Arch. Mech., 50, 3, pp. 405–413, Warszawa 1998
FIFTY YEARS OF THE ARCHIVES OF MECHANICS

Nambu – Poisson dynamics

Dedicated to Prof. Henryk Zorski
on the occasion of his 70-th birthday

R. IBÁÑEZ (BILBAO), M. de LEÓN (MADRID),
J.C. MARRERO (LA LAGUNA) and D.M. de DIEGO (MADRID)

SEVERAL PHYSICAL systems can be described by using multibrackets instead of the usual Poisson or Jacobi brackets. Starting with the original construction of Nambu we give a brief review on recent results on multibrackets on manifolds.

1. Classical Hamiltonian mechanics

THE PHASE-SPACE in classical Hamiltonian mechanics [1, 2, 3, 4] is the cotangent bundle T^*M of the configuration space M which is provided with a canonical symplectic structure ω. If we consider local coordinates $(q^1, \cdots, q^n, p_1, \cdots, p_n)$ on T^*M (q^i are the position coordinates of M and p_i the momentum coordinates), we have

$$\omega = \sum_{i=1}^{n} dq^i \wedge dp_i \,,$$

and the associated *canonical Poisson bracket* (which is non-degenerate) is given by

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right).$$

Hamilton's equations of motion are the first order differential equations

$$\frac{dq^i}{dt} = \{H, q^i\} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = \{H, p_i\} = -\frac{\partial H}{\partial q^i}, \tag{1.1}$$

where H is the Hamiltonian energy of the mechanical system. The Hamilton equations of motion can be expressed in a global way as

$$\frac{df}{dt} = \{H, f\}. \tag{1.2}$$

However, the study of some mechanical systems, particularly systems with symmetries or constraints, may lead to more general Poisson brackets (degenerate brackets).

A *Poisson bracket* [19, 5, 6, 7] on a smooth manifold M is a bilinear operation $\{\,,\}$ on $C^\infty(M,\mathbb{R})$ satisfying the following properties:
i) $\{f,g\} = -\{g,f\}$ (skew-symmetry),
ii) $\{f,gh\} = \{f,g\}h + g\{f,h\}$ (Leibniz rule),
iii) $\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0$ (Jacobi's identity),
for all $f,g,h \in C^\infty(M,\mathbb{R})$.

Given a function $H \in C^\infty(M,\mathbb{R})$, we associate a vector field X_H (called the Hamiltonian vector field) defined by

$$X_H(g) = \{H,g\}.$$

Thus, the solutions of the Hamilton's equations of motion (1.2) are just the integral curves of the Hamiltonian vector field X_H.

EXAMPLE 1. Newton's second law

The Newton's second law states that a particle of mass $m > 0$ moving under the influence of a potential $V(q)$, $q = (q^1,q^2,q^3) \in \mathbb{R}^3$, moves along a curve $q(t)$ in $\mathbb{R}^3$ in such a way that

$$F = ma, \tag{1.3}$$

where $a = d^2q/dt^2$ is the acceleration and $F(q) = -\operatorname{grad} V(q)$ is the force acting on the particle (the force field is conservative).

If we introduce the momemtum coordinates $p_i = m(dq^i/dt)$, then the phase-space is $\mathbb{R}^6$ with coordinates $(q^1,q^2,q^3,p_1,p_2,p_3)$ and canonical Poisson bracket. Therefore the Newton second law (1.3) is equivalent to the Hamilton equations (1.1) with respect to the total energy of the system $H(q,p) = K(p)+V(q)$, where $K(p) = \left(\frac{1}{2m}\right)\|p\|^2$ is the kinetic energy.

EXAMPLE 2. Rigid body

We will consider the motion of a free rigid body around a fixed point [3]. The Euler equations of rigid body dynamics in the absence of external forces are usually written as follows:

$$\begin{aligned} I_1\frac{d\Omega_1}{dt} &= (I_2 - I_3)\Omega_2\Omega_3\,, \\ I_2\frac{d\Omega_2}{dt} &= (I_3 - I_1)\Omega_3\Omega_1\,, \\ I_3\frac{d\Omega_3}{dt} &= (I_1 - I_2)\Omega_1\Omega_2\,, \end{aligned} \tag{1.4}$$

where $\Omega = (\Omega_1,\Omega_2,\Omega_3)$ is the body angular velocity vector and I_1,I_2,I_3 are the moments of inertia.

To see the Hamiltonian structure of the rigid body equations, one can use the description in terms of the Euler angles θ,ψ,ϕ and their conjugate momenta p_θ,p_ψ,p_ϕ (the configuration space is $SO(3)$ and the phase-space is $T^*(SO(3))$), relative to which the equations are in canonical Hamiltonian form (1.1). However, this procedure requires using six equations instead of the three equations (1.4).

We introduce the angular momenta $\Pi = I\Omega \in so(3)^* \cong \mathbb{R}^3$, that is, $\Pi_i = I_i\Omega_i$ $(i = 1, 2, 3)$, so that Eqs. (1.4) become

$$\begin{aligned} \frac{d\Pi_1}{dt} &= \frac{I_2 - I_3}{I_2 I_3}\Pi_2\Pi_3\,, \\ \frac{d\Pi_2}{dt} &= \frac{I_3 - I_1}{I_3 I_1}\Pi_3\Pi_1\,, \\ \frac{d\Pi_3}{dt} &= \frac{I_1 - I_2}{I_1 I_2}\Pi_1\Pi_2\,, \end{aligned} \tag{1.5}$$

that is, $d\Pi/dt = \Pi \times \Omega$.

Introduce the following Poisson bracket on functions of the Π:

$$\{F, G\}(\Pi) = \Pi\cdot(\nabla F(\Pi) \times \nabla G(\Pi)),$$

where ∇f is the gradient of the function f. Notice that this bracket is the Lie-Poisson bracket of the dual $so(3)^*$ of the Lie algebra $so(3)$. If the Hamiltonian H is

$$H(\Pi) = \frac{1}{2}\left(\frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3}\right),$$

then Eqs. (1.5) are the Lie – Poisson equations. Finally, we notice that the kinetic energy and the total angular momentum

$$C(\Pi) = \frac{1}{2}(\Pi_1^2 + \Pi_2^2 + \Pi_3^2), \tag{1.6}$$

are integrals of motion of the Hamiltonian vector field X_H.

2. Nambu mechanics

In 1973 Y. NAMBU [8] proposed a generalization of classical Hamiltonian mechanics to a Hamiltonian system defined on a 3-dimensional phase-space with respect to a ternary Poisson bracket and two Hamiltonian functions.

Nambu considered the 3-dimensional phase-space $\mathbb{R}^3$ with coordinates x_1, x_2, x_3 and the canonical Nambu bracket defined for three arbitrary functions f_1, f_2, f_3 by

$$\{f_1, f_2, f_3\} = \frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)}.$$

Then the *Nambu equations* of motion are given by

$$\frac{dx_1}{dt} = \frac{\partial(H_1, H_2)}{\partial(x_2, x_3)}, \qquad \frac{dx_2}{dt} = \frac{\partial(H_1, H_2)}{\partial(x_3, x_1)}, \qquad \frac{dx_3}{dt} = \frac{\partial(H_1, H_2)}{\partial(x_1, x_2)}, \tag{2.1}$$

or in a more general way

$$\frac{df}{dt} = \{H_1, H_2, f\}, \tag{2.2}$$

where H_1, H_2 are two Hamiltonian functions on $\mathbb{R}^3$.

Now, the solutions of Eqs. (2.2) are the integral curves of the Hamiltonian vector field $X_{H_1H_2}$, where $X_{H_1H_2}(g) = \{H_1, H_2, g\}$ for $g \in C^\infty(\mathbb{R}^3, \mathbb{R})$.

Nambu posed the following question: Are there real physical systems which may be described in this way?

The example considered by himself was the following.

EXAMPLE 3. Rigid body II

Nambu observed [8] that Eqs. (1.5) are nothing else but the Nambu equations (2.2) with respect to the canonical Nambu bracket for the coordinates Π_1, Π_2, Π_3 and the two Hamiltonians

$$H_1 = \frac{1}{2}(\Pi_1^2 + \Pi_2^2 + \Pi_3^2),$$
$$H_2 = \frac{1}{2}\left(\frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3}\right),$$

that is, total angular momemtum and the kinetic energy.

EXAMPLE 4. Static $SU(2)$-monopoles

The Nahm's system in the theory of static $SU(2)$-monopoles [9, 10] is given by the following equations of motion:

$$\frac{dx_1}{dt} = x_2x_3, \qquad \frac{dx_2}{dt} = x_1x_3, \qquad \frac{dx_3}{dt} = x_1x_2\,.$$

The above equations can be written in Nambu form (2.1), where

$$H_1 = \frac{1}{2}(x_1^2 - x_2^2), \qquad H_2 = \frac{1}{2}(x_1^2 - x_3^2).$$

Other examples of Nambu dynamical systems are the $SU(n)$-isotropic harmonic oscillator and the $SO(4)$-Kepler problem studied in [11] or the rigid body with a single torque about a major axis [12], among others.

3. Nambu – Poisson and generalized Poisson manifolds

After the publication of NAMBU'S paper [8], Nambu mechanics has been discussed by many authors, but it was almost forgotten for many years. A recent paper by TAKHTAJAN [10] gave a new interest to this subject by introducing a geometrical setting for Nambu brackets. He considered brackets of n functions satisfying a generalization of the Jacobi identity, the so-called fundamental identity. More recently, DE AZCÁRRAGA, PEREMOLOV and PÉREZ BUENO [13, 14, 15] have introduced an alternative generalization of the Jacobi identity, the so-called generalized Jacobi identity. Both kinds of multibrackets are natural generalizations of the ordinary Poisson brackets, and the fundamental and generalized Jacobi identities are the corresponding integrability conditions which extend the Jacobi identity.

In order to give a unified setting, we have introduced in [16, 17, 18] the following definition.

A *generalized almost Poisson bracket* of order n on a smooth manifold M is an n-linear mapping $\{\, ,\dots ,\, \} : C^\infty(M,\mathbb{R}) \times \cdots \times C^\infty(M,\mathbb{R}) \longrightarrow C^\infty(M,\mathbb{R})$ satisfying the following properties:

1) $\{f_1,\dots,f_n\} = (-1)^{\epsilon(\sigma)}\{f_{\sigma(1)},\dots,f_{\sigma(n)}\}$, (*skew-symmetry*)

2) $\{f_1 g_1,\dots,f_n\} = f_1\{g_1,\dots,f_n\} + g_1\{f_1,\dots,f_n\}$, (*Leibniz rule*)

for all $f_1,\dots,f_n, g_1 \in C^\infty(M,\mathbb{R})$ and $\sigma \in \mathrm{Symm}\,(n)$, where $\mathrm{Symm}\,(n)$ is a symmetric group of n elements and $\epsilon(\sigma)$ is the parity of σ.

An alternative way to define an n-bracket of functions is to consider the skew-symmetric tensor Λ of type $(n,0)$ given by

$$\Lambda_x(df_1(x),\dots,df_n(x)) = \{f_1,\dots,f_n\}(x)\ ,$$

for all $f_1,\dots,f_n \in C^\infty(M,\mathbb{R})$ and $x \in M$. A manifold M with such a structure is called *generalized almost Poisson manifold* of order n [16].

In local coordinates $(x_1,\cdots,x_m)$ on M, the tensor Λ can be written as follows:

$$\Lambda = \frac{1}{n!}\sum_{i_1,\cdots,i_n=1}^{m} c_{i_1\cdots i_n}\frac{\partial}{\partial x_{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x_{i_n}}, \tag{3.1}$$

where the functions $c_{i_1\cdots i_n} = \{x_{i_1},\cdots,x_{i_n}\} = \Lambda(x_{i_1},\cdots,x_{i_n})$ are skew-symmetric.

With $n-1$ functions $H_1,\cdots,H_{n-1} \in C^\infty(M,\mathbb{R})$, we associate a vector field $X_{H_1\cdots H_{n-1}}$ called the *Hamiltonian vector field* and defined by

$$X_{H_1\cdots H_{n-1}}(g) = \{H_1,\cdots,H_{n-1},g\}, \qquad \text{for all} \quad g \in C^\infty(M,\mathbb{R}).$$

In addition, we can consider an integrability condition. A generalized almost Poisson manifold (M,Λ) of order n is said to be

- *Nambu–Poisson* [10, 16] if it satisfies the *fundamental identity*

$$\{f_1,\dots,f_{n-1},\{g_1,\dots,g_n\}\} = \sum_{i=1}^{n}\{g_1,\dots,\{f_1,\dots,f_{n-1},g_i\},\dots,g_n\}\,, \tag{3.2}$$

for all $f_1,\dots,f_{n-1},g_1,\dots,g_n$ on M;

- *generalized Poisson manifold* [13, 16] if n is even and it satisfies the *generalized Jacobi identity*

$$\mathrm{Alt}\,(\{f_1,\dots,f_{n-1},\{g_1,\dots,g_n\}\}) = 0, \tag{3.3}$$

for all $f_1,\dots,f_{n-1},g_1,\dots,g_n$ on M.

We notice that the generalized Jacobi identity is equivalent to the condition $[\Lambda, \Lambda] = 0$, where $[\ ,\ ]$ is the Schouten–Nijenhuis bracket [5].

REMARK. If in the definition of a Poisson bracket we only assume that the bilinear operator $\{\ ,\ \}$ is of local nature, we obtain a Jacobi bracket (see [20, 21]). The corresponding generalizations to multibrackets have been recently discussed in [22, 23, 24].

EXAMPLE 5. Volume form

Let M be an oriented n-dimensional manifold and v_M a volume form. Given n functions $f_1, \dots, f_n$ on M, we define a Nambu–Poisson bracket by the formula [25]

$$df_1 \wedge \cdots \wedge df_n = \{f_1, \dots, f_n\} v_M \,.$$

Notice that the associated Nambu–Poisson tensor Λ is non-zero everywhere: $\Lambda(x) \neq 0$, for all $x \in M$.

Conversely, each Nambu–Poisson tensor $\Lambda \neq 0$ of order n on a smooth manifold M of dimension n follows from a volume form [16]. If we take $M = \mathbb{R}^n$, and v_M is the standard volume form $v_M = dx_1 \wedge \cdots \wedge dx_n$, we recover the example originally discussed by Nambu [8].

The Nambu–Poisson structure associated with a volume form (for $n \geq 3$) can be considered as the analogue of the symplectic structure in Poisson geometry (see [16]).

EXAMPLE 6. Constant Nambu–Poisson structures

A generalized almost Poisson structure of order n on $\mathbb{R}^m$ is given by the expression (3.1) and this structure is constant if the coefficients $c_{i_1 \cdots i_n} \in \mathbb{R}$. GAUTHERON proved in [25] that the constant Nambu–Poisson structures of order n on $\mathbb{R}^m$ are just the decomposable n-vectors. Notice that, in contrast, all the constant generalized almost Poisson structures are trivially generalized Poisson structures.

4. Foliation of Nambu–Poisson manifolds

As is well known, a Poisson manifold possesses a symplectic foliation [6, 5]. Given a Nambu–Poisson manifold (M, Λ) of order $n \geq 3$, if for each point $x \in M$ we consider the subspace D_x of T_xM spanned by the Hamiltonian vector fields $X_{f_1 \cdots f_{n-1}}$ evaluated at x, then we obtain a generalized distribution D on M (called *characteristic distribution*). By using the results in [25] we have proved in [16] that the characteristic distribution D is completely integrable and therefore, it defines a foliation on M such that the restriction of Λ to each leaf defines an induced Nambu–Poisson structure. There are two kinds of leaves:

i) for $x \in M$ such that $\Lambda(x) \neq 0$, the leaf passing through x has dimension n, and the induced Nambu–Poisson structure on it comes from a volume form;

ii) for $x \in M$ such that $\Lambda(x) = 0$, the leaf passing through x reduces to the point x, and the induced Nambu–Poisson structure is trivial.

5. Local structure of Nambu–Poisson manifolds

The local structure of a Poisson manifold was elucidated in [6]. For a Nambu-Poisson manifold was proved a sort of Darboux theorem (see [25, 16]):

Let (M, Λ) be a Nambu-Poisson manifold of order $n \geq 3$. Then around each point $x \in M$ such that $\Lambda(x) \neq 0$, there exist local coordinates $(x_1, \cdots, x_n, x_{n+1}, \cdots, x_m)$ such that

$$\Lambda = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}.$$

This result shows that the Nambu-Poisson manifolds are extremely rigid.

A consequence of this result is that every Nambu-Poisson manifold of even order is generalized Poisson, but the converse does not hold, as we have observed in Example 6.

6. Other properties of Nambu-Poisson manifolds

Let (M, Λ) be a Nambu-Poisson manifold of order n, then:

i. A function $f \in C^\infty(M, \mathbb{R})$ is an *integral of motion* of a Nambu system with Hamiltonians $H_1, \cdots, H_{n-1}$ if $\{H_1, \cdots, H_{n-1}, f\} = 0$. By using the fundamental identity, one can prove that the Nambu-Poisson bracket of n integrals of motion is again an integral of motion.

ii. If $f \in C^\infty(M, \mathbb{R})$ then the tensor $\Lambda_f = i(df)\Lambda$ is also a Nambu-Poisson tensor of order $n - 1$ on M. In general, $\Lambda_{f_1 \cdots f_r}$ is a Nambu-Poisson tensor of order $n - r$, for r functions $f_1, \cdots, f_r \in C^\infty(M, \mathbb{R})$.

iii. Every Hamiltonian vector field $X_{F_1 \cdots F_{n-1}}$ is an infinitesimal automorphism of Λ, that is,

$$\mathcal{L}_{X_{F_1 \cdots F_{n-1}}} \Lambda = 0.$$

Then its flow consists of Nambu-Poisson morphisms (see [16]).

EXAMPLE 7. Completely integrable systems

Let (M, ω, H) be a completely integrable Hamiltonian system, that is, M is a $2n$-dimensional symplectic manifold with symplectic form ω such that there exist $f_1, \ldots, f_n$ independent functions pairwise in involution, i.e., $\{f_i, f_j\} = 0$ for $i, j = 1, \ldots, n$. Then the associated Hamiltonian vector fields X_{f_i} commute, that is, $[X_{f_i}, X_{f_j}] = 0$. Thus, $\Lambda = X_{f_1} \wedge \ldots \wedge X_{f_n}$ is a Nambu-Poisson tensor on M of order n (see [17]).

Since the system is completely integrable, there exists (at least, locally) a family of independent conjugate functions $g_1, \ldots, g_n$, that is, we have

$$\{f_i, g_j\} = \delta_{ij}\,, \ \{g_i, g_j\} = 0.$$

A direct computation shows that

$$X_{g_1 \cdots g_{n-1}} = X_{f_n}\,.$$

Therefore, if the symplectic dynamical system is given by a Hamiltonian function $H = f_n$, then it is also a Nambu – Poisson dynamical system for the Hamiltonians $g_1, \dots, g_{n-1}$.

Example 8. Compatible Poisson structures

Let M be a differentiable manifold and Λ_1, Λ_2 two compatible Poisson structures, that is, $[\Lambda_1, \Lambda_2] = 0$; then $\Lambda_1 \wedge \Lambda_2$ is a generalized Poisson structure of order 4 (see [17]).

Acknowledgments

This work has been partially supported through grants DGICYT (Spain) Projects PB94-0106, PB94-0633-C02-02, Consejería de Educación y Cultura de la Comunidad de Madrid, Universidad de La Laguna, and UNED (Spain).

References

1. R. Abraham and J.E. Marsden, *Foundations of mechanics*, Addison-Wesley, 1978.
2. V.I. Arnold, *Mathematical methods of classical mechanics*, Springer-Verlag, New-York 1978.
3. J.E. Marsden and T.S. Ratiu, *Introduction to mechanics and symmetry*, Springer-Verlag, New-York 1994.
4. M. de León and P.R. Rodrigues, *Methods of differential geometry in analytical mechanics*, North-Holland, Amsterdam 1989.
5. I. Vaisman, *Lectures on the geometry of Poisson manifolds*, Birkhauser, Basel 1994.
6. A. Weinstein, *The local structure of Poisson manifolds*, J. Differential Geometry, **18**, 523–557 1983; Errata et addenda: J. Differential Geometry, **22**, 255, 1985.
7. P.J. Olver, *Applications of Lie groups to differential equations*, Graduate Texts in Math., 107, Springer-Verlag, 1986.
8. Y. Nambu, *Generalized Hamiltonian dynamics*, Physical Review D, **7**, 8, 2405–2412, 1973.
9. L. Takhtajan, *Modular forms as tau-function for certain integrable reductions of the self-dual Yang-Mills equations*, Preprint PAM no. **121**, Univ. Colorado, 1991.
10. L. Takhtajan, *On foundations of the generalized Nambu mechanics*, Comm. Math. Phys., **160**, 295–315, 1994.
11. R. Chatterjee, *Dynamical symmetries and Nambu mechanics*, Letters in Mathematical Physics, **36**, 117–126, 1996.
12. P. Guha, *Volume-preserving integrable systems and Nambu-Poisson manifolds*, Class. Quantum Grav., **14**, L37–L43, 1997.
13. J.A. de Azcárraga, A.M. Perelomov and J.C. Pérez Bueno, *New generalized Poisson structures*, J. Phys. A: Math. Gen., **29**, L151–157, 1996.
14. J.A. de Azcárraga, A.M. Perelomov and J.C. Pérez Bueno, *The Schouten-Nijenhuis bracket, cohomology and generalized Poisson structures*, J. Phys. A: Math. Gen., **29**, 7993–8009 1996.
15. J.A. de Azcárraga, J.M. Izquierdo and J.C. Pérez Bueno, *On the higher-order generalizations of Poisson structures*, J. Phys. A: Math. Gen., **30**, L607-616, 1997.

16. R. IBÁÑEZ, M. DE LEÓN, J.C. MARRERO and D. MARTÍN DE DIEGO, *Dynamics of generalized Poisson and Nambu–Poisson brackets*, J. Math. Phys., **38**, 5, 2332–2344, 1997.

17. R. IBÁÑEZ, M. DE LEÓN, J.C. MARRERO and D. MARTÍN DE DIEGO, *Reduction of generalized Poisson and Nambu–Poisson manifolds*, Reports on Math. Phys., **41**, 3, 325–344, 1998.

18. R. IBÁÑEZ, M. DE LEÓN and J.C. MARRERO, *Homology and cohomology on generalized Poisson manifolds*, J. Phys. A: Math. Gen., **31**, 1253–1266, 1998.

19. A. LICHNEROWICZ, *Les variétés de Poisson et les algébres de Lie associées*, J. Differential Geometry, **12**, 253–300, 1977.

20. A. LICHNEROWICZ, *Les variétés de Jacobi et leurs algébres de Lie associées*, J. Math. Pures et Appl., **57**, 453–488, 1978.

21. A. KIRILLOV, *Local Lie algebras*, Russian Math. Surveys, **31**, 4, 55–75, 1976.

22. G. MARMO, G. VILASI and A.M. VINOGRADOV, *The local structure of n-Poisson and n-Jacobi manifolds*, J. Geom. Phys., **25**, 141–182, 1998.

23. J.C. PÉREZ BUENO, *Generalized Jacobi structures*, J. Phys. A: Math. Gen., **30**, 6509–6515, 1997.

24. R. IBÁÑEZ, M. DE LEÓN, J.C. MARRERO and E. PADRÓN, *Nambu-Jacobi and generalized Jacobi manifolds*, J. Phys. A: Math. Gen., **31**, 1267–1286, 1998.

25. PH. GAUTHERON, *Some remarks concerning Nambu mechanics*, Letters in Mathematical Physics, **37**, 103–116, 1996.

DEPARTAMENTO DE MATEMÁTICAS,
FACULTAD DE CIENCIAS, UNIVERSIDAD DEL PAIS VASCO,
Apartado 644, 48080 Bilbao, SPAIN,
e-mail: mtpibtor@lg.ehu.es

INSTITUTO DE MATEMÁTICAS Y FÍSICA FUNDAMENTAL,
CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS,
Serrano 123, 28006 Madrid, SPAIN,
e-mail: mdeleon@fresno.csic.es

DEPARTAMENTO DE MATEMÁTICA FUNDAMENTAL,
FACULTAD DE MATEMÁTICAS, UNIVERSIDAD DE LA LAGUNA,
La Laguna, Tenerife, Canary Islands, SPAIN,
e-mail: jcmarrer@ull.es

DEPARTAMENTO DE ECONOMÍA APLICADA
CUANTITATIVA, FACULTAD DE CIENCIAS ECONÓMICAS Y EMPRESARIALES,
UNED, 28040 Madrid, SPAIN,
e-mail: dmartin@sr.uned.es

Received October 7, 1997.