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Internal variables in macrodynamics of two-dimensional periodic cellular media

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A NEW CONTINUUM MODEL for studying the plane strain problems in elastodynamics of a cellular medium having a periodic structure is proposed. The model is based on the concept of macro-internal variables, [17], being capable of describing structures of an arbitrary complex lay-out. The continuum model equations constitute a certain generalization of the plane Cosserat continuum equations coupled with the ordinary differential equations for internal variables. The results are applied to the analysis of free vibration and wave propagation problems. The physical correctness of the model proposed is shown by comparing the obtained solutions to the exact ones.

Notations

Subscripts i, j, k, l run over 1, 2 and are related to a Cartesian orthogonal coordinate system on the plane $0x_1x_2$. Superscripts $a, b, ...$ and $A, B, ...$ run over $1, ..., n$ and $1, ..., N$, respectively, where n is the number of nodes and N is the number of rods in a cell which is assumed to be representative for a periodic structure under consideration. Superscripts α, β run over $1, ..., \nu$ being related to the description of micro-oscillations occurring inside every cell. Summation convention holds for all aforementioned indices unless otherwise stated. Points on the plane $0x_1x_2$ are denoted by $\mathbf{x} = (x_1, x_2)$ and t is the time coordinate.

1. Introduction

THE OBJECTIVE OF THIS CONTRIBUTION is the formulation and application of a continuum model to study linearized elastodynamics for a cellular medium of an arbitrary periodic structure in $0x_1x_2$-plane. Examples of cross-sections for such media are shown in Fig. 1. The considerations are restricted to the plane strain problems.

It is assumed that the length dimensions of a representative cell of the periodic structure are small compared with the minimum characteristic length dimension of the whole medium, and that the mass distribution can be approximated by assigning a concentrated mass and an inertia moment to every nodal point of a periodic lattice. Hence the medium under consideration is represented by a certain plane periodic system of mutually interacting rigid joints. For mass-point systems, problems of this kind were studied in a series of papers which will be not discussed here. An overview of the results, related mainly to the wave propagation and vibration problems, can be found in the known book by BRILLOUIN and PARODI [2].

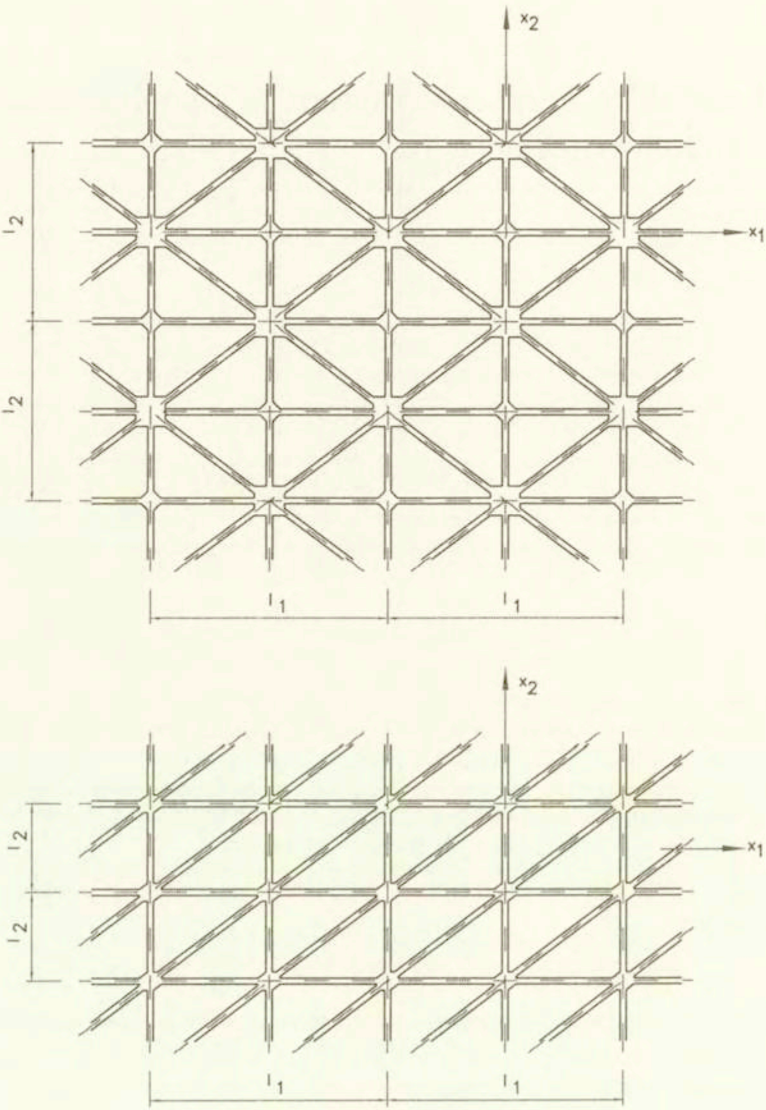

FIG. 1.

It is known that a direct approach to dynamics of periodic systems with a very large number of interacting mass-points or rigid bodies leads to computational difficulties due to a large number of ordinary differential equations describing the problem under consideration. That is why different averaged continuum models of periodic mass-point and rigid-body systems have been proposed in order to reduce the number of basic unknowns and to simplify the analysis of particular problems. From many results obtained in this manner, let us mention those related to periodic structures [5], cellular materials [6], perforated plates [7], and frame-type lattice structures, summarized in [15], where the list of references can be found. Main attention in [14, 15, 16] was focussed on investigation of engineering problems for framed structures (not only periodic) but the analysis was restricted to static problems. A more sophisticated modelling approach, based on the asymptotic procedures of the homogenization theory (for details see the recent book [8], and the references therein) can be found in a book [1]; a similar approach was also applied in [4].

It has to be emphasized that the asymptotic approaches to the formulation of continuum (homogenized) models for periodic structures neglect the effect of the unit cell size on the global behaviour of discrete medium, because in the course of modelling all length dimensions of every unit cell tend to zero (and number of the cells tends to infinity). On the other hand, the aforementioned effect plays an important role in many dynamic problems being responsible, for example, for dispersion of waves propagating across a periodic system made of interacting mass-points or rigid elements. We can mention here a series of papers [10 – 13] related to the analysis of hexagonal gridworks, where Rogula - Kunin's approach was applied, cf. [9].

An alternative nonasymptotic continuum model of periodic mass-point systems was formulated in [3]. This model was based on the concepts of refined macrodynamics of composite materials proposed in [17] and then developed in a series of papers (for references cf. [18]). The idea of refined macrodynamics lies in the description of microdynamic effects on the global composite body behaviour in terms of so-called macro-internal variables which are governed by a system of ordinary differential equations and hence do not enter boundary conditions [17, 18].

In this paper, the line of approach is similar to that leading to the refined macrodynamics of composites, but it has been modified in order to derive a continuum model of the plane problem for an arbitrary periodic cellular structure, with nodal joints as rigid elements interconnected by means of linear-elastic thin plates subjected to cylindrical bending. This model also describes a plane framed structure with rigid joints and linear-elastic beams. It is assumed that the length dimensions in $0x_1x_2$ plane of every rigid nodal element are negligibly small as compared with the spans of interconnecting plates or beams.

The present paper constitutes a generalization of the approach to dynamics of periodic trusses proposed in [3] where only axial forces in the interconnecting rods and concentrated masses at the nodal points of the system were taken into account. The aim of this contribution is to formulate a refined continuum model of the medium under consideration (i.e. a nonasymptotic model which describes the effect of size of the representative periodicity cell on the global body behaviour) which can be applied to the analysis of linear elastic plane cellular media with an arbitrary complex lay-out of the representative cell. The main feature of the model is its relatively simple analytical form given by the partial differential equations of the plane Cosserat continuum coupled with the system of ordinary differential equations involving only second order time derivatives of internal variables. An example is used to show that the proposed model yields physically correct solutions.

2. Preliminaries

Let $\Delta = (-0.5l_1, 0.5l_1) \times (-0.5l_2, 0.5l_2)$ represent a cell which is assumed as representative for of a periodic network on the plane $0x_1, x_2$, cf. Fig. 3. It means that Δ contains the representative structural element for the cellular or framed periodic medium. It has to be emphasized that the choice of this element is not unique and depends not only on the geometry of the network but also on the class of micro-motions we are to investigate using the model proposed in the sequel. It is assumed that the representative element is made of n rigid nodes N^a, $a = 1, ..., n$, interconnected by N linear-elastic homogeneous thin plates R^A, $A = 1, ..., N$ subjected to cylindrical bending in $0x_1x_2$-plane. For framed structures R^A are linear-elastic beams and it is assumed that $0x_1x_2$ is a symmetry plane, both for every beam and every rigid node treated as certain spatial (3-dimensional) elements. By a region occupied by a periodic medium under consideration we shall mean the plane region Ω obtained by a union of all repeated cells. Denoting by L the smallest characteristic length dimension of Ω and setting $l := \sqrt{(l_1)^2 + (l_2)^2}$, it will be assumed that $l/L \ll 1$. This is why l will be referred to as the microstructure length parameter of the medium. It has to be remembered that the periodic structure of the whole medium can be disturbed in the structural elements situated near the boundary $\partial\Omega$ of Ω. An example of a lattice with the representative cell Δ is shown in Fig. 3 but in general, no restrictions are imposed on the form of the lattice and a choice of a representative cell, provided that condition $l/L \ll 1$ holds.

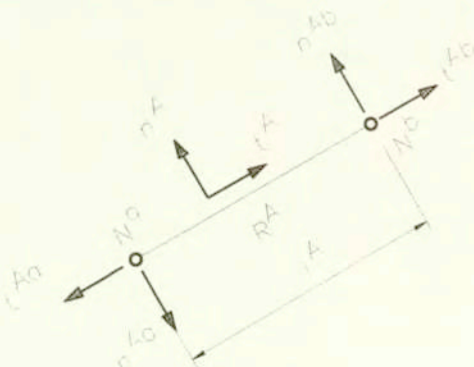

Fig. 2.

Significant properties of a thin plate or a beam R^A will be given by the flexural stiffness B^A in plane $0x_1x_2$, the axial stiffness D^A and the span l^A.

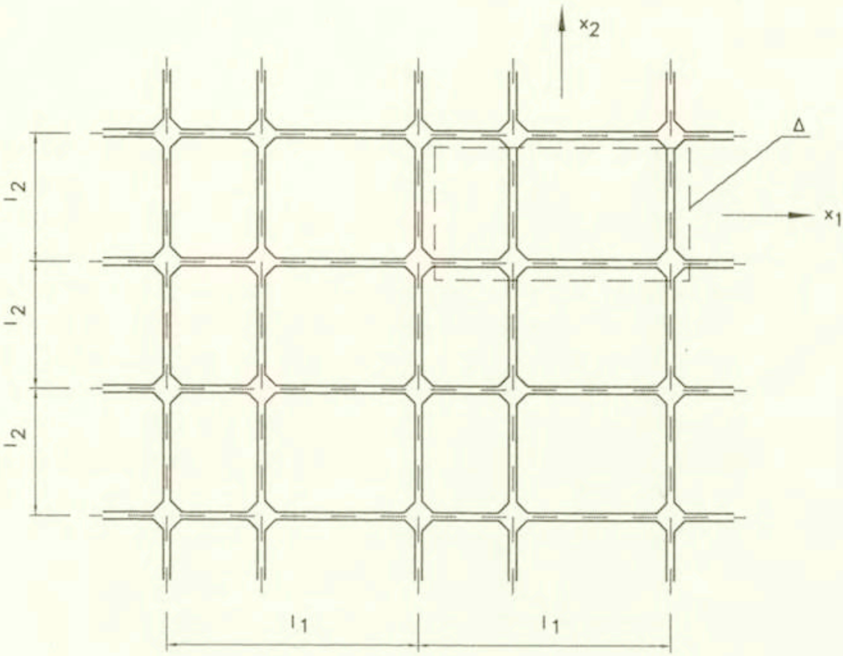

FIG. 3.

The concentrated mass and the rotational moment of inertia (related to the axis normal to the plane $0x_1x_2$) assigned to node N^a will be denoted by M^a, J^a, respectively. Orientation of the plate or the beam R^A interconnecting nodes N^a, N^b will be described by unit vectors $\mathbf{t}^A$, $\mathbf{n}^A$, $\mathbf{t}^{Aa}$, $\mathbf{n}^{Aa}$ shown in Fig. 2. We shall assume that if a plate or a beam R^A is not supported at the nodal point N^a then by definition $\mathbf{t}^{Aa} \equiv \mathbf{0}$, $\mathbf{n}^{Aa} \equiv \mathbf{0}$. Denoting by u_i^a, φ^a displacements and rotation of the node N^a, respectively, let us define

$$(2.1) \qquad \Delta_A u_i := \frac{u_i^b - u_i^a}{l^A}, \qquad \Delta_A \varphi := \varphi^b - \varphi^a, \qquad \varphi_A := \frac{\varphi^a + \varphi^b}{2}.$$

Let us also assume that every plate or beam R^A can be considered in the framework of the Kirchhoff plate theory or the Euler–Bernoulli beam theory, respectively. Then the strain components related to R^A can be taken in the form (no summation over A in formulae (2.2)–(2.4)!)

$$(2.2) \qquad \varepsilon^A := (\Delta_A u_i) t_i^A, \qquad \widetilde{\varepsilon}^A := (\Delta_A u_i) n_i^A - \varphi_A, \qquad \kappa^A := \Delta_A \varphi,$$

and at additional notations

$$(2.3) \qquad \Lambda^A := D^A l^A, \qquad \widetilde{\Lambda}^A := 12 B^A / l^A, \qquad K^A := B^A / l^A,$$

the strain energy σ^A assigned to plate or beam R^A is equal to

$$\sigma^A = \frac{1}{2}\Lambda^A(\varepsilon^A)^2 + \frac{1}{2}\widetilde{\Lambda}^A(\widetilde{\varepsilon}^A)^2 + K^A(\kappa^A)^2. \tag{2.4}$$

It has to be remembered that all the aforementioned denotations and formulae are related to an arbitrary but fixed structural element of the periodic lattice under consideration (possibly except some elements situated near boundary $\partial\Omega$ of Ω). Let us denote by $\mathcal{L}$ a set of all points on the plane $0x_1x_2$ which are centers of all disjoined cells constituting the region Ω. Then the displacement vector and rotation of the node N^a belonging to a cell with center $\mathbf{z}$, $\mathbf{z} \in \mathcal{L}$, at an arbitrary instant t, will be denoted by $\mathbf{u}^a(\mathbf{z},t)$, $\varphi^a(\mathbf{z},t)$, respectively. All external loads acting on the medium will be applied exclusively to the centers of the nodal joints. The resultant force and moment applied to the node N^a in a cell with a center $\mathbf{z} \in \mathcal{L}$ will be denoted by $\mathbf{f}^a(\mathbf{z},t)$ and $m^a(\mathbf{z},t)$, respectively. Introducing the action functional $\mathcal{A} = \mathcal{T} - \mathcal{K} - \mathcal{W}$, where

$$\begin{aligned}
\mathcal{T} &= \frac{1}{2}\sum_{\mathbf{z}\in\mathcal{L}}\sum_{A=1}^{N}\left[\Lambda^A(\varepsilon^A(\mathbf{z},t))^2 + \widetilde{\Lambda}^A(\widetilde{\varepsilon}^A(\mathbf{z},t))^2 + K^A(\kappa^A(\mathbf{z},t))^2\right],\\
\mathcal{K} &= \sum_{\mathbf{z}\in\mathcal{L}}\sum_{a=1}^{n}\left[M^a(\dot{\mathbf{u}}^a(\mathbf{z},t))^2 + J^a(\dot{\varphi}^a(\mathbf{z},t))^2\right],\\
\mathcal{W} &= \sum_{\mathbf{z}\in\mathcal{L}}\sum_{a=1}^{n}\left[\mathbf{f}^a(\mathbf{z},t)\mathbf{u}^a(\mathbf{z},t) + m^a(\mathbf{z},t)\varphi^a(\mathbf{z},t)\right],
\end{aligned} \tag{2.5}$$

and taking into account formulae (2.1), (2.2), from the principle of stationary action we derive equations of motion for $\mathbf{u}^a(\mathbf{z},t)$, $\varphi^a(\mathbf{z},t)$, $\mathbf{z} \in \mathcal{L}$, $a = 1,...,n$. These equations represent a discrete model of a periodic lattice-type structure but are not convenient in investigations of its global dynamic behaviour since the number of points of $\mathcal{L}$ is very large. That is why relations (2.1), (2.2), (2.5) together with assumptions formulated in Sec. 3 will be treated only as a basis for deriving a continuum model of the cellular medium under consideration.

3. Modelling assumptions

In order to formulate the modelling assumptions leading from the discrete model of the periodic medium under consideration to a certain refined (non-asymptotic) continuum model, first of all we have to choose a cell Δ and then to introduce two auxiliary concepts.

The first is the concept of a macro-function related to the choice of a cell Δ. Let $F(\cdot,t)$ be a real-valued function defined on Ω and depending on time t, the values of which, from the computational viewpoint, have to be calculated within the known error ε_F. For a given value of Δ, the microstructure length parameter

l is also known. Function $F(\cdot, t)$ will be called a macro-function (related to ε_F and l) if for every $\mathbf{x}, \mathbf{y} \in \Omega$ such that $\|\mathbf{x} - \mathbf{y}\| < l$ and for every t, condition $|F(\mathbf{x}, t) - F(\mathbf{y}, t)| < \varepsilon_F$ holds. Moreover, if $F(\cdot, t)$ is a differentiable function and similar conditions hold also for all derivatives of F (including time-derivatives), then F will be called a regular macro-function. In the sequel we tacitly assume that every (regular) macro-function F is related to certain parameters ε_F, l.

The second auxiliary concept is that of an oscillation-shape matrix. Define $\nu := n - 1$ and let $h^{a\alpha}$, $g^{a\alpha}$, $\alpha = 1, ..., \nu$ be the real numbers which are elements of $n \times \nu$ matrices of rank ν, satisfying conditions

$$M^a h^{a\alpha} = 0, \qquad J^a g^{a\alpha} = 0, \qquad \alpha = 1, ..., \nu. \tag{3.1}$$

The aforementioned matrices will be referred to as the oscillation-shape matrices. The physical meaning of this concept will be explained below.

The first modelling assumption makes it possible to represent displacement $\mathbf{u}^a(\mathbf{z}, t)$ and rotation $\varphi^a(\mathbf{z}, t)$ of an arbitrary node N^a in a cell with center $\mathbf{z}$, $\mathbf{z} \in \mathcal{L}$, in terms of certain regular macro-functions $U_i(\cdot, t)$, $Q_i^\alpha(\cdot, t)$, $\Phi(\cdot, t)$, $R^\alpha(\cdot, t)$, defined on Ω for every t. This assumption will be referred to as *the macro-kinematic hypothesis* given by

$$\begin{aligned} u_i^a(\mathbf{z}, t) &= U_i(\mathbf{x}, t) + l h^{a\alpha} Q_i^\alpha(\mathbf{x}, t), \\ \varphi^a(\mathbf{z}, t) &= \Phi(\mathbf{x}, t) + l g^{a\alpha} R^\alpha(\mathbf{x}, t), \qquad \mathbf{z} \in \mathcal{L}, \end{aligned} \tag{3.2}$$

where $\mathbf{x}$ is a position vector of the node N^a in a cell with the center $\mathbf{z}$. Because of $|U_i(\mathbf{x}, t) - U_i(\mathbf{z}, t)| < \varepsilon_U$, $|\Phi(\mathbf{x}, t) - \Phi(\mathbf{z}, t)| < \varepsilon_\Phi$, etc., and bearing in mind Eqs. (3.1), we obtain

$$\begin{aligned} U_i(\mathbf{z}, t) &= \frac{M^a u_i^a(\mathbf{z}, t)}{\sum M^a} + \mathcal{O}(\varepsilon_U) + \mathcal{O}(\varepsilon_Q), \\ \Phi(\mathbf{z}, t) &= \frac{J^a \varphi^a(\mathbf{z}, t)}{\sum J^a} + \mathcal{O}(\varepsilon_\Phi) + \mathcal{O}(\varepsilon_R), \qquad \mathbf{z} \in \mathcal{L}. \end{aligned} \tag{3.3}$$

It follows that $U_i(\mathbf{z}, t)$ and $\Phi(\mathbf{z}, t)$ are approximations of weighted averaged displacements and rotations, respectively, within an arbitrary cell. Fields $U_i(\cdot, t)$ and $\Phi(\cdot, t)$ will be called macro-displacements and macro-rotation, respectively, at an instant t. Taking into account Eqs. (3.1) it can be seen that $Q_i^\alpha(\mathbf{z}, t)$, $R^\alpha(\mathbf{z}, t)$ describe oscillations of displacements and rotations at time t within an arbitrary cell. Fields $Q_i^\alpha(\cdot, t)$, $R^\alpha(\cdot, t)$ will be called macro-internal variables; the meaning of this term will be explained in the subsequent section. Roughly speaking, in order to use the macro-kinematic hypothesis we have to choose a certain representative cell Δ, and then to specify oscillation-shape matrices and hence to restrict our considerations to a certain class of motions which can be expected in the problem under consideration. Such situation is met in many dynamic problems of interest in the analysis of special classes of motions.

Let us observe that restrictions imposed on the class of motions under consideration reduce to the requierement that $U_i(\cdot,t)$, $Q_i^\alpha(\cdot,t)$, $\Phi(\cdot,t)$, $R^\alpha(\cdot,t)$ have to be regular macro-functions for every t. Let us also observe that the oscillation-shape matrices are not uniquely determined but their choice is irrelevant.

The second modelling assumption is related to the concept of macro-function and will be called *the macro-approximation hypothesis.* This hypothesis consists of two kinds of postulated approximations:

(i) Finite differences of regular macro-functions within every cell can be approximated by the values of appropriate derivatives.

(ii) Increments of all macro-functions inside an arbitrary cell will be neglected in calculation of averages over this cell.

From (ii) it follows that term $\mathcal{O}(\varepsilon_U)$, $\mathcal{O}(\varepsilon_Q)$, $\mathcal{O}(\varepsilon_\Phi)$, $\mathcal{O}(\varepsilon_R)$ in Eqs. (3.3) in the course of modelling can be neglected. Hence macro-displacements and macro-rotation in the center of an arbitrary cell can be treated as weighted averages of nodal displacements and rotations, respectively, in this cell. From (i) we obtain the formulae for strain components in an arbitrary plate or beam belonging to a cell with the center $\mathbf{z}$. To this end, define $\lambda^A := l/l^A$, $g_A^\alpha := \dfrac{1}{2}(g^{a\alpha}+g^{b\alpha})$, $g^{A\alpha} := (g^{b\alpha}-g^{a\alpha})$ if nodes N^a, N^b belong to the rod R^A, and

$$
\begin{aligned}
E_{ij}(\mathbf{x},t) &:= U_{i,j}(\mathbf{x},t) + \epsilon_{ij}\Phi(\mathbf{x},t),\\
\Phi_i(\mathbf{x},t) &:= \Phi_{,i}(\mathbf{x},t), \qquad \mathbf{x}\in\Omega,
\end{aligned}
\tag{3.4}
$$

where ϵ_{ij} is the Ricci symbol. After simple calculations we arrive at (no summation over A!)

$$
\begin{aligned}
\varepsilon^A(\mathbf{z},t) &= t_i^A t_j^A E_{ij}(\mathbf{z},t) + \lambda^A t_i^{Aa} h^{a\alpha} Q_i^\alpha(\mathbf{z},t) + \mathcal{O}(\varepsilon),\\
\widetilde{\varepsilon}^A(\mathbf{z},t) &= n_i^A t_j^A E_{ij}(\mathbf{z},t) + \lambda^A n_i^{Aa} h^{a\alpha} Q_i^\alpha(\mathbf{z},t) - l g_A^\alpha R^\alpha + \mathcal{O}(\varepsilon),\\
\kappa^A(\mathbf{z},t) &= l^A t_i^A \Phi_i(\mathbf{z},t) + l g^{A\alpha} R^\alpha(\mathbf{z},t) + \mathcal{O}(\varepsilon),
\end{aligned}
\tag{3.5}
$$

where $\mathbf{z}\in\mathcal{L}$ and terms $\mathcal{O}(\varepsilon)$ depend on increments of the pertinent macro-functions inside a cell with center $\mathbf{z}$. Here and in the sequel it has to be remembered that $t_i^{Aa} := 0$, $n_i^{Aa} := 0$ if the node N^a does not belong to the rod R^A.

4. Governing equations

After substituting to (2.5) the right-hand sides of Eqs. (3.2), (3.5) and taking into account the macro-approximation hypothesis (related to calculations of averages), it can be seen that the finite sum over $\mathcal{L}$ can be approximated by an integral over Ω. Setting $|\Delta| = l_1 l_2$ let us introduce the notations

$$
\begin{aligned}
A_{ijkl} &:= |\Delta|^{-1} \sum_{A=1}^{N} \left(\Lambda^A t_i^A t_k^A + \widetilde{\Lambda}^A n_i^A n_k^A \right) t_j^A t_l^A , \\
B_{ijk}^{\alpha} &:= |\Delta|^{-1} \sum_{A=1}^{N} \lambda^A \left(\Lambda^A t_i^A t_k^{Aa} + \widetilde{\Lambda}^A n_i^A n_k^{Aa} \right) t_j^A h^{a\alpha}, \\
C_{ij}^{\alpha\beta} &:= |\Delta|^{-1} \sum_{A=1}^{N} (\lambda^A)^2 \left(\Lambda^A t_i^{Aa} t_j^{Ab} + \widetilde{\Lambda}^A n_i^{Aa} n_j^{Ab} \right) h^{a\alpha} h^{b\beta}, \\
D_{ij} &:= |\Delta|^{-1} \sum_{A=1}^{N} (\lambda^A)^{-2} K^A t_i^A t_j^A , \\
G_i^{\alpha} &:= |\Delta|^{-1} \sum_{A=1}^{N} (\lambda^A)^{-1} K^A t_i^A g^{A\alpha}, \\
F^{\alpha\beta} &:= |\Delta|^{-1} \sum_{A=1}^{N} \left(\widetilde{\Lambda}^A g_A^{\alpha} g_A^{\beta} + K^A g^{A\alpha} g^{A\beta} \right), \\
B_{ij}^{\alpha} &:= -|\Delta|^{-1} \sum_{A=1}^{N} \widetilde{\Lambda}^A n_i^A t_j^A g_A^{\alpha} , \\
C_i^{\alpha\beta} &:= -|\Delta|^{-1} \sum_{A=1}^{N} \lambda^A \widetilde{\Lambda}^A n_i^{Aa} h^{a\alpha} g_A^{\beta} , \\
\chi &:= |\Delta|^{-1} \sum_{a=1}^{n} J^a, \qquad \varrho := |\Delta|^{-1} \sum_{a=1}^{n} M^a, \\
\chi^{\alpha\beta} &:= |\Delta|^{-1} \sum_{a=1}^{n} J^a g^{a\alpha} g^{a\beta}, \qquad \varrho^{\alpha\beta} := |\Delta|^{-1} \sum_{a=1}^{n} M^a h^{a\alpha} h^{a\beta}.
\end{aligned}
\tag{4.1}
$$

Under the extra assumption that there exist continuous macro-functions $f_i(\cdot, t)$, $f_i^{\alpha}(\cdot, t)$, $m(\cdot, t)$, $m^{\alpha}(\cdot, t)$ defined on Ω for every t, such that the conditions

$$
\begin{aligned}
f_i(\mathbf{z}, t) &= |\Delta|^{-1} \sum_{a=1}^{n} f_i^a(\mathbf{z}, t) + \mathcal{O}(\varepsilon_f), \\
f_i^{\alpha}(\mathbf{z}, t) &= |\Delta|^{-1} \sum_{a=1}^{n} f_i^a(\mathbf{z}, t) h^{a\alpha} + \mathcal{O}(\varepsilon_f), \\
m(\mathbf{z}, t) &= |\Delta|^{-1} \sum_{a=1}^{n} m^a(\mathbf{z}, t) + \mathcal{O}(\varepsilon_m), \\
m^{\alpha}(\mathbf{z}, t) &= |\Delta|^{-1} \sum_{a=1}^{n} m^a(\mathbf{z}, t) g^{a\alpha} + \mathcal{O}(\varepsilon_m),
\end{aligned}
\tag{4.2}
$$

hold for every $\mathbf{z} \in \mathcal{L}$ we arrive, after some manipulations, at the integral form of the action functional $\mathcal{A} = \mathcal{T} - K - \mathcal{W}$, where now

$$
\begin{aligned}
\mathcal{T} = \int\limits_{\Omega} \Big(& \frac{1}{2} A_{ijkl} E_{ij} E_{kl} + B^{\alpha}_{ijk} E_{ij} Q^{\alpha}_{k} + l B^{\alpha}_{ij} E_{ij} R^{\alpha} \\
& + \frac{1}{2} C^{\alpha\beta}_{ij} Q^{\alpha}_{i} Q^{\beta}_{j} + l C^{\alpha\beta}_{i} Q^{\alpha}_{i} R^{\beta} + l^2 D_{ij} \Phi_i \Phi_j + l^2 G^{\alpha}_{i} \Phi_i R^{\alpha} \\
& + l^2 \frac{1}{2} F^{\alpha\beta} R^{\alpha} R^{\beta} \Big)\, da, \qquad (4.3)\\
K = \int\limits_{\Omega} \Big(& \frac{1}{2} \varrho \dot{U}_i \dot{U}_i + \frac{1}{2} l^2 \varrho^{\alpha\beta} \dot{Q}^{\alpha}_{i} \dot{Q}^{\beta}_{i} + \frac{1}{2} \chi \dot{\Phi} \dot{\Phi} + \frac{1}{2} l^2 \varrho^{\alpha\beta} \dot{R}^{\alpha} \dot{R}^{\beta} \Big)\, da, \\
\mathcal{W} = \int\limits_{\Omega} & \left(f_i U_i + l f^{\alpha}_{i} Q^{\alpha}_{i} + m \Phi + l m^{\alpha} R^{\alpha} \right) da, \qquad da = dx_1 dx_2 .
\end{aligned}
$$

From the principle of stationary action we obtain the following equations of motion for macro-displacements U_i and a macro-rotation Φ:

$$
\begin{aligned}
A_{ijkl} E_{kl,j} + B^{\alpha}_{ijk} Q^{\alpha}_{k,j} + l B^{\alpha}_{ij} R^{\alpha}_{,j} - \varrho \ddot{U}_i + f_i &= 0, \\
l^2 D_{ij} \Phi_{i,j} + l^2 G^{\alpha}_{i} R^{\alpha}_{,i} - \epsilon_{ij} \left(A_{ijkl} E_{kl} + B^{\alpha}_{ijk} Q^{\alpha}_{k} + l B^{\alpha}_{ij} R^{\alpha} \right) - \chi \ddot{\Phi} + m &= 0
\end{aligned}
\tag{4.4}
$$

coupled with equations for macro-internal variables Q^{α}_{i}, R^{α}:

$$
\begin{aligned}
l^2 \varrho^{\alpha\beta} \ddot{Q}^{\beta}_{i} + C^{\alpha\beta}_{ij} Q^{\beta}_{j} + l C^{\alpha\beta}_{i} R^{\beta} + B^{\alpha}_{kji} E_{kj} &= l f^{\alpha}_{i} , \\
l^2 \chi^{\alpha\beta} \ddot{R}^{\beta} + l^2 F^{\alpha\beta} R^{\beta} + l C^{\beta\alpha}_{i} Q^{\beta}_{i} + l^2 G^{\alpha}_{i} \Phi_i + l B^{\alpha}_{ij} E_{ij} &= l m^{\alpha},
\end{aligned}
\tag{4.5}
$$

where E_{ij}, Φ_i are defined by Eqs. (3.4). The obtained equations have to be satisfied for every t in region Ω of the plane $0x_1x_2$, and represent a continuum model of the cellular periodic medium under consideration.

The governing equations (4.4), (4.5) can be also written in the alternative form given by:

(i) Equations of motion

$$
\begin{aligned}
T_{ij,j} - \varrho \ddot{U}_i + f_i &= 0, \\
M_{i,i} - \epsilon_{ij} T_{ij} - \chi \ddot{\Phi} + m &= 0.
\end{aligned}
\tag{4.6}
$$

(ii) Dynamic evolution equations

$$
\begin{aligned}
l^2 \varrho^{\alpha\beta} \ddot{Q}^{\beta}_{i} + S^{\alpha}_{i} &= l f^{\alpha}_{i} , \\
l^2 \chi^{\alpha\beta} \ddot{R}^{\beta} + H^{\alpha} &= l m^{\alpha}.
\end{aligned}
\tag{4.7}
$$

(iii) Constitutive equations

$$
(4.8)\qquad
\begin{bmatrix} T_{ij} \\ M_i \\ S_i^\alpha \\ H^\alpha \end{bmatrix}
=
\begin{bmatrix}
A_{ijkl} & 0 & B_{ijk}^\beta & lB_{ij}^\beta \\
0 & l^2 D_{ik} & 0 & l^2 G_i^\beta \\
B_{kli}^\alpha & 0 & C_{ik}^{\alpha\beta} & lC_i^{\alpha\beta} \\
lB_{kl}^\alpha & l^2 G_k^\alpha & lC_k^{\beta\alpha} & l^2 F^{\alpha\beta}
\end{bmatrix}
\begin{bmatrix} E_{kl} \\ \Phi_k \\ Q_k^\beta \\ R^\beta \end{bmatrix},
$$

which have to be considered together with the notations (3.4). From a formal viewpoint, Eqs. (4.6) are similar to the known plane Cosserat media equations of motion. Hence T_{ij} and M_i can be called components of a macro-stress tensor and a macro-couple-stress vector, respectively. However, contrary to the Cosserat media, we also deal here with the dynamic evolution equations (4.7) which are coupled with the Cosserat equations (4.6) *via* the constitutive equations (4.8).

The first characteristic feature of the obtained model is that it describes, in one formal scheme, a wide class of plane problems for elastic periodic cellular media. Moreover, the effect of the cell size on the dynamic behaviour of the structure is taken into account. This fact is caused by the nonasymptotic modelling procedure which leads to the occurence of the microstructure length parameter l in Eqs. (4.7), related to the coefficients at the second-order time derivatives of macro-internal variables. At the same time, all coefficients in equations (4.4), (4.5) defined by formulae (4.1) are independent of the cell size and hence, the dependence of these equations on l has an explicit form.

The second characteristic feature of the obtained continuum model is that equations (4.5) for Q_i^α, R^α are ordinary differential equations involving exclusively time-derivatives of these fields. Hence functions Q_i^α, R^α are not restricted by boundary conditions, which have to be formulated only for macro-displacements U_i and a macro-rotation Φ. That is why functions Q_i^α, R^α were called macro-internal variables. Thus, in formulation of initial-boundary value problems, Eqs. (4.4), (4.5) have to be considered together with both the boundary and initial conditions for U_i, Φ and only the initial conditions for Q_i^α, R^α. This fact is essential in investigations of special problems since in the framework of the continuum model, only boundary conditions for U_i and Φ have a physical motivation, being independent of possible disturbances of the periodic structure of a medium near the boundary of a region Ω.

It has to be emphasized that solutions to problems described by Eqs. (4.3) – (4.5) together with boundary and initial conditions, have a physical sense only if solutions U_i, Φ, Q_i^α, R^α to these problems are represented by regular macro-functions.

In order to evaluate the cell size effect on the dynamic behaviour of the medium, together with the refined model we can also consider its asymptotic approximation obtained by a formal passage $l \to 0$ in (4.3). In this case we arrive at a special case of results obtained in [3].

At the end of this section let us consider some special cases of the proposed continuum models for discrete periodic structures described in Sec. 3. Firstly, assume that $B^{\alpha}_{ijk} = 0$, $G^{\alpha}_{i} = 0$ and $C^{\alpha\beta}_{i} = 0$. Such situation takes place if x_α are elastic symmetry axes for constitutive equations (4.8). In this case the model will be called macro-ortothropic. Moreover, if $B^{\beta}_{ij} = 0$ then constitutive equations (4.8) reduce to the form

$$T_{ij} = A_{ijkl}E_{kl}\,, \qquad M_i = D_{ij}\Phi_j\,, \tag{4.9}$$

and dynamic evolution equations (4.7) yield

$$\begin{aligned} l^2\varrho^{\alpha\beta}\,\ddot{Q}^{\beta}_{i} + C^{\alpha\beta}_{ij}Q^{\beta}_{j} &= lf^{\alpha}_{i}\,,\\ l^2\chi^{\alpha\beta}\,\ddot{R}^{\beta} + l^2F^{\alpha\beta}R^{\beta} &= lm^{\alpha}. \end{aligned} \tag{4.10}$$

Hence, if $f^{\alpha}_{i} = 0$, $m^{\alpha} = 0$ then under homogeneous initial conditions for macro-internal variables Q^{α}_{i}, R^{α}, we obtain $Q^{\alpha}_{i} = 0$, $R^{\alpha} = 0$. It follows that for this class of problems, there are no oscillations in displacements and rotations within every cell of a structure provided that the external loadings are not oscillating on a cell (i.e. $f^{\alpha}_{i} = 0$, $m^{\alpha} = 0$), and the initial values of macro-internal variables Q^{α}_{i}, R^{α} are equal to zero. Macro-ortothropic models in which $B^{\beta}_{ij} = 0$, will be called uncoupled and their continuum models are represented by equations of motion (4.6) and constitutive equations (4.9) (together with notations (3.4)) for U_i, Φ and indepedently, by the evolution equations (4.10) for Q^{β}_{i}, R^{β}. Now assume that the choice of a cell Δ implies $\nu = n-1 = 0$ and hence there are no macro-internal variables in the considered model. This special discrete medium will be referred to as having a simple lay-out structure [12]; an example of this medium is shown at the bottom of Fig. 1. The detailed analysis of different problems for these media was given in [15] where also the list of references can be found. However, also in the case of media with a simple lay-out structure we can analyse more complicated motions by introducing another cell for which $\nu > 0$. Let us also observe that for quasi-stationary processes, the macro-internal variables can be eliminated from (4.4) by means of (4.5).

5. Example

General results of the previous section will be now illustrated by the analysis of a wave propagation in a periodic medium with a rectangular network shown in Fig. 3. In order to introduce the simplest continuum model of this medium, let the cell Δ be assumed in the form given in Fig. 3. This cell has two nodal points; in this case $n = 2$ and $\nu = n - 1 = 1$, i.e., the oscillation-shape matrices reduce to vectors with components h^{11}, h^{21} and g^{11}, g^{21}. Moreover, let the directions of network be paralell to the coordinate axes x_1, x_2, and let x_1 be a symmetry axis for the constitutive equations (4.8) which have to be invariant under transformation $x_2 \to -x_2$. We are to show, under what conditions a longitudinal harmonic wave can propagate across such medium and how the dispersion relation depends on the unit cell size.

Taking into account the aforementioned properties of a medium and assuming that all unknown functions depend only on x_1 and time t, from Eqs. (4.4), (4.5), (3.4), bearing in mind definitions (4.1) and neglecting external loadings, we obtain two independent systems of equations. The first system is related to U_1 and Q_1:

$$\begin{aligned} &A_{1111}U_{1,11} + B_{111}Q_{1,1} - \varrho\,\ddot{U}_1 = 0, \\ &l^2\varrho^{11}\,\ddot{Q}_1 + C_{11}Q_1 + B_{111}U_{1,1} = 0, \end{aligned} \tag{5.1}$$

and the second one involves U_2, Φ, Q_2 and R:

$$\begin{aligned} &A_{2121}U_{2,11} + (A_{2112} - A_{2121})\Phi_{,1} + B_{212}Q_{2,1} - \varrho\,\ddot{U}_2 = 0, \\ &l^2D_{11}\Phi_{,11} + (A_{2112} + A_{1221} - A_{1212} - A_{2121})\Phi \\ &\qquad + (A_{2121} - A_{1221})U_{2,1} + (B_{212} - B_{122})Q_2 + l^2G_1R_{,1} - \chi\,\ddot{\Phi} = 0, \\ &l^2\varrho^{11}\,\ddot{Q}_2 + C_{22}Q_2 + (B_{122} - B_{212})\Phi + B_{212}U_{2,1} = 0, \\ &l^2\chi^{11}\,\ddot{R} + l^2FR + l^2G_1\Phi_{,1} = 0, \end{aligned} \tag{5.2}$$

where the superscripts related to macro-internal variables were neglected, i.e., $Q_1 = Q_1^1$, $Q_2 = Q_2^1$, $R = R^1$ etc.

In this paper investigations will be restricted to longitudinal waves. Substituting $U_1 = U(x_1 - ct)$, $Q_1 = Q(x_1 - ct)$ to Eqs. (5.1), where c is the propagation speed, we obtain

$$\begin{aligned} &(A_{1111} - \varrho c^2)U_{,11} + B_{111}Q_{,1} = 0, \\ &l^2\varrho^{11}c^2Q + C_{11}Q + B_{111}U_{,1} = 0. \end{aligned} \tag{5.3}$$

Define

$$\bar{c}^2 := \frac{A_{1111}}{\varrho}, \qquad \tilde{c}^2 := \frac{1}{\varrho}\left(A_{1111} - \frac{(B_{111})^2}{C_{11}}\right). \tag{5.4}$$

It can be shown that $\bar{c}^2 > \tilde{c}^2 > 0$. Analysis of Eqs. (5.1) leads to the three following cases:

CASE 1. If $c < \tilde{c}$ or $\bar{c} > c$ then across a medium under consideration, a harmonic longitudinal wave can propagate. Under the notation

$$(k(c))^2 := \frac{C_{11}}{l^2\varrho^{11}}\,\frac{\tilde{c}^2 - c^2}{c^2(\bar{c}^2 - c^2)}, \qquad (k(c))^2 > 0, \tag{5.5}$$

solution to Eqs. (5.3) will be given by

$$U = A_U \sin[k(c)(x_1 - ct)], \qquad Q = A_Q \cos[k(c)(x_1 - ct)].$$

CASE 2. If $\widetilde{c} < c < \overline{c}$ then we shall deal with the exponential wave. Setting

$$(\kappa(c))^2 := -\frac{C_{11}}{l^2\varrho^{11}} \frac{\widetilde{c}^2 - c^2}{c^2(\overline{c}^2 - c^2)}, \qquad (\kappa(c))^2 > 0, \tag{5.6}$$

we obtain

$$U = A_U \exp[-\kappa(c)(x_1 - ct)] + B_U \exp[\kappa(c)(x_1 - ct)]$$

and

$$Q = -\varrho(B_{111})^{-1}(\overline{c}^2 - c^2)U_{,1}.$$

CASE 3. If $c = \widetilde{c}$ then we deal with a degenerate case in which $U_{,11} = 0$, $Q_{,1} = 0$. At the same time, the propagation speed $c = \overline{c}$ cannot be realized in the medium under consideration.

Let $k = 2\pi/L$, where L is the wavelength, be the wave number for a harmonic longitudinal wave. Taking into account (5.5) we obtain the dispersion relation

$$-l^2\varrho^{11}k^2c^4 + \left(C_{11} + l^2\varrho^{11}c^2k^2\right)c^2 - C_{11}\widetilde{c}^2 = 0. \tag{5.7}$$

In the framework of applicability of the proposed continuum model, the wavelength L has to be much longer than the microstructure length parameter; only in this case the obtained solutions to Eqs. (5.3) are represented by macro-functions. Hence $\varepsilon := kl$ is a small parameter, $\varepsilon \ll 1$. For harmonic waves we obtain the following asymptotic formulae related to the lower $c = c_L$ and higher $c = c_H$ propagation speeds

$$\begin{aligned} c_L^2 &= \widetilde{c}^2 - l^2\frac{\widetilde{c}^2}{C_{11}}\left(\overline{c}^2 - \widetilde{c}^2\right)\varrho^{11}k^2 + \mathcal{O}(\varepsilon^4), \\ c_H^2 &= \frac{C_{11}}{l^2\varrho^{11}k^2} + \overline{c}^2 - \widetilde{c}^2 + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{5.8}$$

where $c_L < \widetilde{c}$ and $c_H > \overline{c}$. In the case of an exponential wave, i.e., for $c \in (\widetilde{c}, \overline{c})$, we obtain from (5.3) the dispersion relation

$$c^2 = \widetilde{c}^2 + l^2\frac{\widetilde{c}^2}{C_{11}}\left(\overline{c}^2 - \widetilde{c}^2\right)\varrho^{11}\kappa^2 + \mathcal{O}(\varepsilon^4), \tag{5.9}$$

where $\kappa := \delta^{-1}$ and δ is called the length of the exponential wave; here $\varepsilon := \kappa l$ is a small parameter $\varepsilon \ll 1$.

The above results were obtained by using the proposed continuum model of a periodic discrete medium. In the framework of the asymptotic model, by formal passage with l to zero, we obtain from (5.3) only one propagation velocity $c = \widetilde{c}$.

6. Reliability of the model

In order to investigate the physical correctness of the model we restrict ourselves to equations (5.1). Denoting by M^1, M^2 the masses assigned to nodal points, by l^1, l^2 the distances between them (where $l^1 + l^2 = l_1$, cf. Fig. 3) and by D^1, D^2 the axial stiffnesses of the corresponding plates, under notation $\mu = M^2/M^1$ and assuming $h^{11} = -\mu$, $h^{21} = 1$, from formulae (3.6) we obtain

$$
\begin{gathered}
A_{1111} = \frac{1}{l_1 l_2}\left(D^1 l^1 + D^2 l^2\right), \qquad C_{11} = \frac{l_1}{l_2}(1+\mu)^2\left(\frac{D^1}{l^1} + \frac{D^2}{l^2}\right), \\
B_{111} = \frac{1}{l_2}(1+\mu)(D^1 - D^2), \qquad \varrho = (1+\mu)\frac{M^1}{l_1 l_2}, \qquad \varrho^{11} = \mu(1+\mu)\frac{M^1}{l_1 l_2}.
\end{gathered} \tag{6.1}
$$

Setting $A := A_{1111}$, $B := B_{111}$, $C := C_{11}$ and $U := U_1$, $Q := Q_1$, we shall rewrite Eqs. (5.1) to the form

$$
\begin{gathered}
AU_{,11} + BQ_{,1} - \varrho\,\ddot{U} = 0, \\
l^2\mu\varrho\,\ddot{Q} + CQ + BU_{,1} = 0
\end{gathered} \tag{6.2}
$$

and look for solutions to (6.2) in the well known form

$$
U = a_U \exp i(\omega t - \kappa x_1), \qquad Q = a_Q \exp i(\omega t - \kappa x_1), \tag{6.3}
$$

where $\kappa = 2\pi/L$ and L is the wavelength. Substituting the right-hand sides of (6.3) into (6.2) and denoting $l := l_1 = l^1 + l^2$, $k := \kappa l = 2\pi l/L$, for the free vibration frequency ω we obtain the dispersion relation

$$
(l)^2\mu\varrho^2\omega^4 - (Ak^2\mu + C)\varrho\omega^2 + \kappa^2(AC - B^2) = 0. \tag{6.4}
$$

The model is applicable only if (6.3) are macro-functions; hence the value k has to be sufficiently small compared to 1 and solutions ω_+, ω_- to (6.4), by taking into account (6.1) and performing some manipulations, will be given by

$$
\begin{aligned}
(\omega_+)^2 &= \left(\frac{1}{M^1} + \frac{1}{M^2}\right)\left(\frac{D^1}{l^1} + \frac{D^2}{l^2}\right) \\
&\qquad + \frac{(D^1 - D^2)^2}{(M^1 + M^2)\left(\dfrac{D^1}{l^1} + \dfrac{D^2}{l^2}\right)}\left(\frac{k}{l}\right)^2 + o(k^2), \\
(\omega_-)^2 &= \left[\frac{D^1 l^1 + D^2 l^2}{M^1 + M^2} + \frac{(D^1 - D^2)^2}{(M^1 + M^2)\left(\dfrac{D^1}{l^1} + \dfrac{D^2}{l^2}\right)}\right]\left(\frac{k}{l}\right)^2 + o(k^2).
\end{aligned} \tag{6.5}
$$

Exact solution of the discrete problem under consideration is known for $D^1 = D^2$ and can be found in [2]. For the long wave approximation and after neglecting terms k^2 as small compared to 1, it can be easily shown that the model solution (6.5) coincides with the long wave approximation of the exact solution given in [2]. Hence we conclude that the proposed model in the problem under consideration yields the physically correct solutions and can be treated as a reliable one.

7. Conclusions

From the illustrative example given in Sec. 5 and its physical correctness discussed in Sec. 6 it follows that the continuum model of a cellular periodic medium,which was proposed in this contribution, constitutes a useful tool for investigations of certain dynamic problems. We also conclude that the proposed continuum model describes dynamics of a whole class of discrete periodic media in one formal scheme given by equations of motion (4.6), dynamic evolution equations (4.7) and constitutive equations (4.8). It has to be emphasized that the microdynamic phenomena (micro-oscillations of mass-elements inside an arbitrary periodicity cell) are described by the macro-internal variables governed by the ordinary differential equations and not entering the boundary conditions; hence the proposed continuum model have certain features typical for the dynamics of discrete media. Moreover, accuracy of the model proposed can be verified *a posteriori* by evaluations of the computational accuracy parameters ε_U, ε_Q, $\varepsilon_{\nabla U}, \ldots$ related to the obtained macro-functions $U(\cdot)$, $Q_i^\alpha(\cdot), \ldots,$ which constitute a solution to the problem under consideration.

Application of Eqs. (4.6) – (4.8) to the dynamic analysis of honeycomb cellular structures as well as comparison of the obtained data with those resulting from the known models will be given in the subsequent paper.

For the sake of simplicity, the proposed model was restricted to the plane problems for periodic cellular media or framed linear-elastic structures. However, the proposed general line of modelling can be also applied to spatial periodic systems, to problems with damping and to non-linear dynamic problems. Formulation of these models and their applications will be considered separately.

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