velocity they are satified in an integral form;
- the additional disturbances of the stationary and oscillatory types are finite as compared with the additional steady-state velocity fields, responsible for realistic variability of the fibre geometry.

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