Arch. Mech., **50**, 2, pp. 281–319, Warszawa 1998
FIFTY YEARS OF THE ARCHIVES OF MECHANICS

On anisotropic functions of vectors and second order tensors – all subgroups of the transverse isotropy group $C_{\infty h}$

H. XIAO (BOCHUM and BEIJING)

A UNIFIED PROCEDURE for constructing both the generating sets and the functional bases is suggested, which reduces the representation problem for anisotropic functions of any finite number of vector and second order tensor variables under any subgroup $g \subset C_{\infty h}$, to that for the same types of anisotropic functions of not more than two vector and/or second order tensor variables. By using this procedure and new results for isotropic extension of anisotropic functions, simple irreducible generating sets and functional bases in unified forms are presented to determine general reduced forms of scalar-, vector-, and second order tensor-valued anisotropic functions of any finite number of vectors and second order tensors, under all kinds of subgroups of the transverse isotropy group $C_{\infty h}$. The results given are derived in the sense of nonpolynomial representation.

1. Introduction

IN CONTINUUM PHYSICS, scalar-, vector- and second order tensor-valued functions of vector and second order tensor[1] variables serve as mathematical models of macroscopic physical behaviours of materials. Such tensor functions are required to possess form-invariance under the action of the material symmetry group due to material objectivity and material symmetry. Let $f(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$, $\mathbf{h}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$ and $\mathbf{F}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)$ be, respectively, scalar-, vector- and second order tensor-valued functions of the a vector variables $\mathbf{u}_i$, the b skewsymmetric second order tensor variables $\mathbf{W}_\sigma$, and the c symmetric second order tensor variables $\mathbf{A}_L$, where $i = 1, \dots, a$; $\sigma = 1, \dots, b$ and $L = 1, \dots, c$. Moreover, let g be a subgroup of the full orthogonal group Orth, which may serve as a material symmetry group for solid materials. The tensor functions f, $\mathbf{h}$ and $\mathbf{F}$ are invariant or form-invariant under the group g, respectively, if for every orthogonal tensor $\mathbf{Q} \in g$,

$$
\begin{aligned}
f(\mathbf{Q}\mathbf{u}_i, \mathbf{Q}\mathbf{W}_\sigma\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{A}_L\mathbf{Q}^{\mathrm{T}}) &= f(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L),\\
\mathbf{h}(\mathbf{Q}\mathbf{u}_i, \mathbf{Q}\mathbf{W}_\sigma\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{A}_L\mathbf{Q}^{\mathrm{T}}) &= \mathbf{Q}\mathbf{h}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L),\\
\mathbf{F}(\mathbf{Q}\mathbf{u}_i, \mathbf{Q}\mathbf{W}_\sigma\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{A}_L\mathbf{Q}^{\mathrm{T}}) &= \mathbf{Q}\mathbf{F}(\mathbf{u}_i, \mathbf{W}_\sigma, \mathbf{A}_L)\mathbf{Q}^{\mathrm{T}}.
\end{aligned}
$$

General reduced forms of the tensor functions f, $\mathbf{h}$ and $\mathbf{F}$ under the above invariance restrictions are called *representations* for f, $\mathbf{h}$ and $\mathbf{F}$ under the group

[1] Throughout the paper, vector and tensor mean three-dimensional vectors and tensors.

g. It has been known (see PIPKIN and WINEMAN [15], WINEMAN and PIPKIN [30]) that finding representations for the tensor functions f, $\mathbf{h}$ and $\mathbf{F}$ under the group g is equivalent to determining *functional bases* (for f) and *generating sets* (for $\mathbf{h}$ and $\mathbf{F}$) under the group g. Moreover, both the functional bases and the generating sets to be used are further required to be *irreducible* in order to arrive at compact representations. For detail, refer to the definitions given later.

Proper subgroups of the full orthogonal group Orth include the proper orthogonal group Orth^+, the five classes of cubic crystal groups, the two classes of icosahedral groups, the five classes of transverse isotropy groups and the denumerably infinitely many classes of subgroups of the latter (see, e.g., SPENCER [26], VAINSHTEIN [28]). Of them, the 32 classes of crystallographic point groups, the five classes of transverse isotropy groups and Orth (see SPENCER [26]) are related to common solid materials in engineering, while the others are associated with quasi-crystalline solids and texture materials (see VAINSHTEIN [28]), etc. In the past decades, many efforts were devoted to finding representations for various kinds of tensor functions under the aforementioned orthogonal subgroups, and many significant results were obtained. Here, we would not reproduce the large number of the related references. For details, refer to TRUESDELL and NOLL [27], SPENCER [26], KIRAL and ERINGEN [10], SMITH [21] and the recent reviews by BETTEN [4], RYCHLEWSKI and ZHANG [17], and ZHENG [43] *et al.*, and to the references therein. Although now many results in many cases, mainly in the sense of polynomial representation, are available (see SPENCER [26], KIRAL and ERINGEN [10], and SMITH [21]), general aspect of representation problems, mainly in the sense of nonpolynomial representation, remains open, except for some particular cases such as isotropic, orthotropic and transversely isotropic functions etc. (see ADKINS [1–3], PIPKIN and RIVLIN [14], WANG [29], SMITH [19], BOEHLER [5], and PENNISI and TROVATO [13], SMITH [20], ZHENG [41], JEMIOŁO and TELEGA [8], *et al.*). Applying the isotropic extension method for anisotropic functions, initiated by LOKHIN and Sedov [12] and BOEHLER [6, 7] and LIU [11] (see also RYCHLEWSKI [16], ZHANG and RYCHLEWSKI [40], ZHENG and SPENCER [44]) and further developed recently by the author (see XIAO and GUO [31], XIAO [35, 39]), as well as the general representation theorems given in XIAO [33, 34], we shall derive general irreducible representations for scalar-, vector- and second order tensor-valued anisotropic functions of any finite number of vectors and second order tensors, under various kinds of orthogonal subgroups, in a series of works. In this paper, we shall confine ourselves to anisotropic functions under subgroups of the transverse isotropy group $C_{\infty h}$ (see PIPKIN and RIVLIN [14], SMITH and RIVLIN [23, 24], SMITH, SMITH and RIVLIN [25], SMITH and KIRAL [22], SMITH [18, 20, 21], SPENCER [26], KIRAL and SMITH [9], ZHENG [41, 42, 43], XIAO [32, 36, 37], *et al.* for related results in some cases; in particular, see ZHENG [42] for the counterpart of the two-dimmentional case of the problem considered here).

At the end of this introduction, we state some facts that will be used.

Throughout the paper, V, Skw and Sym are used to represent the vector space, the skewsymmetric and symmetric second order tensor spaces, respectively. $\mathbf{u}$, $\mathbf{v}$, $\mathbf{r}$ etc., $\mathbf{W}$, $\mathbf{H}$ etc., $\mathbf{A}$, $\mathbf{B}$ etc. are used to denote vectors, skewsymmetric second order tensors and symmetric second order tensors, respectively. On the other hand, $g(X)$ is the symmetry group of the set X of vectors and tensors (see XIAO [33]) and for any subgroup $g \subset \mathrm{Orth}$, $M(g)$ is the g-subspace of the space $M \in \{V, \mathrm{Skw}, \mathrm{Sym}\}$ (see XIAO [32, 33]). The former consists of the orthogonal tensors preserving X, and the latter includes the elements in M that are invariant under the action of the group g.

A finite set of scalar-valued functions that are invariant under the group $g \subset \mathrm{Orth}$, i.e. invariants under g, is a functional basis under g if each invariant under g is expressible as a real single-valued function of these invariants. On the other hand, a finite set of vector-valued (resp. second order tensor-valued) functions that are form-invariant under the group $g \subset \mathrm{Orth}$, i.e. vector (resp. second order tensor) generators under g, is a generating set under g if each vector-valued (resp. second order tensor-valued) function that is form-invariant under g is expressible as a linear combination of these generators whose coefficients are invariants under g. Furthermore, a functinal basis (resp. a generating set) under the group $g \subset \mathrm{Orth}$ is irreducible if none of its proper subsets is again a functional basis (resp. a generating set) under the group g. A criterion for generating set is as follows (see XIAO [33]).

The vector-valued or skewsymmetric tensor-valued or symmetric tensor-valued functions $\boldsymbol{\psi}_1, \cdots, \boldsymbol{\psi}_r$ that are form-invariant under the group $g \subset \mathrm{Orth}$ form a generating set under g iff

$$\forall X \in V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c : \ \mathrm{rank}\{\boldsymbol{\psi}_1(X), \cdots, \boldsymbol{\psi}_r(X)\} \geq \mathrm{dim} M(g \cap g(X)), \tag{1.1}$$

where $M = V$, Skw, Sym, respectively, when $\boldsymbol{\psi}_i$ is vector-valued, skewsymmetric tensor-valued and symmetric tensor-valued, respectively. Here, for any set S of vectors or tensors and any subspace L, the notation rank$S$ and dim$L$ are used to represent the number of the linearly independent elements in the set S and the dimension of the subspace L, respectively.

To facilitate the application of the above criterion, we list the following facts for the subgroups $g \subset C_{\infty h}$.

$$\dim V(g) = \begin{cases} 1, & g \subset \mathrm{Orth}^+, \quad g \neq C_1, \\ 2, & g = C_{1h}, \\ 3, & g = C_1, \\ 0, & \text{otherwise}; \end{cases} \tag{1.2}$$

$$\dim \mathrm{Skw}(g) = \begin{cases} 3, & g = C_1, S_2, \\ 1, & \text{otherwise}; \end{cases} \tag{1.3}$$

$$\dim \mathrm{Sym}(g) = \begin{cases} 6, & g = C_1, S_2\,, \\ 4, & g = C_2, C_{1h}, C_{2h}\,, \\ 2, & \text{otherwise}. \end{cases} \tag{1.4}$$

The groups appearing above will be given later. On the other hand, a criterion for functional bases is as follows (see PIPKIN and WINEMAN [15, 30]; see also XIAO [33]).

The invariants $f_1, \cdots, f_r$ under the group $g \subset \mathrm{Orth}$ form a functional basis under g iff

$$f_1(\overline{X}) = f_1(X), \cdots, f_r(\overline{X}) = f_r(X) \Longrightarrow \exists\, \mathbf{Q} \in g: \ \overline{X} = \mathbf{Q} \star X\,, \tag{1.5}$$

for $\overline{X}$, $X \in V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$. The latter means that $\overline{X}$ and X pertain to the same g-orbit. Thus, the variable X is determined to within an orthogonal tensor pertaining to the group g.

To check the irreducibility of a given functional basis, we shall use the following fact.

A functional basis I under the group $g \subset \mathrm{Orth}$ is irreducible iff for any given element $f_0 \in I$ there exist X, $X' \in V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$, which belong to two different g-orbits, such that

$$f_0(X) \neq f_0(X') \ \ \& \ \ f(X) = f(X') \quad \text{for all } \ f \in I/\{f_0\}\,. \tag{1.6}$$

In fact, for any given element f_0, the proper subset $I/\{f_0\}$ can not be a functional basis under g if (1.6) holds, or else according to the aforementioned criterion for functional bases, X and X' must pertain to the same g-orbit (see (1.5)) and hence $f_0(X) = f_0(X')$, which contradicts $(1.6)_1$.

Let S be a functional basis or generating set under the group g. We shall speak of the irreducibility of an element in S. By this we mean the following fact: an invariant or a generator $\chi \in S$ is irreducible if the proper subset $S/\{\chi\}$ fails to supply a functional basis or generating set under g. Obviously, a functional basis or generating set is irreducible iff each element of it is irreducible.

By means of the Schoenflies symbol we list the transverse isotropy groups $C_{\infty h}$ and all its finite subgroups as follows.

$$\begin{aligned}
C_{\infty h}(\mathbf{n}) &= \{\pm \mathbf{R}_{\mathbf{n}}^{\theta} \mid \theta \in R\}, \\
C_{\infty}(\mathbf{n}) &= C_{\infty h}(\mathbf{n}) \cap \mathrm{Orth}^+; \\
C_1 &= \{\mathbf{I}\}, \qquad S_2 = \{\pm \mathbf{I}\}, \\
C_{1h}(\mathbf{n}) &= \{\mathbf{I}, -\mathbf{R}_{\mathbf{n}}^{\pi}\}, \qquad C_2(\mathbf{n}) = \{\mathbf{I}, \mathbf{R}_{\mathbf{n}}^{\pi}\}, \qquad C_{2h}(\mathbf{n}) = \{\pm \mathbf{I}, \pm \mathbf{R}_{\mathbf{n}}^{\pi}\}; \\
S_{4m+2}(\mathbf{n}) &= \{\pm \mathbf{R}_{\mathbf{n}}^{2k\pi/2m+1} \mid k = 1, \cdots, 2m+1\}, \\
C_{2mh}(\mathbf{n}) &= \{\pm \mathbf{R}_{\mathbf{n}}^{k\pi/m} \mid k = 1, \cdots, 2m\}, \\
C_{2m+1}(\mathbf{n}) &= S_{4m+2}(\mathbf{n}) \cap \mathrm{Orth}^+, \qquad C_{2m}(\mathbf{n}) = C_{2mh}(\mathbf{n}) \cap \mathrm{Orth}^+;
\end{aligned}$$

$$S_{4m}(\mathbf{n}) = \{(-\mathbf{R}_{\mathbf{n}}^{\pi/2m})^k \mid k = 1, \cdots, 4m\},$$
$$C_{2m+1h}(\mathbf{n}) = \{(-\mathbf{R}_{\mathbf{n}}^{\pi/2m+1})^k \mid k = 1, \cdots, 4m+2\}.$$

Here and hereafter $\mathbf{n}$ is a given unit vector and $\mathbf{I}$ is the identity tensor. Henceforth, we shall quote the above subgroups by dropping out the defining vector $\mathbf{n}$ when no confusion arises.

For $\mathbf{u}, \mathbf{v}, \mathbf{r} \in V$ and $\mathbf{A} \in \mathrm{Sym}$, we introduce the following notations.

$$\begin{aligned} \overset{\circ}{\mathbf{u}} &= \mathbf{u} - (\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \\ \overset{\circ}{\mathbf{A}} &= \mathbf{A} - (\mathbf{n}\cdot\mathbf{A}\mathbf{n})\mathbf{n}\otimes\mathbf{n}; \end{aligned} \tag{1.7}$$

$$\mathbf{q}(\mathbf{A}) = \frac{1}{2}(\mathbf{e}\cdot\mathbf{A}\mathbf{e} - \mathbf{e}'\cdot\mathbf{A}\mathbf{e}')\mathbf{e} + (\mathbf{e}\cdot\mathbf{A}\mathbf{e}')\mathbf{e}'. \tag{1.8}$$

Throughout, $\mathbf{e}$ is any given unit vector in the $\mathbf{n}$-plane and

$$\mathbf{e}' = \mathbf{n} \times \mathbf{e}. \tag{1.9}$$

Hence, the triplet $(\mathbf{n}, \mathbf{e}, \mathbf{e}')$ is an orthonormal system. It is evident that $(1.7)_{1,2}$ define two linear functions of $\mathbf{u}$ and $\mathbf{A}$ that are form-invariant under the maximal transverse isotropy group $D_{\infty h}(\mathbf{n})$.

Moreover, we denote

$$\begin{aligned} \mathbf{u}\wedge\mathbf{v} &= \mathbf{u}\otimes\mathbf{v} - \mathbf{v}\otimes\mathbf{u}, \\ \mathbf{u}\vee\mathbf{v} &= \mathbf{u}\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{u}, \\ [\mathbf{u},\mathbf{v},\mathbf{r}] &= \mathbf{u}\cdot(\mathbf{v}\times\mathbf{r}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{r} = \mathbf{v}\cdot(\mathbf{r}\times\mathbf{u}), \end{aligned} \tag{1.10}$$

for vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{r}$. Throughout, $\mathbf{u}\cdot\mathbf{v}$ and $\mathbf{u}\times\mathbf{v}$ are used to designate the inner product and the vector product of $\mathbf{u}$ and $\mathbf{v}$, respectively.

Let $<\mathbf{u}, \mathbf{v}>$ denote the angle between two vectors $\mathbf{u}$ and $\mathbf{v}$. For any vector $\mathbf{z}$ in the $\mathbf{n}$-plane and for each positive integer r, the following notations are used:

$$\boldsymbol{\rho}_r(\mathbf{z}) = |\mathbf{z}|^r(\mathbf{e}\cos r <\mathbf{z},\mathbf{e}> - \mathbf{e}'\sin r <\mathbf{z},\mathbf{e}>); \tag{1.11}$$

$$\alpha_r(\mathbf{z}) = |\mathbf{z}|^r \cos r <\mathbf{z},\mathbf{e}>, \qquad \beta_r(\mathbf{z}) = |\mathbf{z}|^r \sin r <\mathbf{z},\mathbf{e}>. \tag{1.12}$$

When $\mathbf{z} = \mathbf{0}$, the angle $<\mathbf{z}, \mathbf{e}>$ is assumed to be zero. By means of the formulas

$$\begin{aligned} \cos <\mathbf{z},\mathbf{e}> &= \frac{\mathbf{z}\cdot\mathbf{e}}{|\mathbf{z}|}, \\ \sin <\mathbf{z},\mathbf{e}> &= \frac{\mathbf{z}\cdot\mathbf{e}'}{|\mathbf{z}|}, \end{aligned} \tag{1.13}$$

and Tschebysheff polynomials, it can easily be proved that each tensor function defined by (1.11) – (1.12) is a polynomial of the components $\mathbf{z}\cdot\mathbf{e}$ and $\mathbf{z}\cdot\mathbf{e}'$.

2. A unified scheme of constructing functional bases and generating sets

Usually, quite different methods are used to derive functional bases and generating sets separately, which are generally cumbersome. In this section we shall describe a simple, unified scheme for constructing both functional bases and generating sets. Such a scheme enables us to derive generating sets for general vector-valued and second order tensor-valued anisotropic functions only from those for the same types of vector-valued and second order tensor-valued anisotropic functions with not more than three variables (see XIAO [33]) (two variables for the anisotropic functions considered here; see XIAO [34]) and, especially, at the same time it enables us to obtain functional bases for scalar-valued anisotropic functions directly using the generating sets obtained for vector-valued and second order tensor-valued functions. For the sake of definiteness, in the subsequent account of such a unified scheme we shall consider only the anisotropy groups of interest in this paper.

Let g be any subgroup of $C_{\infty h}$. Henceforth, we denote the domain $V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$ by $\mathcal{D}$. For any given set of vectors and second order tensors, $X = (\mathbf{u}_1, \cdots, \mathbf{u}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{W}_c) \in \mathcal{D}$, by means of the following facts

$$g(\mathbf{W}) \cap C_{\infty h}(\mathbf{n}) = \begin{cases} C_{1h}(\mathbf{n}), & \mathbf{Wn} = \mathbf{0}\,, \\ C_1, & \mathbf{Wn} \neq \mathbf{0}\,, \end{cases} \tag{2.1}$$

$$g(\mathbf{A}) \cap C_{\infty h}(\mathbf{n}) = \begin{cases} C_{\infty h}(\mathbf{n}), & \mathbf{A} = x\mathbf{I} + y\mathbf{n}\otimes\mathbf{n}\,, \\ C_{2h}(\mathbf{n}), & \overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{0}, \quad \mathbf{q}(\mathbf{A}) \neq \mathbf{0}\,, \\ S_2, & \overset{\circ}{\mathbf{A}}\mathbf{n} \neq \mathbf{0}\,, \end{cases} \tag{2.2}$$

$$g(\mathbf{u}) \cap C_{\infty h}(\mathbf{n}) = \begin{cases} C_{\infty h}(\mathbf{n}), & \mathbf{u} = \mathbf{0}\,, \\ C_{\infty}(\mathbf{n}), & \mathbf{u} = x\mathbf{n}, \quad x \neq 0\,, \\ C_{1h}(\mathbf{n}), & \mathbf{u}\cdot\mathbf{n} = 0, \quad \mathbf{n}\times\mathbf{u} \neq \mathbf{0}\,, \\ C_1, & (\mathbf{u}\cdot\mathbf{n})(\mathbf{n}\times\mathbf{u}) \neq \mathbf{0}\,, \end{cases} \tag{2.3}$$

it can be proved (see §2.2 in XIAO [34]) that there are $X_0 \subset X$, where $X_0 \in \{(\mathbf{u}, \mathbf{v}), (\mathbf{u}, \mathbf{W}), (\mathbf{u}, \mathbf{A})\}$ is a subset of X with two elments, such that

$$g \cap g(X) = g \cap g(X_0)\,, \tag{2.4}$$

and accordingly, that irreducible generating sets for general anisotropic vector-valued and second order tensor-valued functions of the variables X relative to the group $g \subset C_{\infty h}$ can be formed by union of irreducible generating sets for the same types of anisotropic functions of not more than two variables (see Theorem 2.4 in XIAO [34]). Generally, the just-stated fact implies that if $G(\mathbf{u}, \mathbf{v})$, $G(\mathbf{u}, \mathbf{W})$ and $G(\mathbf{u}, \mathbf{A})$ are irreducible generating sets for vector-valued or skewsymmetric tensor-valued or symmetric tensor-valued anisotropic functions of the two variables $(\mathbf{u}, \mathbf{v})$, $(\mathbf{u}, \mathbf{W})$ and $(\mathbf{u}, \mathbf{A})$ relative to the subgroup $g \subset C_{\infty h}$ respectively, then the union

$$\bigcup_{i,j=1}^{a} \bigcup_{\alpha=1}^{b} \bigcup_{\sigma=1}^{c} (G(\mathbf{u}_i, \mathbf{u}_j) \cup G(\mathbf{u}_i, \mathbf{W}_\alpha) \cup G(\mathbf{u}_i, \mathbf{A}_\sigma))$$

furnishes an irreducible generating set for vector-valued or skewsymmetric tensor-valued or symmetric tensor-valued anisotropic functions of the vectors $\mathbf{u}_1, \cdots, \mathbf{u}_a$, the skewsymmetric tensors $\mathbf{W}_1, \cdots, \mathbf{W}_b$ and the symmetric tensors $\mathbf{A}_1, \cdots, \mathbf{A}_c$ relative to the subgroup $g \subset C_{\infty h}$. In the above, each generating set of two variables may be constructed by applying the related results for isotropic extension of anisotropic functions given in XIAO [35, 39]. Moreover, in constructing generating sets for one and two variables we can, as mentioned before, derive functional bases for general scalar-valued anisotropic functions of the variables X, directly utilizing the results for the generating sets obtained. To realize this goal the following fact is essential.

Let $X \in \mathcal{D}$ be a set of vectors and second order tensors with a proper subset X_0 satisfying (2.4). Moreover, let $I(X_0)$ and

$$V(X_0) = \{\mathbf{h}_1(X_0), \cdots, \mathbf{h}_r(X_0)\},$$
$$\mathrm{Skw}(X_0) = \{\mathbf{\Omega}_1(X_0), \cdots, \mathbf{\Omega}_s(X_0)\},$$
$$\mathrm{Sym}(X_0) = \{\mathbf{\Psi}_1(X_0), \cdots, \mathbf{\Psi}_t(X_0)\},$$

be a functional basis and generating sets for vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables X_0 relative to the group g, respectively. For a generic vector $\mathbf{r} \in X/X_0$, a generic skewsymmetric tensor $\mathbf{H} \in X/X_0$ and a generic symmetric tensor $\mathbf{B} \in X/X_0$, construct the following invariants under g:

$$\{\mathbf{h}_1(X_0)\cdot\mathbf{r}, \cdots, \mathbf{h}_r(X_0)\cdot\mathbf{r}\},$$
$$\{\mathrm{tr}\,\mathbf{\Omega}_1(X_0)\mathbf{H}, \cdots, \mathrm{tr}\,\mathbf{\Omega}_s(X_0)\mathbf{H}\},$$
$$\{\mathrm{tr}\,\mathbf{\Psi}_1(X_0)\mathbf{B}, \cdots, \mathrm{tr}\,\mathbf{\Psi}_t(X_0)\mathbf{B}\}.$$

The union of the above invariants for all $\mathbf{r}, \mathbf{H}, \mathbf{B} \in X/X_0$ is denoted by $I^0(X)$. Then X can be determined to within an orthogonal tensor pertaining to the group g by $I(X_0)$ and $I^0(X)$.

The proof is as follows. First, from (2.4) we deduce

$$g \cap g(X_0) \subset g(\mathbf{z}),\ \forall\, \mathbf{z} \in X\,. \tag{2.5}$$

By virtue of this and the obvious fact:

$$g_1 \subset g_2 \implies M(g_2) \subset M(g_1)$$

for any two subgroups $g_1, g_2 \subset \mathrm{Orth}$ and $M \in \{V, \mathrm{Skw}, \mathrm{Sym}\}$, we infer

$$\mathbf{r} \in V(g(\mathbf{r})) \subset V(g \cap g(X_0)) = \operatorname{span} V(X_0),$$

$$\mathbf{H} \in \mathrm{Skw}(g(\mathbf{H})) \subset \mathrm{Skw}(g \cap g(X_0)) = \operatorname{span} \mathrm{Skw}(X_0),$$

$$\mathbf{B} \in \mathrm{Sym}(g(\mathbf{B})) \subset \mathrm{Sym}(g \cap g(X_0)) = \operatorname{span} \mathrm{Sym}(X_0),$$

for any vector $\mathbf{r} \in X/X_0$, any skewsymmetric tensor $\mathbf{H} \in X/X_0$ and any symmetric tensor $\mathbf{B} \in X/X_0$, where the last equality in each of the above three expressions can be derived from (2.13), (2.16) and (2.15) in XIAO [33]. From the above three expressions we know that X/X_0 is determined by the union $I^0(X)$ indicated before if the three generating sets $V(X_0)$, $\mathrm{Skw}(X_0)$ and $\mathrm{Sym}(X_0)$ are known. On the other hand, X_0 is determined to within an orthogonal tensor $\mathbf{Q} \in g$ by a functional basis $I(X_0)$ of X_0 under g. Since each generator is form-invariant under g, we infer that the three generating sets just mentioned can be determined to within an orthogonal tensor $\mathbf{Q} \in g$ by the functional basis $I(X_0)$.

Combining the above facts we conclude that the aforementioned fact is true.

Owing to the fact proved above, in the process of constructing generating sets for vector-valued and second order tensor-valued functions of the two variables $X_0 \subset X$, we can obtain general functional bases of the variables X merely by forming the corresponding inner product between each generic variable $\mathbf{x} \in X/X_0$ and each presented generator and moreover, by constructing functional bases of the two variables $X_0 \subset X$.

The above process of constructing generating sets and functional bases may be simplified due to the fact that some or even all the three lists of two variables, $(\mathbf{u}, \mathbf{v})$, $(\mathbf{u}, \mathbf{W})$ and $(\mathbf{u}, \mathbf{A})$, may be further reduced to some lists of a single variable. Generally, simplification concerning each list X_0 of two variables is possible (see (2.6) below). Specifically, we design the following scheme.

1. For each variable $\mathbf{x} \in \{\mathbf{u}, \mathbf{W}, \mathbf{A}\}$, construct irreducible generating sets $V^0(\mathbf{x})$, $\mathrm{Skw}^0(\mathbf{x})$ and $\mathrm{Sym}^0(\mathbf{x})$ for vector-valued and skewsymmetric and symmetric tensor-valued functions, respectively. Then form the corresponding inner product between each generic variable $\mathbf{y} \in \{\mathbf{r}, \mathbf{H}, \mathbf{B}\}$ and each presented vector, skewsymmetric tensor, and symmetric tensor generator and moreover, construct an irreducible functional basis of the single variable $\mathbf{x}$.

2. For each list $X_0 = (\mathbf{u}, \mathbf{x})$ of two variables, where $\mathbf{x} \in \{\mathbf{v}, \mathbf{W}, \mathbf{A}\}$, consider the union $G(\mathbf{u}) \cup G(\mathbf{x})$, where $G(\mathbf{u})$ and $G(\mathbf{x})$ are the generating sets for vector-, skewsymmetric tensor-, and symmetric tensor-valued functions of the single variables $\mathbf{u}$ and $\mathbf{x}$ respectively, constructed in the first step. This union obeys the criterion (1.1) for the cases

$$g(\mathbf{z}) \cap g = g(X_0) \cap g, \qquad \mathbf{z} \in \{\mathbf{u}, \mathbf{x}\}.$$

Thus, it suffices to treat the case other than the above cases, which is specified by the conditions

$$g(\mathbf{z}) \cap g \neq g(\mathbf{u}, \mathbf{x}) \cap g, \qquad \mathbf{z} = \mathbf{u}, \mathbf{x}\,. \tag{2.6}$$

Then, analyze the latter case and judge whether or not the aforementioned union also obeys the criterion (1.1) for the latter case. If not, then add some generators with two variables $\mathbf{u}$ and $\mathbf{x}$ into this union so that an irreducible generating set for the two variables $(\mathbf{u}, \mathbf{x})$ is formed. If yes, then the aforementioned union is already the desired result. Moreover, form the corresponding inner product between each variable $\mathbf{y} \in \{\mathbf{r}, \mathbf{H}, \mathbf{B}\}$ and each presented vector, skewsymmetric tensor, and symmetric tensor generator with two variables (the generators with a single variable have been covered in the first step) and moreover, construct an irreducible functional basis of the two variables $(\mathbf{u}, \mathbf{x})$ specified by (2.6).

3. Combine all vector generators, skewsymmetric tensor generators, symmetric tensor generators and invariants obtained, respectively, and let each generic variable concerned run over the set X. Then the desired general irreducible representations are available (see Theorem 2.4 in XIAO [34]).

In the above procedures, the related results for isotropic extension of anisotropic functions given in XIAO [35, 39] may be applied to construct irreducible generating sets for one or two variables. Furthermore, according to Theorem 3.7 in XIAO [34], for skewsymmetric and symmetric tensor-valued functions we need only to consider two vector variables $(\mathbf{u}, \mathbf{v})$, a single skewsymmetric tensor variable $\mathbf{W}$, and a single symmetric tensor variable $\mathbf{A}$, respectively. As to the scalar-valued and vector-valued functions of two variables X_0, the condition (2.6) usually leads to such X_0 that construction of functional bases and generating sets for X_0 can be considerably simplified.

Following the above scheme, in the succeeding sections we shall construct general irreducible representations for anisotropic functions of the variables $X \in \mathcal{D}$ under all subgroups $g \in C_{\infty h}$. In view of the fact stated above, in the second step above we shall omit generating sets for skewsymmetric and symmetric tensor-valued functions as well as the invariants obtained by forming the corresponding inner product between each variable $\mathbf{x} \in \{\mathbf{r}, \mathbf{H}, \mathbf{B}\}$ and each generator with one variable.

Finally, it should be pointed out that the above construction scheme in itself does not mean the irreducibility of each invariant obtained by means of the inner product. As a result, additiontial proof for the latter is needed. To this end, one may contruct the pair (X, X') fulfilling (1.6).

3. The triclinic and monoclinic crystal classes C_1, S_2 and C_{1h}

3.1. The triclinic group C_1

Henceforth, $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an orthonormal basis of V. Trivially,

V	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$
Skw	$\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_3 \wedge \mathbf{e}_1$
Sym	$\mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_2 \otimes \mathbf{e}_2, \mathbf{e}_3 \otimes \mathbf{e}_3, \mathbf{e}_1 \vee \mathbf{e}_2, \mathbf{e}_2 \vee \mathbf{e}_3, \mathbf{e}_3 \vee \mathbf{e}_1$
R	$\mathbf{r}\cdot\mathbf{e}_i;\ \mathbf{e}_i\cdot\mathbf{H}\mathbf{e}_j;\ \mathbf{e}_i\cdot\mathbf{B}\mathbf{e}_j;\ i,j = 1,2,3,$

where $\mathbf{r} = \mathbf{u}_1, \cdots, \mathbf{u}_a$, $\mathbf{H} = \mathbf{W}_1, \cdots, \mathbf{W}_b$ and $\mathbf{B} = \mathbf{A}_1, \cdots, \mathbf{A}_c$, provide irreducible representations for scalar-. vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables $X \in \mathcal{D}$ relative to the triclinic group C_1.

3.2. The triclinic group S_2

It is evident that for any $X \in \mathcal{D}$ there is a vector $\mathbf{u} \in X$ such that

$$g(X) \cap S_2 = g(\mathbf{u}) \cap S_2 \,.$$

Accordingly we construct the following table.

V	$\mathbf{u}, \mathbf{u}\times\mathbf{e}_1, \mathbf{u}\times\mathbf{e}_2, \mathbf{u}\times\mathbf{e}_3$
Skw	$\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_3 \wedge \mathbf{e}_1$
Sym	$\mathbf{e}_1 \otimes \mathbf{e}_1, \mathbf{e}_2 \otimes \mathbf{e}_2, \mathbf{e}_3 \otimes \mathbf{e}_3, \mathbf{e}_1 \vee \mathbf{e}_2, \mathbf{e}_2 \vee \mathbf{e}_3, \mathbf{e}_3 \vee \mathbf{e}_1$
R	$\mathbf{u}\cdot\mathbf{r}, [\mathbf{u}, \mathbf{r}, \mathbf{e}_i];\ \mathbf{e}_i\cdot\mathbf{H}\mathbf{e}_j;\ \mathbf{e}_i\cdot\mathbf{B}\mathbf{e}_j;\ <(\mathbf{u}\cdot\mathbf{e}_i)(\mathbf{u}\cdot\mathbf{e}_j)>;\ i,j = 1,2,3$

Here and hereafter the invariants in the angle brackets supply an irreducible functional basis for one variable or two variables under consideration. The other invarants are obtained by forming the corresponding inner product between each generator given and a variable $\mathbf{x} \in \{\mathbf{r}, \mathbf{H}, \mathbf{B}\}$.

Thus, we obtain the main result of this subsection as follows.

THEOREM 3.1. *The table given above, together with* $\mathbf{u}, \mathbf{r} = \mathbf{u}_1, \cdots, \mathbf{u}_a, \mathbf{u} \neq \mathbf{r}$; $\mathbf{H} = \mathbf{W}_1, \cdots, \mathbf{W}_b$; $\mathbf{B} = \mathbf{A}_1, \cdots, \mathbf{A}_c$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X \in \mathcal{D}$ *under the triclinic group* S_2.

REMARK. An irreducible functional basis of two vector variables under S_2 was derived by LIU [11], in which 18 invariants are employed. Here only 16 invariants are used.

3.3. The monoclinic crystal class C_{1h}

Since C_{1h} has only two subgroups, i.e. C_1 and C_{1h}, we infer that for any $X \in \mathcal{D}$ there exists a single vector or second order tensor $\mathbf{x} \in X$ such that

$$g(X) \cap C_{1h}(\mathbf{n}) = g(\mathbf{x}) \cap C_{1h}(\mathbf{n}).$$

Consequently, it suffices to treat the cases for a single variable. As mentioned before, $\mathbf{e}$ and $\mathbf{e}'$ are two orthonormal vectors in the $\mathbf{n}$-plane. Henceforth we denote

$$\mathbf{N} = \mathbf{e} \wedge \mathbf{e}' = \mathbf{E}\mathbf{n}\,. \tag{3.1}$$

Throughout, $\mathbf{E}$ is used to denote the third-order Eddington tensor, i.e. the permutation tensor. It is evident that $\mathbf{N}$ is invariant under $C_{\infty h}(\mathbf{n})$ and independent of the choice of the orthonornormal vectors $\mathbf{e}$ and $\mathbf{e}'$ in the $\mathbf{n}$-plane.

CASE 1. A single vector variable $\mathbf{u}$

V	$\mathbf{e}, \mathbf{e}', (\mathbf{u}\cdot\mathbf{n})\mathbf{n}$
Skw	$\mathbf{N}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\mathbf{e}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\mathbf{e}'$
Sym	$\mathbf{n}\otimes\mathbf{n}, \mathbf{e}\otimes\mathbf{e}, \mathbf{e}'\otimes\mathbf{e}', \mathbf{e}\vee\mathbf{e}', (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\mathbf{e}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\mathbf{e}'$
R	$\mathbf{r}\cdot\mathbf{e}, \mathbf{r}\cdot\mathbf{e}', (\mathbf{r}\cdot\mathbf{n})(\mathbf{u}\cdot\mathbf{n})$; $\operatorname{tr}\mathbf{HN}, (\mathbf{u}\cdot\mathbf{n})(\mathbf{n}\cdot\mathbf{He}), (\mathbf{u}\cdot\mathbf{n})(\mathbf{n}\cdot\mathbf{He}')$; $\mathbf{n}\cdot\mathbf{Bn}, \mathbf{e}\cdot\mathbf{Be}, \mathbf{e}'\cdot\mathbf{Be}', \mathbf{e}\cdot\mathbf{Be}', (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{Be}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{Be}'$; $<\mathbf{u}\cdot\mathbf{e}, \mathbf{u}\cdot\mathbf{e}', (\mathbf{u}\cdot\mathbf{n})^2>$.

CASE 2. A single skewsymmetric tensor variable $\mathbf{W}$

V	$\mathbf{e}, \mathbf{e}', (\mathbf{n}\cdot\mathbf{We})\mathbf{n}, (\mathbf{n}\cdot\mathbf{We}')\mathbf{n}$
Skw	$\mathbf{N}, \mathbf{n}\wedge\mathbf{Wn}, \mathbf{n}\wedge(\mathbf{n}\times\mathbf{Wn})$
Sym	$\{\mathbf{n}\otimes\mathbf{n}, \mathbf{e}\otimes\mathbf{e}, \mathbf{e}'\otimes\mathbf{e}', \mathbf{e}\vee\mathbf{e}', \mathbf{n}\vee\mathbf{Wn}, \mathbf{n}\vee(\mathbf{n}\times\mathbf{Wn})\}$ $(= \mathrm{Sym}_0(\mathbf{W}))$
R	$\mathbf{r}\cdot\mathbf{e}, \mathbf{r}\cdot\mathbf{e}', (\mathbf{r}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{We}, (\mathbf{r}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{We}'$; $\operatorname{tr}\mathbf{HN}, \mathbf{n}\cdot\mathbf{WHn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{Hn}]$; $\mathbf{n}\cdot\mathbf{Bn}, \mathbf{e}\cdot\mathbf{Be}, \mathbf{e}'\cdot\mathbf{Be}', \mathbf{e}\cdot\mathbf{Be}', \mathbf{n}\cdot\mathbf{WBn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{Bn}]$; $<\operatorname{tr}\mathbf{WN}, (\mathbf{n}\cdot\mathbf{We})^2, (\mathbf{n}\cdot\mathbf{We}')^2, (\mathbf{n}\cdot\mathbf{We})(\mathbf{n}\cdot\mathbf{We}')>$ $(= I_0(\mathbf{W}))$.

It can be readily verified that the first three sets given above are irreducible generating sets for vector-valued, skewsymmetric and symmetric tensor-valued anisotropic functions of the skewsymmetric tensor variable $\mathbf{W}\in\mathrm{Skw}$ relative to $C_{1h}(\mathbf{n})$, respectively, by means of (1.2) – (1.4) and the fact

$$g(\mathbf{W})\cap C_{1h}(\mathbf{n}) = \begin{cases} C_{1h}(\mathbf{n}), & \mathbf{Wn}=\mathbf{0}, \\ C_1, & \mathbf{Wn}\neq\mathbf{0}. \end{cases}$$

Moreover, by means of criterion (1.5) it can be proved easily that the four invariants of $\mathbf{W}$ listed in the angle brackets form an irreducible functional basis of the variable $\mathbf{W}\in\mathrm{Skw}$ under $C_{1h}(\mathbf{n})$.

CASE 3. A single symmetric tensor variable $\mathbf{A}$

V	$\mathbf{e}, \mathbf{e}', (\mathbf{n}\cdot\mathbf{Ae})\mathbf{n}, (\mathbf{n}\cdot\mathbf{Ae}')\mathbf{n}$
Skw	$\mathbf{N}, \mathbf{n}\wedge\mathbf{An}, \mathbf{n}\wedge(\mathbf{n}\times\mathbf{An})$
Sym	$\{\mathbf{n}\otimes\mathbf{n}, \mathbf{e}\otimes\mathbf{e}, \mathbf{e}'\otimes\mathbf{e}', \mathbf{e}\vee\mathbf{e}', \mathbf{n}\vee\mathbf{An}, \mathbf{n}\vee(\mathbf{n}\times\mathbf{An})\}$ $(= \mathrm{Sym}_0(\mathbf{A}))$
R	$\mathbf{r}\cdot\mathbf{e}, \mathbf{r}\cdot\mathbf{e}', (\mathbf{r}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{Ae}, (\mathbf{r}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{Ae}'$; $\operatorname{tr}\mathbf{HN}, \mathbf{n}\cdot\mathbf{HAn}, [\mathbf{n}, \mathbf{An}, \mathbf{Hn}]$; $\mathbf{n}\cdot\mathbf{Bn}, \mathbf{e}\cdot\mathbf{Be}, \mathbf{e}'\cdot\mathbf{Be}', \mathbf{e}\cdot\mathbf{Be}', \mathbf{n}\cdot\mathbf{ABn}, [\mathbf{n}, \mathbf{An}, \mathbf{Bn}]$; $<\mathbf{n}\cdot\mathbf{An}, \mathbf{e}\cdot\mathbf{Ae}, \mathbf{e}'\cdot\mathbf{Ae}', \mathbf{e}\cdot\mathbf{Ae}', (\mathbf{n}\cdot\mathbf{Ae})^2, (\mathbf{n}\cdot\mathbf{Ae}')^2,$ $(\mathbf{n}\cdot\mathbf{Ae})(\mathbf{n}\cdot\mathbf{Ae}')>$ $(= I_0(\mathbf{A}))$.

It can be readily proved that the first three sets given above are irreducible generating sets for vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the symmetric tensor variable $\mathbf{A} \in \mathrm{Sym}$ relative to $C_{1h}(\mathbf{n})$, respectively, by means of the criterion (1.1), (1.2) – (1.4) and the fact

$$g(\mathbf{A}) \cap C_{1h}(\mathbf{n}) = \begin{cases} C_{1h}(\mathbf{n}), & \mathring{\mathbf{A}}\mathbf{n} = \mathbf{0}\,, \\ C_1, & \mathring{\mathbf{A}}\mathbf{n} \neq \mathbf{0}\,. \end{cases}$$

Moreover, the given irreducible functional basis of the symmetric tensor variable $\mathbf{A} \in \mathrm{Sym}$ under $C_{1h}(\mathbf{n})$ can be found in XIAO [36].

Henceforth, we denote the irreducible functional bases and generating sets for scalar-valued and symmetric tensor-valued functions of the single variables $\mathbf{W}$ and $\mathbf{A}$, given in the two tables for Case 2 and Case 3 respectively, by $I_0(\mathbf{W})$, $I_0(\mathbf{A})$, $\mathrm{Sym}_0(\mathbf{W})$, and $\mathrm{Sym}_0(\mathbf{A})$, respectively.

Combining the above three cases, we arrive at the main result of this subsection as follows.

THEOREM 3.2. *The four sets given by*

$I_0(\mathbf{W}); I_0(\mathbf{A}); \mathbf{u}\cdot\mathbf{e}, \mathbf{u}\cdot\mathbf{e}', (\mathbf{u}\cdot\mathbf{n})^2;$
$(\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n});$
$(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{W}\mathbf{e}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{W}\mathbf{e}';$
$(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{A}\mathbf{e}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\cdot\mathbf{A}\mathbf{e}';$
$\mathbf{n}\cdot\mathbf{W}\mathbf{H}\mathbf{n}, [\mathbf{n}, \mathbf{W}\mathbf{n}, \mathbf{H}\mathbf{n}];$
$\mathbf{n}\cdot\mathbf{A}\mathbf{B}\mathbf{n}, [\mathbf{n}, \mathbf{A}\mathbf{n}, \mathbf{B}\mathbf{n}];$
$\mathbf{n}\cdot\mathbf{A}\mathbf{W}\mathbf{n}, [\mathbf{n}, \mathbf{W}\mathbf{n}, \mathbf{A}\mathbf{n}];$

and

$\mathbf{e}, \mathbf{e}', (\mathbf{u}\cdot\mathbf{n})\mathbf{n};$
$(\mathbf{n}\cdot\mathbf{W}\mathbf{e})\mathbf{n}, (\mathbf{n}\cdot\mathbf{W}\mathbf{e}')\mathbf{n};$
$(\mathbf{n}\cdot\mathbf{A}\mathbf{e})\mathbf{n}, (\mathbf{n}\cdot\mathbf{A}\mathbf{e}')\mathbf{n};$

and

$\mathbf{N}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\mathbf{e}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\mathbf{e}';$
$\mathbf{n}\wedge\mathbf{W}\mathbf{n}, \mathbf{n}\wedge(\mathbf{n}\times\mathbf{W}\mathbf{n});$
$\mathbf{n}\wedge\mathbf{A}\mathbf{n}, \mathbf{n}\wedge(\mathbf{n}\times\mathbf{A}\mathbf{n});$

and

$\mathrm{Sym}_0(\mathbf{W}), \mathrm{Sym}_0(\mathbf{A}), (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\mathbf{e}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\mathbf{e}';$

where $\mathbf{u}, \mathbf{v} = \mathbf{u}_1, \cdots, \mathbf{u}_a$; $\mathbf{W}, \mathbf{H} = \mathbf{W}_1, \cdots, \mathbf{W}_b$; $\mathbf{A}, \mathbf{B} = \mathbf{A}_1, \cdots, \mathbf{A}_c$; $\mathbf{u} \neq \mathbf{v}$, $\mathbf{W} \neq \mathbf{H}$, $\mathbf{A} \neq \mathbf{B}$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X \in \mathcal{D}$ *under the monoclinic group* $C_{1h}(\mathbf{n})$, *respectively.*

4. The classes C_{2mh} and C_{2m}

The classes C_{2mh} and C_{2m} include the monoclinic crystal classes C_{2h} and C_2, the tetrahedral crystal classes C_{4h} and C_4, and the hexagonal crystal classes

C_{6h} and C_6 as the particular cases when $m = 1, 2, 3$. Moreover, the transverse isotropy groups $C_{\infty h}$ and C_∞ can be treated as the particular case of the classes at issue when $m = \infty$. It should be pointed out that the former can not be regarded as the particular cases of the classes $S_{4m+2}(\mathbf{n})$ and $C_{2m+1}(\mathbf{n})$ when $m = \infty$, since the element $\mathbf{R}_{\mathbf{n}}^{\pi}$ is not contained in either of the latter.

4.1. The classes C_{2mh}

Six cases will be discussed.

CASE 1. A single vector variable $\mathbf{u}$

From the criterion (1.1) and the formula (1.2) – (1.4) as well as

$$(4.1)\qquad g(\mathbf{u}) \cap C_{2mh}(\mathbf{n}) = \begin{cases} C_{2mh}(\mathbf{n}), & \mathbf{u} = \mathbf{0}, \\ C_{2m}(\mathbf{n}), & \overset{\circ}{\mathbf{u}} = \mathbf{0}, \quad \mathbf{u}\cdot\mathbf{n} \neq 0, \\ C_{1h}(\mathbf{n}), & \mathbf{u}\cdot\mathbf{n} = 0, \quad \overset{\circ}{\mathbf{u}} \neq \mathbf{0}, \\ C_1, & (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}} \neq \mathbf{0}, \end{cases}$$

$$(4.2)\qquad g(\mathbf{u}) \cap g(\mathbf{N}) = \begin{cases} C_{\infty h}(\mathbf{n}), & \mathbf{u} = \mathbf{0}, \\ C_{\infty}(\mathbf{n}), & \overset{\circ}{\mathbf{u}} = \mathbf{0}, \quad \mathbf{u}\cdot\mathbf{n} \neq 0, \\ C_{1h}(\mathbf{n}), & \mathbf{u}\cdot\mathbf{n} = 0, \quad \overset{\circ}{\mathbf{u}} \neq \mathbf{0}, \\ C_1, & (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}} \neq \mathbf{0}, \end{cases}$$

we deduce that the following fact holds: generating sets for vector-valued (for $m \geq 1$), skewsymmetric tensor-valued (for $m \geq 1$) and symmetric tensor-valued (for $m \geq 2$) anisotropic functions of the vector variable $\mathbf{u}$ under $C_{2mh}(\mathbf{n})$ can be derived from those for vector-, skewsymmetric and symmetric tensor-valued isotropic functions of the extended variables $(\mathbf{u}, \mathbf{N})$, respectively. Moreover, with the aid of the fact that $(-\mathbf{I})\mathbf{u} = -\mathbf{u}$, $-\mathbf{I} \in C_{2mh}(\mathbf{n})$ and the invariance condition stated at the start of the introduction, it may be readily understood that functional bases for scalar-valued anisotropic functions of the vector variable $\mathbf{u}$ under $C_{2mh}(\mathbf{n})$ can be obtained from those for scalar-valued anisotropic functions of the symmetric tensor variable $\mathbf{u} \otimes \mathbf{u} \in \mathrm{Sym}$ under $C_{2mh}(\mathbf{n})$. The latter can be found in XIAO [36] for $m = 1$ and in Case 3 below for $m \geq 2$.

Applying the above facts, we construct the following table:

$V \qquad \{\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}} \times \mathbf{n}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\}\ (= V^0(\mathbf{u}))$

$\mathrm{Skw} \qquad \{\mathbf{N}, (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}} \wedge \mathbf{n}, (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}} \wedge (\overset{\circ}{\mathbf{u}} \times \mathbf{n})\}\ (= \mathrm{Skw}^0(\mathbf{u}))$

$\mathrm{Sym} \qquad \{\mathbf{n} \otimes \mathbf{n}, \delta_{1m}\mathbf{e} \otimes \mathbf{e}, \delta_{1m}\mathbf{e}' \otimes \mathbf{e}', \delta_{1m}\mathbf{e} \vee \mathbf{e}', (1-\delta_{1m})\mathbf{I}, (1-\delta_{1m})\overset{\circ}{\mathbf{u}} \otimes \overset{\circ}{\mathbf{u}},$
$(1-\delta_{1m})\overset{\circ}{\mathbf{u}} \vee (\overset{\circ}{\mathbf{u}} \times \mathbf{n}); (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}} \vee \mathbf{n}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \vee (\overset{\circ}{\mathbf{u}} \times \mathbf{n})\}\ (= \mathrm{Sym}^0_m(\mathbf{u}))$

$$
\begin{array}{ll}
R & \mathring{\mathbf{u}}\cdot\mathring{\mathbf{r}}, [\mathbf{n}, \mathring{\mathbf{u}}, \mathring{\mathbf{r}}], (\mathbf{u}\cdot\mathbf{n})(\mathbf{r}\cdot\mathbf{n}); \\
& \operatorname{tr}\mathbf{HN}, (\mathbf{u}\cdot\mathbf{n})\mathring{\mathbf{u}}\cdot\mathbf{Hn}, (\mathbf{u}\cdot\mathbf{n})[\mathbf{n}, \mathring{\mathbf{u}}, \mathbf{Hn}]; \\
& \mathbf{n}\cdot\mathbf{Bn}, \delta_{1m}\mathbf{e}\cdot\mathbf{Be}, \delta_{1m}\mathbf{e}'\cdot\mathbf{Be}', \delta_{1m}\mathbf{e}\cdot\mathbf{Be}', (1-\delta_{1m})\operatorname{tr}\mathbf{B}, \\
& (1-\delta_{1m})\mathring{\mathbf{u}}\cdot\mathbf{B}\mathring{\mathbf{u}}, (1-\delta_{1m})[\mathbf{n}, \mathring{\mathbf{u}}, \mathbf{B}\mathring{\mathbf{u}}], (\mathbf{u}\cdot\mathbf{n})\mathring{\mathbf{u}}\cdot\mathbf{Bn}, (\mathbf{u}\cdot\mathbf{n})[\mathbf{n}, \mathbf{Bn}, \mathring{\mathbf{u}}]; \\
& <(\mathbf{u}\cdot\mathbf{n})^2, \delta_{1m}(\mathbf{u}\cdot\mathbf{e})^2, \delta_{1m}(\mathbf{u}\cdot\mathbf{e}')^2, \delta_{1m}(\mathbf{u}\cdot\mathbf{e})(\mathbf{u}\cdot\mathbf{e}'), \\
& (1-\delta_{1m})\alpha_{2m}(\mathring{\mathbf{u}}), (1-\delta_{1m})\beta_{2m}(\mathring{\mathbf{u}})> \ (= I_m^0(\mathbf{u})).
\end{array}
$$

Henceforth, the first three sets consisting of vector generators, skewsymmetric tensor generators and symmetric tensor generators, respectively, are denoted by $V^0(\mathbf{u})$, $\operatorname{Skw}^0(\mathbf{u})$, and $\operatorname{Sym}_m^0(\mathbf{u})$ respectively. We have $\operatorname{Sym}_1^0 = \operatorname{Sym}_0(\mathbf{u})$ for $m = 1$ and

$$
\begin{aligned}
(4.3)\qquad \operatorname{Sym}_m^0(\mathbf{u}) = \{\mathbf{I},\ \mathbf{n}\otimes\mathbf{n},\ \mathring{\mathbf{u}}\otimes\mathring{\mathbf{u}},\ \mathring{\mathbf{u}}\vee(\mathbf{n}\times\mathring{\mathbf{u}}),& \\
(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\mathring{\mathbf{u}},\ (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee(\mathbf{n}\times\mathring{\mathbf{u}})\} &\equiv \operatorname{Sym}^0(\mathbf{u})
\end{aligned}
$$

for $m \geq 2$. Moreover, the set of invariants given in the angle brackets is signified by $I_m^0(\mathbf{u})$.

It should be pointed out that in the above table, each element with the coefficients δ_{1m} or $(1-\delta_{1m})$ comes into play only when $m = 1$ or $m \geq 2$. Here and hereafter δ_{rs} is used to represent the Kronecker delta. Such a difference between C_{2h} and C_{2mh} for $m \geq 2$, which will also appear in the next five cases, arises from the fact

$$
\dim \operatorname{Sym}(C_{2mh}) = \begin{cases} 4, & m = 1, \\ 2, & \dim\operatorname{Sym}(C_{2mh}) = 2, \quad m \geq 2. \end{cases}
$$

CASE 2. A single skewsymmetric tensor **W**

Every vector-valued function of the variable $\mathbf{W} \in \operatorname{Skw}$ that is form-invariant under C_{2mh} vanishes. Moreover, from the criterion (1.1) and the formula (1.3)–(1.4) as well as

$$
(4.4)\qquad g(\mathbf{W}) \cap C_{2mh}(\mathbf{n}) = \begin{cases} C_{2mh}(\mathbf{n}), & \mathbf{Wn} = \mathbf{0}, \\ S_2, & \mathbf{Wn} \neq \mathbf{0}, \end{cases}
$$

$$
(4.5)\qquad g(\mathbf{W}) \cap g(\mathbf{N}) = \begin{cases} C_{\infty h}(\mathbf{n}), & \mathbf{Wn} = \mathbf{0}, \\ S_2, & \mathbf{Wn} \neq \mathbf{0}, \end{cases}
$$

we infer that the following fact holds: generating sets for skewsymmetric tensor-valued (for $m \geq 1$) and symmetric tensor-valued (for $m \geq 2$) anisotropic functions of the variable $\mathbf{W} \in \operatorname{Skw}$ under $C_{2mh}(\mathbf{n})$ can be derived from those for skewsymmetric and symmetric tensor-valued isotropic functions of the extended variables $(\mathbf{W}, \mathbf{N})$. Thus, we construct the following table.

Skw	$\{\mathbf{N}, \mathbf{n}\wedge\mathbf{Wn}, \mathbf{n}\wedge(\mathbf{n}\times\mathbf{Wn})\}$ $(=\mathrm{Skw}^0(\mathbf{W}))$
Sym	$\{\mathbf{n}\otimes\mathbf{n}, \delta_{1m}\mathbf{e}\otimes\mathbf{e}, \delta_{1m}\mathbf{e}'\otimes\mathbf{e}', \delta_{1m}\mathbf{e}\vee\mathbf{e}', (1-\delta_{1m})\mathbf{I}, (1-\delta_{1m})\mathbf{Wn}\otimes\mathbf{Wn},$ $(1-\delta_{1m})\mathbf{Wn}\vee(\mathbf{n}\times\mathbf{Wn}), \mathbf{n}\vee\mathbf{Wn}, \mathbf{n}\vee(\mathbf{n}\times\mathbf{Wn})\}$ $(=\mathrm{Sym}^0_m(\mathbf{W}))$
R	$\mathrm{tr}\,\mathbf{HN}, \mathbf{n}\cdot\mathbf{WHn}, [\mathbf{n},\mathbf{Wn},\mathbf{Hn}]$; $\mathbf{n}\cdot\mathbf{Bn}, \delta_{1m}\mathbf{e}\cdot\mathbf{Be}, \delta_{1m}\mathbf{e}'\cdot\mathbf{Be}', \delta_{1m}\mathbf{e}\cdot\mathbf{Be}',$ $(1-\delta_{1m})\mathbf{n}\cdot\mathbf{WBWn}, (1-\delta_{1m})[\mathbf{n},\mathbf{Wn},\mathbf{WBn}], \mathbf{n}\cdot\mathbf{WBn}, [\mathbf{n},\mathbf{Wn},\mathbf{Bn}]$; $<\mathrm{tr}\,\mathbf{WN}, \delta_{1m}(\mathbf{n}\cdot\mathbf{We})^2, \delta_{1m}(\mathbf{n}\cdot\mathbf{We}')^2, \delta_{1m}(\mathbf{n}\cdot\mathbf{We})(\mathbf{n}\cdot\mathbf{We}'),$ $(1-\delta_{1m})\alpha_{2m}(\mathbf{Wn}), (1-\delta_{1m})\beta_{2m}(\mathbf{Wn})>$ $(=I^0_m(\mathbf{W}))$.

Henceforth, the two generating sets in the above table are denoted by $\mathrm{Skw}^0(\mathbf{W})$ and $\mathrm{Sym}^0_m(\mathbf{W})$ and the functional basis given in the angle brackets is denoted by $I^0_m(\mathbf{W})$.

We have $\mathrm{Sym}^0_1(\mathbf{W}) = \mathrm{Sym}_0(\mathbf{W})$ and

$$\mathrm{Sym}^0_m(\mathbf{W}) = \{\mathbf{I},\ \mathbf{n}\otimes\mathbf{n},\ \mathbf{Wn}\otimes\mathbf{Wn},\ \mathbf{Wn}\vee(\mathbf{n}\times\mathbf{Wn}),\ \mathbf{n}\vee\mathbf{Wn},\ \mathbf{n}\vee(\mathbf{n}\times\mathbf{Wn})\} \equiv \mathrm{Sym}^0(\mathbf{W}), \tag{4.6}$$

for each $m \geq 2$.

We need only to show that the set $I^0_m(\mathbf{W})$ is an irreducible functional basis of $\mathbf{W}$ under $C_{2mh}(\mathbf{n})$. To this end, we prove that $I^0_m(\mathbf{W})$ obeys the criterion (1.5). The proof for $m = 1$ is easy. Let $m \geq 2$. Observing the fact that the last three invariants in the above table form a functional basis of the vector $\mathbf{Wn}$ in the $\mathbf{n}$-plane (see Case 1), we infer that for $\overline{\mathbf{W}}, \mathbf{W} \in \mathrm{Skw}$,

$$I^0_m(\overline{\mathbf{W}}) = I^0_m(\mathbf{W}) \implies \exists\, \mathbf{Q} \in C_{2mh}(\mathbf{n}) :\ \overline{\mathbf{W}}\mathbf{n} = \mathbf{Q}(\mathbf{Wn}),\ \mathrm{tr}\,\overline{\mathbf{W}}\mathbf{N} = \mathrm{tr}\,\mathbf{WN}.$$

In the above, we can assume $\mathbf{Q} \in C_{2m}(\mathbf{n})$, since $\mathbf{Wn}$ lies on the $\mathbf{n}$-plane. Thus, by means of the identity

$$\mathbf{W} = \frac{1}{2}(\mathrm{tr}\,\mathbf{WN})\mathbf{N} + (\mathbf{Wn})\wedge\mathbf{n} \tag{4.7}$$

as well as the facts: $\mathbf{QNQ}^{\mathrm{T}} = \mathbf{N}$, $\mathbf{Qn} = \mathbf{n}$ for every $\mathbf{Q} \in C_{2m}(\mathbf{n})$, we deduce

$$I^0_m(\overline{\mathbf{W}}) = I^0_m(\mathbf{W}) \implies \exists\, \mathbf{Q} \in C_{2m}(\mathbf{n}) :\ \overline{\mathbf{W}} = \mathbf{QWQ}^{\mathrm{T}}.$$

Thus, we conclude that $I^0_m(\mathbf{W})$ obeys the criterion (1.5). The irreducibility of this basis is evident.

CASE 3. A single symmetric tensor variable $\mathbf{A}$

Every vector-valued function of the variable $\mathbf{A} \in \mathrm{Sym}$ that is form-invariant under $C_{2mh}(\mathbf{n})$ vanishes. Moreover, from the facts

$$g(\mathbf{A}) \cap C_{2mh}(\mathbf{n}) = \begin{cases} C_{2mh}(\mathbf{n}), & \mathbf{A} = x\mathbf{I} + y\mathbf{n}\otimes\mathbf{n}, \\ C_{2h}(\mathbf{n}), & \overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{0}, \quad \mathbf{q}(\mathbf{A}) \neq \mathbf{0}, \\ S_2, & \overset{\circ}{\mathbf{A}}\mathbf{n} \neq \mathbf{0}, \end{cases} \tag{4.8}$$

$$(4.9)\qquad g(\mathbf{A})\cap g(\mathbf{N}) = \begin{cases} C_{\infty h}(\mathbf{n}), & \mathbf{A} = x\mathbf{I} + y\mathbf{n}\otimes\mathbf{n}\,, \\ C_{2h}(\mathbf{n}), & \overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{0}, \quad \mathbf{q}(\mathbf{A}) \neq \mathbf{0}\,, \\ S_2, & \overset{\circ}{\mathbf{A}}\mathbf{n} \neq \mathbf{0}\,, \end{cases}$$

and the criterion (1.1) as well as the formula (1.3) – (1.4), we infer that the following fact holds: generating sets for skewsymmetric tensor-valued (for $m \geq 1$) and symmetric tensor-valued (for $m \geq 2$) anisotropic functions of the variable $\mathbf{A} \in \mathrm{Sym}$ under $C_{2mh}(\mathbf{n})$ can be derived from those for skewsymmetric and symmetric tensor-valued isotropic functions of the extended variables $(\mathbf{A}, \mathbf{N})$. Using these facts, we construct the following table:

Skw	$\{\mathbf{N}, \mathbf{n}\wedge\mathbf{A}\mathbf{n}, \mathbf{n}\wedge(\mathbf{n}\times\mathbf{A}\mathbf{n})\}$ $(= \mathrm{Skw}^0(\mathbf{A}))$
Sym	$\{\mathbf{n}\otimes\mathbf{n}, \delta_{1m}\mathbf{e}\otimes\mathbf{e}, \delta_{1m}\mathbf{e}'\otimes\mathbf{e}', \delta_{1m}\mathbf{e}\vee\mathbf{e}', (1-\delta_{1m})\mathbf{I}, (1-\delta_{1m})\overset{\circ}{\mathbf{A}}\mathbf{n}\otimes\overset{\circ}{\mathbf{A}}\mathbf{n},$ $(1-\delta_{1m})\overset{\circ}{\mathbf{A}}\mathbf{n}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{A}}\mathbf{n}), \mathbf{A}, \mathbf{A}\mathbf{N}-\mathbf{N}\mathbf{A}\}$ $(= \mathrm{Sym}^0_m(\mathbf{A}))$
R	$\mathrm{tr}\,\mathbf{H}\mathbf{N}, \mathbf{n}\cdot\mathbf{A}\mathbf{H}\mathbf{n}, [\mathbf{n}, \mathbf{A}\mathbf{n}, \mathbf{H}\mathbf{n}]$; $\mathbf{n}\cdot\mathbf{B}\mathbf{n}, \mathrm{tr}\,\mathbf{A}\mathbf{B}, \mathrm{tr}\,\mathbf{A}\mathbf{B}\mathbf{N}, \delta_{1m}\mathbf{e}\cdot\mathbf{B}\mathbf{e}, \delta_{1m}\mathbf{e}'\cdot\mathbf{B}\mathbf{e}', \delta_{1m}\mathbf{e}\cdot\mathbf{B}\mathbf{e}',$ $(1-\delta_{1m})\mathrm{tr}\,\mathbf{B}, (1-\delta_{1m})\mathbf{n}\cdot\overset{\circ}{\mathbf{A}}\mathbf{B}\overset{\circ}{\mathbf{A}}\mathbf{n}, (1-\delta_{1m})[\mathbf{n}, \overset{\circ}{\mathbf{A}}\mathbf{n}, \mathbf{B}\overset{\circ}{\mathbf{A}}\mathbf{n}]$; $< I_0(\mathbf{A}) >$ (for $m = 1$); $< I^0_m(\mathbf{A}) >$ (for $m \geq 2$).

It can be readily shown that the presented set for skewsymmetric tensor-valued functions, denoted by $\mathrm{Skw}^0(\mathbf{A})$ henceforth, obeys the criterion (1.1), and it is evident that this set is irreducible. The presented set for symmetric tensor-valued functions is denoted by $\mathrm{Sym}^0_m(\mathbf{A})$ henceforth. $\mathrm{Sym}^0_1(\mathbf{A})$ can be found in XIAO [32] and $\mathrm{Sym}^0_m(\mathbf{A})$ for $m \geq 2$ is an equivalent form of the minimal generating set given in XIAO [36]. We have $\mathrm{Sym}^0_1(\mathbf{A}) = \mathrm{Sym}_0(\mathbf{A})$ (cf. Sec. 3.3) and

$$(4.10)\qquad \mathrm{Sym}^0_m(\mathbf{A}) = \{\mathbf{I},\ \mathbf{n}\otimes\mathbf{n},\ \mathbf{A},\ \mathbf{A}\mathbf{N}-\mathbf{N}\mathbf{A},\ \overset{\circ}{\mathbf{A}}\mathbf{n}\otimes\overset{\circ}{\mathbf{A}}\mathbf{n}, \\ \overset{\circ}{\mathbf{A}}\mathbf{n}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{A}}\mathbf{n})\} \equiv \mathrm{Sym}^0(\mathbf{A})\,,$$

for each $m \geq 2$.

Moreover, in the above table, the basis $I_0(\mathbf{A})$ for $m = 1$ is given in Sec. 3.3 and the basis $I^0_m(\mathbf{A})$ for $m \geq 2$ is given by (cf. XIAO [38])

$$(4.11)\qquad I^0_m(\mathbf{A}) = \{\mathbf{n}\cdot\mathbf{A}\mathbf{n}, \mathrm{tr}\,\mathbf{A}, \mathrm{tr}\,\mathbf{A}^3, [\mathbf{n}, \mathbf{A}\mathbf{n}, \mathbf{A}^2\mathbf{n}], \\ \alpha_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n}), \beta_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n}), \alpha_m(\mathbf{q}(\mathbf{A})), \beta_m(\mathbf{q}(\mathbf{A}))\}.$$

CASE 4. Two vector variables $(\mathbf{u}, \mathbf{v})$

The condition (2.6) yields

$$(4.12)\qquad g(\mathbf{z})\cap C_{2mh}(\mathbf{n}) \neq g(\mathbf{u}, \mathbf{v})\cap C_{2mh}(\mathbf{n}), \qquad \mathbf{z} = \mathbf{u}, \mathbf{v}\,.$$

Hence,

$$g(\mathbf{z}) \cap C_{2mh}(\mathbf{n}) \neq C_1 , \qquad \mathbf{z} = \mathbf{u}, \mathbf{v}.$$

By using (4.1) and the latter we infer

$$(\mathbf{z}\cdot\mathbf{n})\mathbf{z} \times \mathbf{n} = \mathbf{0}, \qquad \mathbf{z} = \mathbf{u}, \mathbf{v}.$$

The above result and (4.12) yield

$$\mathbf{u}\cdot\mathbf{n} = 0, \quad \mathbf{v} \times \mathbf{n} = \mathbf{0} \qquad \text{or} \qquad \mathbf{v}\cdot\mathbf{n} = 0, \quad \mathbf{u} \times \mathbf{n} = \mathbf{0}, \tag{4.13}$$

where $(\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v}) \neq 0$, and hence

$$g(\mathbf{u}, \mathbf{v}) \cap C_{2mh}(\mathbf{n}) = C_1 .$$

Accordingly, we construct the following table (the first case in (4.13) is considered).

V	$\mathring{\mathbf{u}}, \mathring{\mathbf{u}} \times \mathbf{n}, (\mathbf{v}\cdot\mathbf{n})\mathbf{n}$
Skw	$\mathbf{N}, \mathbf{u} \wedge \mathbf{v}, (\mathbf{u} \times \mathbf{n}) \wedge \mathbf{v} + (\mathbf{v} \times \mathbf{n}) \wedge \mathbf{u}$
Sym	$\mathbf{n} \otimes \mathbf{n}, \delta_{1m}\mathbf{e} \otimes \mathbf{e}, \delta_{1m}\mathbf{e}' \otimes \mathbf{e}', \delta_{1m}\mathbf{e} \vee \mathbf{e}', (1 - \delta_{1m})\mathring{\mathbf{u}} \otimes \mathring{\mathbf{u}},$ $(1 - \delta_{1m})\mathring{\mathbf{u}} \vee (\mathring{\mathbf{u}} \times \mathbf{n}), \mathbf{u} \vee \mathbf{v}, (\mathbf{n} \times \mathbf{u}) \vee \mathbf{v} + (\mathbf{n} \times \mathbf{v}) \vee \mathbf{u}$
R	$\mathbf{u}\cdot\mathbf{H}\mathbf{v}, [\mathbf{n}, \mathbf{u}, \mathbf{H}\mathbf{v}] + [\mathbf{n}, \mathbf{v}, \mathbf{H}\mathbf{u}];$ $\mathbf{u}\cdot\mathbf{B}\mathbf{v}, [\mathbf{n}, \mathbf{u}, \mathbf{B}\mathbf{v}] + [\mathbf{n}, \mathbf{v}, \mathbf{B}\mathbf{u}];$ $< (\mathbf{v}\cdot\mathbf{n})^2, \delta_{1m}(\mathbf{u}\cdot\mathbf{e})^2, \delta_{1m}(\mathbf{u}\cdot\mathbf{e}')^2, \delta_{1m}(\mathbf{u}\cdot\mathbf{e})(\mathbf{u}\cdot\mathbf{e}'),$ $(1 - \delta_{1m})\alpha_{2m}(\mathring{\mathbf{u}}), (1 - \delta_{1m})\beta_{2m}(\mathring{\mathbf{u}}) >$

Owing to (4.13), the above table can easily be constructed.

CASE 5. A vector variable **u** and a skewsymmetric tensor variable **W**
The condition (2.6) yields

$$g(\mathbf{z}) \cap C_{2mh}(\mathbf{n}) \neq g(\mathbf{u}, \mathbf{W}) \cap C_{2mh}(\mathbf{n}), \qquad \mathbf{z} = \mathbf{u}, \mathbf{W} . \tag{4.14}$$

Hence,

$$g(\mathbf{u}) \cap C_{2mh}(\mathbf{n}) \neq C_1, \qquad \mathbf{u} \neq \mathbf{0},$$
$$g(\mathbf{W}) \cap C_{2mh}(\mathbf{n}) \neq C_{2mh}(\mathbf{n}) .$$

From the latter and (4.1) and (4.4) we derive

$$\begin{aligned} &(\mathbf{u}\cdot\mathbf{n})\mathbf{u} \times \mathbf{n} = \mathbf{0}, \qquad \mathbf{u} \neq \mathbf{0}, \\ &g(\mathbf{W}) \cap C_{2mh}(\mathbf{n}) = S_2, \qquad \text{i.e.} \quad \mathbf{W}\mathbf{n} \neq \mathbf{0}, \end{aligned} \tag{4.15}$$

and hence

$$g(\mathbf{u}, \mathbf{W}) \cap C_{2mh}(\mathbf{n}) = C_1 .$$

Thus, we construct the following table for vector generators and invariants (second order tensor generators and related invariants have been covered by Case 2 due to $(4.15)_2$).

V $\quad \overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \overset{\circ}{\mathbf{u}} \times \mathbf{n}, \mathbf{W}\mathbf{u}, \mathbf{W}(\mathbf{u} \times \mathbf{n}) + \mathbf{n} \times \mathbf{W}\mathbf{u}$

R $\quad \mathbf{r}\cdot\mathbf{W}\mathbf{u}, [\mathbf{n}, \mathbf{u}, \mathbf{W}\mathbf{r}] + [\mathbf{n}, \mathbf{r}, \mathbf{W}\mathbf{u}]$;
$< I_m^0(\mathbf{u}), I_m^0(\mathbf{W}), (1-\delta_{1m})(\overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\mathbf{n})^2, (1-\delta_{1m})(\overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\mathbf{n})[\overset{\circ}{\mathbf{u}}, \mathbf{W}\mathbf{n}, \mathbf{n}] >$.

We are in a position to prove that the above two sets, denoted by $V(\mathbf{u},\mathbf{W})$ and $I(\mathbf{u},\mathbf{W})$, are an irreducible generating set and an irreducible functional basis for scalar-valued and vector-valued anisotropic functions of the two variables $(\mathbf{u},\mathbf{W})$ specified by (4.15) relative to the group $C_{2mh}(\mathbf{n})$, respectively. For the case when $\mathbf{u}\times\mathbf{n} = \mathbf{0}$, the proof is easy. In the following, we treat the case when $\mathbf{u}\cdot\mathbf{n} = 0$. We prove that the two sets in question obey the criteria (1.1) and (1.5), respectively. First, we have

$$\operatorname{rank} V(\mathbf{u},\mathbf{W}) = \begin{cases} \operatorname{rank}\{\mathbf{u}, \mathbf{u}\times\mathbf{n}, \mathbf{W}\mathbf{u}\} = 3, & \mathbf{n}\cdot\mathbf{W}\mathbf{u} \neq 0, \\ \operatorname{rank}\{\mathbf{u}, \mathbf{u}\times\mathbf{n}, \mathbf{W}(\mathbf{u}\times\mathbf{n})\} = 3, & \mathbf{n}\cdot\mathbf{W}\mathbf{u} = 0, \end{cases}$$

where in the second equality, the fact

$$\mathbf{n}\cdot\mathbf{W}(\mathbf{u}\times\mathbf{n}) = (\mathbf{W}\mathbf{n})\cdot(\mathbf{u}\times\mathbf{n}) \neq 0$$

is used, which can be derived by using the facts: 1. The three vectors $\mathbf{u}, \mathbf{u}\times\mathbf{n}$ and $\mathbf{W}\mathbf{n}$ lie in the $\mathbf{n}$-plane and the first two are independent, and 2. $\mathbf{n}\cdot\mathbf{W}\mathbf{u} = (\mathbf{W}\mathbf{n})\cdot\mathbf{u} = 0$. From the above, we know that the set $V(\mathbf{u},\mathbf{W})$ obeys (1.1) for the case $\mathbf{u}\cdot\mathbf{n} = 0$.

Next, for $m = 1$, it is readily verified that the set $I(\mathbf{u},\mathbf{W})$, i.e. $I_1^0(\mathbf{u}) \cup I_1^0(\mathbf{W})$, obeys (1.5). For $m \geq 2$, let $I'(\mathbf{u},\mathbf{W}) = I_m^0(\mathbf{u}) \cup I_m^0(\mathbf{W})$. Then, for the pair $(\overline{\mathbf{u}}, \overline{\mathbf{W}})$ and $(\mathbf{u},\mathbf{W})$, where the two vectors lie on the $\mathbf{n}$-plane, we have

$$I'(\overline{\mathbf{u}}, \overline{\mathbf{W}}) = I'(\mathbf{u},\mathbf{W}) \implies \exists\, \mathbf{R}, \mathbf{Q} \in C_{2mh}(\mathbf{n}) :\ \overline{\mathbf{u}} = \mathbf{R}\mathbf{u},\ \ \overline{\mathbf{W}} = \mathbf{Q}\mathbf{W}\mathbf{Q}^{\mathrm{T}}.$$

Denoting $\mathbf{Q}_0 = \mathbf{Q}^{\mathrm{T}}\mathbf{R} = \epsilon\mathbf{R}_{\mathbf{n}}^{\psi}$, $\epsilon^2 = 1$, and noting that both $\overline{\mathbf{u}}$ and $\overline{\mathbf{W}}\mathbf{n}$ are on the $\mathbf{n}$-plane, we infer

$$\begin{aligned} (\overset{\circ}{\overline{\mathbf{u}}}\cdot\overline{\mathbf{W}}\mathbf{n})^2 &= ((\mathbf{Q}_0\overset{\circ}{\mathbf{u}})\cdot\mathbf{W}\mathbf{n})^2 \\ &= |\overset{\circ}{\mathbf{u}}|^2\cdot|\mathbf{W}\mathbf{n}|^2\cos^2(\theta+\psi), \\ (\overset{\circ}{\overline{\mathbf{u}}}\cdot\overline{\mathbf{W}}\mathbf{n})[\overset{\circ}{\overline{\mathbf{u}}}, \overline{\mathbf{W}}\mathbf{n}, \mathbf{n}] &= \frac{1}{2}|\overset{\circ}{\mathbf{u}}|^2|\mathbf{W}\mathbf{n}|^2\sin^2(\theta+\psi), \end{aligned}$$

where θ is the angle between $\overset{\circ}{\mathbf{u}}$ and $\mathbf{W}\mathbf{n}$, and $(\theta+\psi)$ the angle between $\mathbf{R}_{\mathbf{n}}^{\psi}\mathbf{u}$ and $\mathbf{W}\mathbf{n}$. When $\psi = 0$, the above two equalities yield $(\overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\mathbf{n})^2$ and $(\overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\mathbf{n})[\overset{\circ}{\mathbf{u}}, \mathbf{W}\mathbf{n}, \mathbf{n}]$.

Hence, we deduce

$$\begin{cases} (\overset{\circ}{\overline{\mathbf{u}}}\cdot\overline{\mathbf{W}}\mathbf{n})^2 = (\overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\mathbf{n})^2\,, \\ (\overset{\circ}{\overline{\mathbf{u}}}\cdot\overline{\mathbf{W}}\mathbf{n})[\overset{\circ}{\overline{\mathbf{u}}}, \overline{\mathbf{W}}\mathbf{n}, \mathbf{n}] = (\overset{\circ}{\mathbf{u}}\cdot\mathbf{W}\mathbf{n})[\overset{\circ}{\mathbf{u}}, \mathbf{W}\mathbf{n}, \mathbf{n}]\,, \end{cases} \Longrightarrow \begin{cases} \cos 2(\theta+\psi) = \cos 2\theta\,, \\ \sin 2(\theta+\psi) = \sin 2\theta\,, \end{cases}$$

$$\Longrightarrow \psi = k\pi, \quad k = 0, \pm 1, \pm 2, \cdots .$$

Therefore,

$$\mathbf{R} = \mathbf{Q}\mathbf{Q}_0 = \epsilon\mathbf{Q}\mathbf{R}_{\mathbf{n}}^{k\pi}, \qquad \epsilon^2 = 1.$$

Let $\mathbf{R}_0 = (-1)^k\epsilon\mathbf{Q}$. Then, using the facts

$$\mathbf{u}\cdot\mathbf{n} = 0 \Longrightarrow \mathbf{R}_{\mathbf{n}}^{k\pi}\mathbf{u} = (-1)^k\mathbf{u},$$

we deduce

$$\mathbf{R}_0\mathbf{u} = (-1)^k\epsilon\mathbf{Q}\mathbf{u} = \epsilon\mathbf{Q}\mathbf{R}_{\mathbf{n}}^{k\pi}\mathbf{u} = \mathbf{R}\mathbf{u} = \overline{\mathbf{u}},$$
$$\mathbf{R}_0\mathbf{W}\mathbf{R}_0^{\mathrm{T}} = \mathbf{Q}\mathbf{W}\mathbf{Q}^{\mathrm{T}} = \overline{\mathbf{W}},$$

where $\mathbf{R}_0 \in C_{2mh}(\mathbf{n})$.

Thus, we conclude that the set $I(\mathbf{u}, \mathbf{W})$ obeys the criterion (1.5) for $(\mathbf{u}, \mathbf{W})$ specified by (4.15).

CASE 6. A vector variable $\mathbf{u}$ and a symmetric tensor variable $\mathbf{A}$

The condition (2.6) yields

$$g(\mathbf{z}) \cap C_{2mh}(\mathbf{n}) \neq g(\mathbf{u}, \mathbf{A}) \cap C_{2mh}(\mathbf{n}), \qquad \mathbf{z} = \mathbf{u}, \mathbf{A}\,. \tag{4.16}$$

From this we derive

$$\begin{gathered} (\mathbf{u}\cdot\mathbf{n})\mathbf{u}\times\mathbf{n} = \mathbf{0}, \qquad \mathbf{u} \neq \mathbf{0}\,, \\ g(\mathbf{A}) \cap C_{2mh}(\mathbf{n}) = S_2\,, \qquad \text{i.e. } \overset{\circ}{\mathbf{A}}\mathbf{n} \neq \mathbf{0}\,, \end{gathered} \tag{4.17}$$

and

$$g(\mathbf{u}, \mathbf{A}) \cap C_{2mh}(\mathbf{n}) = C_1\,.$$

Thus, we construct the following table for vector generators and invariants (second order tensor generators and related invariants have been covered by Case 3 due to $(4.17)_2$).

V $\quad \overset{\circ}{\mathbf{u}}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \overset{\circ}{\mathbf{u}}\times\mathbf{n}, \mathbf{A}\mathbf{u}, \mathbf{A}(\mathbf{u}\times\mathbf{n}) + \mathbf{n}\times\mathbf{A}\mathbf{u}$

R $\quad \mathbf{r}\cdot\mathbf{A}\mathbf{u}, [\mathbf{n}, \mathbf{u}, \mathbf{A}\mathbf{r}] + [\mathbf{n}, \mathbf{r}, \mathbf{A}\mathbf{u}];$
$\quad < I_m^0(\mathbf{u}), I_m^0(\mathbf{A}), (1-\delta_{1m})(\overset{\circ}{\mathbf{u}}\cdot\mathbf{A}\mathbf{n})^2, (1-\delta_{1m})(\overset{\circ}{\mathbf{u}}\cdot\mathbf{A}\mathbf{n})[\overset{\circ}{\mathbf{u}}, \mathbf{A}\mathbf{n}, \mathbf{n}] >$

It is easy to treat the case when $\mathbf{u}\times\mathbf{n} = \mathbf{0}$. Let $\mathbf{u}\cdot\mathbf{n} = 0$. Following the same procedures used in the last case, we can prove the following facts: 1. The set of vector generators given in the above table is a generating set for vector-valued anisotropic functions of the two variables $(\mathbf{u}, \mathbf{A})$ specified by (4.17) under C_{2mh}, and 2. The set of invariants listed in the angle brackets is a functional basis of the two variables $(\mathbf{u}, \mathbf{A})$ specified by (4.17) under C_{2mh}, $m \geq 2$. Besides, it can be readily verified that the union $I_1^0(\mathbf{u}) \cup I_1^0(\mathbf{A})$ is a functional basis of the two variables $(\mathbf{u}, \mathbf{A})$ at issue under $C_{2h}(\mathbf{n})$. Hence, for each $m \geq 1$, the set listed in the angle brackets gives a functional basis of the two variables $(\mathbf{u}, \mathbf{A})$ specified by (4.17) under C_{2mh}.

Combining the above six cases, we arrive at the desired representations under each subgroup C_{2mh}. Moreover, it can be easily seen that the analysis given also applies to the transverse isotropy group $C_{\infty h}$ if the functional bases $I_m^0(\mathbf{u})$, $I_m^0(\mathbf{W})$ and $I_m^0(\mathbf{A})$ are respectively replaced by three bases of the single variables $\mathbf{u}$, $\mathbf{W}$ and $\mathbf{A}$ under the group $C_{\infty h}$. These facts are summerized as follows.

THEOREM 4.1. *The four sets given by*

$I_m^0(\mathbf{u}); I_m^0(\mathbf{W}); I_m^0(\mathbf{A});\ (\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n}); \mathring{\mathbf{u}}\cdot\mathring{\mathbf{v}}, [\mathbf{n}, \mathring{\mathbf{u}}, \mathring{\mathbf{v}}];$

$(\mathbf{u}\cdot\mathbf{n})\mathring{\mathbf{u}}\cdot\mathbf{Wn}, (\mathbf{u}\cdot\mathbf{n})[\mathbf{n}, \mathbf{Wn}, \mathring{\mathbf{u}}], (1-\delta_{1m})(\mathring{\mathbf{u}}\cdot\mathbf{Wn})^2, (1-\delta_{1m})(\mathring{\mathbf{u}}\cdot\mathbf{Wn})[\mathring{\mathbf{u}}, \mathbf{Wn}, \mathbf{n}];$

$(\mathbf{u}\cdot\mathbf{n})\mathring{\mathbf{u}}\cdot\mathbf{An}, (\mathbf{u}\cdot\mathbf{n})[\mathbf{n}, \mathbf{An}, \mathring{\mathbf{u}}], (1-\delta_{1m})\mathring{\mathbf{u}}\cdot\mathbf{A}\mathring{\mathbf{u}}, (1-\delta_{1m})[\mathbf{n}, \mathring{\mathbf{u}}, \mathbf{A}\mathring{\mathbf{u}}],$

$(1-\delta_{1m})(\mathring{\mathbf{u}}\cdot\mathbf{An})^2, (1-\delta_{1m})(\mathring{\mathbf{u}}\cdot\mathbf{An})[\mathring{\mathbf{u}}, \mathbf{An}, \mathbf{n}];$

$\mathbf{n}\cdot\mathbf{WHn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{Hn}];$

$\mathbf{n}\cdot\mathbf{WAn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{An}], (1-\delta_{1m})\mathbf{n}\cdot\mathbf{WAWn}, (1-\delta_{1m})[\mathbf{n}, \mathbf{Wn}, \mathbf{AWn}];$

$\operatorname{tr}\mathbf{AB}, \operatorname{tr}\mathbf{ABN}, (1-\delta_{1m})\mathbf{n}\cdot\mathring{\mathbf{A}}\mathbf{B}\mathring{\mathbf{A}}\mathbf{n}, (1-\delta_{1m})[\mathbf{n}, \mathring{\mathbf{A}}\mathbf{n}, \mathbf{B}\mathring{\mathbf{A}}\mathbf{n}];$

$\mathbf{u}\cdot\mathbf{Wv}, [\mathbf{n}, \mathbf{u}, \mathbf{Wv}] + [\mathbf{n}, \mathbf{v}, \mathbf{Wu}];$

$\mathbf{u}\cdot\mathbf{Av}, [\mathbf{n}, \mathbf{u}, \mathbf{Av}] + [\mathbf{n}, \mathbf{v}, \mathbf{Au}];$

and

$\mathring{\mathbf{u}}, \mathring{\mathbf{u}} \times \mathbf{n}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n};$

$\mathbf{Wu}, \mathbf{W}(\mathbf{u}\times\mathbf{n}) + \mathbf{n}\times\mathbf{Wu};$

$\mathbf{Au}, \mathbf{A}(\mathbf{u}\times\mathbf{n}) + \mathbf{n}\times\mathbf{Au};$

and

$\operatorname{Skw}^0(\mathbf{u}); \operatorname{Skw}^0(\mathbf{W}); \operatorname{Skw}^0(\mathbf{A}); \mathbf{u}\wedge\mathbf{v}, (\mathbf{u}\times\mathbf{n})\wedge\mathbf{v} + (\mathbf{v}\times\mathbf{n})\wedge\mathbf{u};$

and

$\operatorname{Sym}_m^0(\mathbf{u}); \operatorname{Sym}_m^0(\mathbf{W}); \operatorname{Sym}_m^0(\mathbf{A}); \mathbf{u}\vee\mathbf{v}, (\mathbf{u}\times\mathbf{n})\vee\mathbf{v} + (\mathbf{v}\times\mathbf{n})\vee\mathbf{u};$

where $\mathbf{u}, \mathbf{v} = \mathbf{u}_1, \cdots, \mathbf{u}_a$; $\mathbf{W}, \mathbf{H} = \mathbf{W}_1, \cdots, \mathbf{W}_b$; $\mathbf{A}, \mathbf{B} = \mathbf{A}_1, \cdots, \mathbf{A}_c$; $\mathbf{u} \neq \mathbf{v}$, $\mathbf{W} \neq \mathbf{H}$, $\mathbf{A} \neq \mathbf{B}$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X \in \mathcal{D}$ *under the group* $C_{2mh}(\mathbf{n})$ *for each* $m = 1, 2, \cdots, \infty$, *respectively. For* $m = \infty$, *i.e. for the transverse isotropy group* $C_{\infty h}(\mathbf{n})$, *it is assumed that* $\delta_{1\infty} = 0$ *and moreover that*

$$I_\infty^0(\mathbf{u}) = \{(\mathbf{u}\cdot\mathbf{n})^2, |\mathbf{u}|^2\}, \qquad I_\infty^0(\mathbf{W}) = \{\operatorname{tr}\mathbf{WN}, \operatorname{tr}\mathbf{W}^2\},$$
$$I_\infty^0(\mathbf{A}) = \{\operatorname{tr}\mathbf{A}, \operatorname{tr}\mathbf{A}^2, \operatorname{tr}\mathbf{A}^3, \mathbf{n}\cdot\mathbf{An}, \mathbf{n}\cdot\mathbf{A}^2\mathbf{n}, [\mathbf{n}, \mathbf{An}, \mathbf{A}^2\mathbf{n}]\}.$$

REMARK. Irreducible nonpolynomial representations for transversely isotropic functions of the variables $X \in \mathcal{D}$ relative to $C_{\infty h}$ were first derived by ZHENG [41], which belongs to the case when $m = \infty$ in the above theorem. It can be seen that the new results presented in the above theorem are more compact than those given in ZHENG [41], e.g. the presented functional basis of two symmetric tensor variables and the presented generating set for a single symmetric tensor variable, respectively, consist of eighteen invariants and six generators, respectively, while the corresponding results given in ZHENG [41] include nineteen invariants and eight generators, respectively.

4.2. The classes C_{2m}

Let $I^0(\mathbf{H}_1, \cdots, \mathbf{H}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c)$, $\mathrm{Skw}^0(\mathbf{H}_1, \cdots, \mathbf{H}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c)$ and $\mathrm{Sym}^0(\mathbf{H}_1, \cdots, \mathbf{H}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c)$ be, respectively, an irreducible functional basis and irreducible generating sets for scalar-valued, skewsymmetric and symmetric tensor-valued anisotropic functions of $(a+b)$ skewsymmetric tensor variables and c symmetric tensor variables under an orthogonal subgroup g containing the central inversion $-\mathbf{I}$. Then, according to Theorem 2.1 and 2.2 in XIAO [35], the four sets

$$\begin{aligned}
&I^0(\mathbf{Eu}_1, \cdots, \mathbf{Eu}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c),\\
\mathbf{E}:\ &\mathrm{Skw}^0(\mathbf{Eu}_1, \cdots, \mathbf{Eu}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c),\\
&\mathrm{Skw}^0(\mathbf{Eu}_1, \cdots, \mathbf{Eu}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c),\\
&\mathrm{Sym}^0(\mathbf{Eu}_1, \cdots, \mathbf{Eu}_a; \mathbf{W}_1, \cdots, \mathbf{W}_b; \mathbf{A}_1, \cdots, \mathbf{A}_c),
\end{aligned}$$

supply, respectively, an irreducible functional basis and irreducible generating sets for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of a vector variables, b skewsymmetric tensor variables and c symmetric tensor variables under the rotation subgroup of g, i.e. $g \cap \mathrm{Orth}^+$. Here, the second set above is obtained by forming the double dot product between each skewsymmetric tensor generator and the third order Eddington tensor $\mathbf{E}$. From this fact and Theorem 4.1, we derive the following result.

THEOREM 4.2. *The four sets given by*

$I^0_m(\mathbf{W}), I^0_m(\mathbf{A})$;

$\mathbf{u}\cdot\mathbf{n}, \delta_{m\infty}|\mathbf{u}|^2, \delta_{1m}(\mathbf{u}\cdot\mathbf{e})^2, \delta_{1m}(\mathbf{u}\cdot\mathbf{e}')^2, \delta_{1m}(\mathbf{u}\cdot\mathbf{e})(\mathbf{u}\cdot\mathbf{e}')$,

$(1-\delta_{1m})\alpha_{2m}(\mathring{\mathbf{u}}), (1-\delta_{1m})\beta_{2m}(\mathring{\mathbf{u}})$;

$\mathring{\mathbf{u}}\cdot\mathring{\mathbf{v}}, [\mathbf{n}, \mathring{\mathbf{u}}, \mathring{\mathbf{v}}]$;

$[\mathbf{n}, \mathbf{u}, \mathbf{Wn}], \mathbf{u}\cdot\mathbf{Wn}$;

$[\mathbf{n}, \mathbf{u}, \mathbf{An}], \mathbf{u}\cdot\mathbf{An}, (1-\delta_{1m})\mathring{\mathbf{u}}\cdot\mathbf{A}\mathring{\mathbf{u}}, (1-\delta_{1m})[\mathbf{n}, \mathbf{u}, \mathbf{Au}]$;

$\mathbf{n}\cdot\mathbf{WHn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{Hn}]$;

$\mathbf{n}\cdot\mathbf{WAn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{An}], (1-\delta_{1m})\mathbf{n}\cdot\mathbf{WAWn}, (1-\delta_{1m})[\mathbf{n}, \mathbf{Wn}, \mathbf{AWn}]$;

$\mathrm{tr}\,\mathbf{AB}, \mathrm{tr}\,\mathbf{ABN}, (1-\delta_{1m})\mathbf{n}\cdot\mathring{\mathbf{A}}\mathbf{B}\mathring{\mathbf{A}}\mathbf{n}, (1-\delta_{1m})[\mathbf{n}, \mathring{\mathbf{A}}\mathbf{n}, \mathbf{B}\mathring{\mathbf{A}}\mathbf{n}]$;

and

$\mathbf{n}, \mathring{\mathbf{u}}, \mathring{\mathbf{u}} \times \mathbf{n}$;

$\mathbf{Wn}, \mathbf{n} \times \mathbf{Wn}$;

$\mathbf{An}, \mathbf{n} \times \mathbf{An}$;

and

$\mathbf{N}, \mathbf{n} \wedge \mathring{\mathbf{u}}, \mathbf{n} \wedge (\mathbf{n} \times \mathring{\mathbf{u}}); \mathrm{Skw}^0(\mathbf{W}); \mathrm{Skw}^0(\mathbf{A})$;

and

$\mathrm{Sym}^0_m(\mathbf{Eu}), \mathrm{Sym}^0_m(\mathbf{W}), \mathrm{Sym}^0_m(\mathbf{A})$;

where $\mathbf{u}, \mathbf{v} = \mathbf{u}_1, \cdots, \mathbf{u}_a$; $\mathbf{W}, \mathbf{H} = \mathbf{W}_1, \cdots, \mathbf{W}_b$; $\mathbf{A}, \mathbf{B} = \mathbf{A}_1, \cdots, \mathbf{A}_c$; $\mathbf{u} \neq \mathbf{v}$, $\mathbf{W} \neq \mathbf{H}$, $\mathbf{A} \neq \mathbf{B}$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X \in \mathcal{D}$ *under the group* $C_{2m}(\mathbf{n})$ *for each* $m = 1, 2, \cdots, \infty$, *respectively. When* $m = \infty$, *the bases* $I^0_\infty(\mathbf{W})$ *and* $I^0_\infty(\mathbf{A})$ *are given in Theorem 4.1 and moreover*

$$\delta_{m\infty} = \begin{cases} 1, & m = \infty, \\ 0, & m = 1, 2, \cdots . \end{cases}$$

5. The classes S_{4m}

The classes S_{4m} include the tetrahedral crystal class S_4 as the particular case when $m = 1$.

As pointed out in Sec. 2, for any $X \in \mathcal{D}$ there is $X_0 \subset X$ such that (2.4) holds, where X_0 consists of two vectors or a vector and a second order tensor, i.e. $X_0 = (\mathbf{u}, \mathbf{x})$, $\mathbf{x} \in V \cup \mathrm{Skw} \cup \mathrm{Sym}$. Furthermore, from the facts

$$g(\mathbf{r}) \cap S_{4m}(\mathbf{n}) = \begin{cases} S_{4m}(\mathbf{n}), & \mathbf{u} = \mathbf{0}, \\ C_{2m}(\mathbf{n}), & \mathring{\mathbf{u}} = \mathbf{0}, \quad \mathbf{u} \cdot \mathbf{n} \neq 0, \\ C_1, & \mathring{\mathbf{u}} \neq \mathbf{0}, \end{cases} \tag{5.1}$$

$$g(\mathbf{W}) \cap S_{4m}(\mathbf{n}) = \begin{cases} S_{4m}(\mathbf{n}), & \mathbf{Wn} = \mathbf{0}, \\ C_1, & \mathbf{Wn} \neq \mathbf{0}, \end{cases} \tag{5.2}$$

$$g(\mathbf{A}) \cap S_{4m}(\mathbf{n}) = \begin{cases} S_{4m}(\mathbf{n}), & \mathring{\mathbf{A}} = x\mathbf{I} + y\mathbf{n} \otimes \mathbf{n}, \\ C_2(\mathbf{n}), & \mathring{\mathbf{A}}\mathbf{n} = \mathbf{0}, \quad \mathbf{q}(\mathbf{A}) \neq \mathbf{0}, \\ C_1, & \mathring{\mathbf{A}}\mathbf{n} \neq \mathbf{0}, \end{cases} \tag{5.3}$$

we infer that for any $X \in \mathcal{D}$ there is a single vector or a single second order tensor $\mathbf{z} \in V \cup \mathrm{Skw} \cup \mathrm{Sym}$ such that

$$g(\mathbf{z}) \cap S_{4m}(\mathbf{n}) = g(X) \cap S_{4m}(\mathbf{n}).$$

The above fact indicates that the cases for two variables can further be reduced to the cases for single variables, which are discussed as follows.

Case 1. A single vector variable $\mathbf{u}$

According to the related results in Xiao [35, 39], representations for scalar-, vector-, and second order tensor-valued anisotropic functions of the vector variable $\mathbf{u}$ under the group $S_{4m}(\mathbf{n})$ may be obtained from those for scalar-, vector-, and second order tensor-valued isotropic functions of the extended variables $(\mathbf{u}, \boldsymbol{\phi}_m(\mathbf{u}), \mathbf{N})$ respectively, where

$$\boldsymbol{\phi}_m(\mathbf{u}) = \mathbf{n} \vee \boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{u}}) + \delta_{1m}(\mathbf{u}\cdot\mathbf{n})(\mathbf{e}\otimes\mathbf{e} - \mathbf{e}'\otimes\mathbf{e}'), \tag{5.4}$$

and where $\boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{u}})$ is a vector depending on $\overset{\circ}{\mathbf{u}}$, given by (1.11).

We mention that the second term at the right-hand side of (5.4) comes into play only when the group $S_4(\mathbf{n})$ is concerned. This fact implies the particular property of the group $S_4(\mathbf{n})$, which will be seen below.

Applying the above fact and the related results for isotropic functions and then removing some redundant elements, we construct the following table.

V	$\{\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}}\times\mathbf{n}, \alpha_{2m}(\overset{\circ}{\mathbf{u}})\mathbf{n}, \beta_{2m}(\overset{\circ}{\mathbf{u}})\mathbf{n}\}$ $(= V_m^1(\mathbf{u}))$
Skw	$\{\mathbf{N}, \mathbf{n}\wedge\boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{u}}), \mathbf{n}\wedge(\boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{u}})\times\mathbf{n})\}$ $(= \mathrm{Skw}_m^1(\mathbf{u}))$
Sym	$\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}}\vee(\overset{\circ}{\mathbf{u}}\times\mathbf{n}), \boldsymbol{\phi}_m(\mathbf{u}), \boldsymbol{\phi}_m(\mathbf{u})\mathbf{N} - \mathbf{N}\boldsymbol{\phi}_m(\mathbf{u})\}$ $(= \mathrm{Sym}_m^1(\mathbf{u}))$
R	$\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{r}}, [\mathbf{n}, \overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{r}}], (\mathbf{r}\cdot\mathbf{n})\alpha_{2m}(\overset{\circ}{\mathbf{u}}), (\mathbf{r}\cdot\mathbf{n})\beta_{2m}(\overset{\circ}{\mathbf{u}})$; $\mathrm{tr}\,\mathbf{HN}, \boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{u}})\cdot\mathbf{Hn}, [\mathbf{n}, \mathbf{Hn}, \boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{u}})]$; $\mathrm{tr}\,\mathbf{B}, \mathbf{n}\cdot\mathbf{Bn}, \overset{\circ}{\mathbf{u}}\cdot\mathbf{B}\overset{\circ}{\mathbf{u}}, [\mathbf{n}, \overset{\circ}{\mathbf{u}}, \mathbf{B}\overset{\circ}{\mathbf{u}}], \mathrm{tr}\,\boldsymbol{\phi}_m(\mathbf{u})\mathbf{B}, \mathrm{tr}\,\boldsymbol{\phi}_m(\mathbf{u})\mathbf{BN}$; $<(\mathbf{u}\cdot\mathbf{n})\alpha_{2m}(\overset{\circ}{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\beta_{2m}(\overset{\circ}{\mathbf{u}}), \alpha_{4m}(\overset{\circ}{\mathbf{u}}), \beta_{4m}(\overset{\circ}{\mathbf{u}})>$ $(= I_m^1(\mathbf{u}))$

In the following, we prove that the first three sets given in the above table, denoted by $V_m^1(\mathbf{u})$ and $\mathrm{Skw}_m^1(\mathbf{u})$ and $\mathrm{Sym}_m^1(\mathbf{u})$ henceforth, are irreducible generating sets for vector-valued and skewsymmetric and symmetric tensor-valued anisotropic functions of the vector variable $\mathbf{u}$ under $S_{4m}(\mathbf{n})$, respectively. To this end, we prove that each of these sets obeys the criterion (1.1). In fact, we have

$$\alpha_{2m}(\overset{\circ}{\mathbf{u}}) = \beta_{2m}(\overset{\circ}{\mathbf{u}}) = 0 \iff \boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{u}}) = \mathbf{0} \iff \overset{\circ}{\mathbf{u}} = \mathbf{0}. \tag{5.5}$$

Then, by using (4.1) and the latter, as well as (1.2) and (1.3), we deduce that $V_m^1(\mathbf{u})$ and $\mathrm{Skw}_m^1(\mathbf{W})$ obey (1.1) respectively. For $\mathrm{Sym}_m^1(\mathbf{u})$, the group S_4 and the groups S_{4m} for $m \geq 2$ should be considered separately due to the following particular property of the former:

$$\dim \mathrm{Sym}(g(\mathbf{u})\cap S_4(\mathbf{n})) = \begin{cases} \dim \mathrm{Sym}(S_4(\mathbf{n})) = 2, & \mathbf{u} = \mathbf{0}, \\ \dim \mathrm{Sym}(C_2(\mathbf{n})) = 4, & \overset{\circ}{\mathbf{u}} = \mathbf{0}, \quad \mathbf{u}\cdot\mathbf{n} \neq 0, \\ \mathrm{rank}\,\mathrm{Sym}(C_1) = 6, & \overset{\circ}{\mathbf{u}} \neq \mathbf{0}, \end{cases} \tag{5.6}$$

while for $m \geq 2$,

$$\dim \operatorname{Sym}(g(\mathbf{u}) \cap S_{4m}(\mathbf{n})) = \begin{cases} 2, & \overset{\circ}{\mathbf{u}} = \mathbf{0}\,, \\ 6, & \overset{\circ}{\mathbf{u}} \neq \mathbf{0}\,. \end{cases} \tag{5.7}$$

For $m = 1$, we have

$$\operatorname{rank} \operatorname{Sym}_1^1(\mathbf{u}) = \begin{cases} \operatorname{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}\} = 2, & \mathbf{u} = \mathbf{0}\,, \\ \{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathbf{e}\otimes\mathbf{e} - \mathbf{e}'\otimes\mathbf{e}', \mathbf{e}\vee\mathbf{e}'\} = 4, & \overset{\circ}{\mathbf{u}} = \mathbf{0}, \quad \mathbf{u} \neq \mathbf{0}\,, \\ \{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}}\otimes(\overset{\circ}{\mathbf{u}}\times\mathbf{n}), \mathbf{n}\vee\boldsymbol{\rho}_1(\overset{\circ}{\mathbf{u}}), & \\ \qquad \mathbf{n}\vee(\mathbf{n}\times\boldsymbol{\rho}_1(\overset{\circ}{\mathbf{u}}))\} = 6, & \overset{\circ}{\mathbf{u}} \neq \mathbf{0}, \end{cases}$$

and for $m \geq 2$ we have

$$\operatorname{rank} \operatorname{Sym}_m^1(\mathbf{u}) = \begin{cases} \operatorname{rank}\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}\} = 2, & \overset{\circ}{\mathbf{u}} = \mathbf{0}\,, \\ 6, & \overset{\circ}{\mathbf{u}} \neq \mathbf{0}\,, \end{cases}$$

Thus, we deduce that the set $\operatorname{Sym}_m(\mathbf{u})$ obeys (1.1) for each $m \geq 1$.

It is evident that both $\operatorname{Skw}_m^1(\mathbf{u})$ and $\operatorname{Sym}_m^1(\mathbf{u})$ are irreducible. Moreover, by considering the facts

$$\beta_{2m}(\overset{\circ}{\mathbf{u}}) = 0 \qquad \text{for} \quad \mathbf{u} = \mathbf{e},$$
$$\alpha_{2m}(\overset{\circ}{\mathbf{u}}) = 0 \qquad \text{for} \quad \mathbf{u} = \mathbf{R}_{\mathbf{n}}^{\pi/4m}\mathbf{e},$$

we know that $V_m^1(\mathbf{u})$ is also irreducible.

Next, we prove that the set given in the above table in the angle brackets, denoted by $I_m^1(\mathbf{u})$ henceforth, is an irreducible functional basis of the vector variable $\mathbf{u}$ under $S_{4m}(\mathbf{n})$. First, suppose $\mathbf{u}\cdot\mathbf{n} \neq 0$. Then we infer

$$\begin{aligned} I_m^1(\overline{\mathbf{u}}) = I_m^1(\mathbf{u}) \Longrightarrow |\overset{\circ}{\overline{\mathbf{u}}}| = |\overset{\circ}{\mathbf{u}}|, \quad \overline{\mathbf{u}}\cdot\mathbf{n} = \delta\mathbf{u}\cdot\mathbf{n}, \quad \cos 2m\overline{\theta} = \delta\cos 2m\theta, \\ \sin 2m\overline{\theta} = \delta \sin 2m\theta \\ \Longrightarrow \overline{\theta} = \frac{4p+1-\delta}{4m}\pi + \theta, \quad p = 0, \pm1, \pm2, \cdots; \quad \overline{\mathbf{u}}\cdot\mathbf{n} = \delta\mathbf{u}\cdot\mathbf{n}, \\ |\overset{\circ}{\overline{\mathbf{u}}}| = |\overset{\circ}{\mathbf{u}}|, \end{aligned}$$

where $\delta^2 = 1$, $\overline{\theta} =< \overset{\circ}{\overline{\mathbf{u}}}, \mathbf{e} >$ and $\theta =< \overset{\circ}{\mathbf{u}}, \mathbf{e} >$. Thus, we have $\overline{\mathbf{u}} = \mathbf{Q}\mathbf{u}$, where $\mathbf{Q} \in S_{4m}(\mathbf{n})$ depends on δ, given by

$$\mathbf{Q} = \begin{cases} \mathbf{R}_{\mathbf{n}}^{2p\pi/2m}, & \delta = 1\,, \\ -\mathbf{R}_{\mathbf{n}}^{(2m+2p+1)\pi/2m}, & \delta = -1\,. \end{cases}$$

Hence the set $I_m^1(\mathbf{u})$ obeys (1.5) when $\mathbf{u}\cdot\mathbf{n} \neq 0$.

Second, suppose $\mathbf{u}\cdot\mathbf{n} = 0$, i.e. $\overset{\circ}{\mathbf{u}} = \mathbf{u}$. Since $\mathbf{R}_{\mathbf{n}}^{\pi} \in S_{4m}(\mathbf{n})$, we have

$$f(\mathbf{u}) = f(\mathbf{R}_{\mathbf{n}}^{\pi}\mathbf{u}) = f(-\mathbf{u}) = f((-\mathbf{I})\mathbf{u})$$

for each invariant $f(\mathbf{u})$ under $S_{4m}(\mathbf{n})$. Since the group $S_{4m}(\mathbf{n})$ and the central inversion $-\mathbf{I}$ generate the group $C_{4mh}(\mathbf{n})$, the above fact implies that for the vector variable $\mathbf{u} = \overset{\circ}{\mathbf{u}}$ each invariant of $\mathbf{u}$ under $S_{4m}(\mathbf{n})$ turns out to be an invariant of $\mathbf{u}$ under the larger group $C_{4mh}(\mathbf{n}) \supset S_{4m}(\mathbf{n})$. Thus, for the variable $\mathbf{u}$ obeying $\mathbf{u}\cdot\mathbf{n} = 0$, a functional basis of $\mathbf{u}$ under S_{4m} is provided by a functional basis of $\mathbf{u}$ under C_{4mh}, the latter being formed by the last two invariants listed in the presented table.

From the above, we know that $I_m^1(\mathbf{u})$ is a functional basis of the vector $\mathbf{u}$ under $S_{4m}(\mathbf{n})$. Furthermore, from the following four pairs $(X, X') = (\mathbf{u}, \mathbf{u}')$ fulfilling the condition (1.6) we infer that each of its elements is irreducible.

$$\beta_{4m}(\overset{\circ}{\mathbf{u}}) : \mathbf{u} = \mathbf{R}_{\mathbf{n}}^{\pi/8m}\mathbf{e},\ \mathbf{u}' = 2\mathbf{u};\ \alpha_{4m}(\overset{\circ}{\mathbf{u}}) :\ \mathbf{u} = \mathbf{e},\ \mathbf{u}' = 2\mathbf{e};$$
$$(\mathbf{u}\cdot\mathbf{n})\alpha_{2m}(\overset{\circ}{\mathbf{u}}) : \mathbf{u} = \mathbf{n} + \mathbf{e},\ \mathbf{u}' = 2\mathbf{u};\ \beta_{2m}(\overset{\circ}{\mathbf{u}}) :\ \mathbf{u} = \mathbf{n} + \mathbf{R}_{\mathbf{n}}^{\pi/4m}\mathbf{e},\ \mathbf{u}' = 2\mathbf{u}.$$

CASE 2. A single skewsymmetric tensor variable **W**

Every scalar-valued (resp. second order tensor-valued) anisotropic function of $\mathbf{W}$ under $S_{4m}(\mathbf{n})$ is equivalent to a scalar-valued (resp. second order tensor--valued) anisotropic function of $\mathbf{W}$ under $C_{4mh}(\mathbf{n})$. Irreducible representations for the latter can be obtained by merely replacing m with $2m$ in the table for Case 2 of Sec. 4.1. As a result, we need only to consider vector-valued functions.

Representations for vector-valued anisotropic functions of the skewsymmetric tensor variable $\mathbf{W}$ under $S_{4m}(\mathbf{n})$ can be derived from those for isotropic functions of the extended variables $(\mathbf{W}, \boldsymbol{\rho}_{2m-1}(\mathbf{Wn}), \mathbf{N})$ (see XIAO [35, 39]), where $\boldsymbol{\rho}_{2m-1}(\mathbf{Wn})$ is defined by (1.11). Applying this fact, we derive an irreducible generating set for vector-valued anisotropic functions of the variable $\mathbf{W} \in \mathrm{Skw}$ and the related invariants formed by the inner product between the generic vector variable $\mathbf{r}$ and each vector generator as follows.

V	$\{\boldsymbol{\rho}_{2m-1}(\mathbf{Wn}),\ \mathbf{n}\times\boldsymbol{\rho}_{2m-1}(\mathbf{Wn}),\ (\alpha_{2m}(\mathbf{Wn})\mathbf{n},\ \beta_{2m}(\mathbf{Wn})\mathbf{n}\}\ (= V_m^1(\mathbf{W})),$
R	$(\mathbf{r}\cdot\mathbf{n})\alpha_{2m}(\mathbf{Wn}),\ (\mathbf{r}\cdot\mathbf{n})\beta_{2m}(\mathbf{Wn}).$

In the above table, the generic vector variable $\mathbf{r}$ is treated as being subject to the condition that $\mathbf{u}\times\mathbf{n} = \mathbf{0}$, since the case when there is an $\mathbf{r}$ such that $\mathbf{r}\times\mathbf{n} \neq \mathbf{0}$, i.e. $g(\mathbf{r}) \cap S_{4m}(\mathbf{n}) = C_1$, has been covered by Case 1. The same is true for the next case.

CASE 3. A single symmetric tensor variable **A**

Every scalar-valued (resp. second order tensor-valued) anisotropic function of $\mathbf{A}$ under $S_{4m}(\mathbf{n})$ is equivalent to a scalar-valued (resp. second order tensor-valued) anisotropic function of $\mathbf{W}$ under $C_{4mh}(\mathbf{n})$. Irreducible representations for the latter can be obtained by merely replacing m with $2m$ in the Table for Case 3 of Sec. 4.1. As a result, we only need to consider vector-valued functions and the related invariants. According to XIAO [35, 39], generating sets for vector-valued anisotropic functions of the symmetric tensor variable $\mathbf{A}$ under $S_{4m}(\mathbf{n})$ can be derived from those for isotropic functions of the four extended variables $(\mathbf{N}, \mathbf{A}, \beta_m(\mathbf{q}(\mathbf{A}))\mathbf{n}, \boldsymbol{\rho}'_m(\mathbf{A}))$, where

$$\boldsymbol{\rho}'_m(\mathbf{A}) = \boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{A}}\mathbf{n}) + |\mathbf{q}(\mathbf{A})|^{m-1}\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{n}. \tag{5.8}$$

Basing on the above facts, we construct the following table (refer to the remark at the end of the last case).

V $\{\boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{A}}\mathbf{n}) + |\mathbf{q}(\mathbf{A})|^{m-1}\alpha_m(\mathbf{q}(\mathbf{A}))\mathbf{n}, \mathbf{n}\times\boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{A}}\mathbf{n}) + |\mathbf{q}(\mathbf{A})|^{m-1}\beta_m(\mathbf{q}(\mathbf{A}))\mathbf{n},$
$\alpha_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n})\mathbf{n}, \beta_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n})\mathbf{n}\} (= V^1_m(\mathbf{A})),$

R $(\mathbf{r}\cdot\mathbf{n})\alpha_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n}), (\mathbf{r}\cdot\mathbf{n})\beta_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n}), (\mathbf{r}\cdot\mathbf{n})\alpha_m(\mathbf{q}(\mathbf{A})), (\mathbf{r}\cdot\mathbf{n})\beta_m(\mathbf{q}(\mathbf{A}))$.

We proceed to prove that the set $I^1_m(\mathbf{A})$ given above is an irreducible generating set for the vector-valued anisotropic functions of a symmetric tensor under the group $S_{4m}(\mathbf{n})$. By using (1.2) and the fact (see (5.8) and Case 1)

$$\boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{A}}\mathbf{n})\cdot\overset{\circ}{\mathbf{A}}\mathbf{n} = \left[\mathbf{n}, \overset{\circ}{\mathbf{A}}\mathbf{n}, \boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{A}}\mathbf{n})\right] = 0 \iff \boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{A}}\mathbf{n}) = \mathbf{0} \iff \overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{0}$$

as well as $g(\mathbf{A})\cap S_{4m}(\mathbf{n}) = C_1$ for $\overset{\circ}{\mathbf{A}}\mathbf{n} \neq \mathbf{0}$, we infer that the criterion (1.1) can be satisfied when $\overset{\circ}{\mathbf{A}}\mathbf{n} \neq \mathbf{0}$ and each of the presented generators is irreducible. On the other hand, let $\overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{0}$. Then by using (1.2) and the facts

$$g(\mathbf{A})\cap S_{4m}(\mathbf{n}) = \begin{cases} C_2(\mathbf{n}), & \overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{0},\ \mathbf{q}(\mathbf{A}) \neq \mathbf{0}, \\ S_{4m}(\mathbf{n}), & \overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{q}(\mathbf{A}) = \mathbf{0}, \end{cases}$$

$$\overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{0},\ \mu_m(\overset{\circ}{\mathbf{A}}) = \nu_m(\overset{\circ}{\mathbf{A}}) = 0 \iff \overset{\circ}{\mathbf{A}} = \mathbf{O},$$

we deduce that the criterion (1.1) can also be satisfied when $\overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{O}$, i.e. $\mathbf{n}\times\mathbf{A}\mathbf{n} = \mathbf{0}$.

Combining the above three cases, we arrive at the main result of this section.

THEOREM 5.1. *The four sets given by*

$I^1_m(\mathbf{u}), I^0_{2m}(\mathbf{W}), I^0_{2m}(\mathbf{A})$;

$\overset{\circ}{\mathbf{u}}\cdot\overset{\circ}{\mathbf{v}}, [\mathbf{n}, \overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{v}}], (\mathbf{v}\cdot\mathbf{n})\alpha_{2m}(\overset{\circ}{\mathbf{u}}), (\mathbf{v}\cdot\mathbf{n})\beta_{2m}(\overset{\circ}{\mathbf{u}})$;

$(\mathbf{u}\cdot\mathbf{n})\alpha_{2m}(\mathbf{Wn}), (\mathbf{u}\cdot\mathbf{n})\beta_{2m}(\mathbf{Wn}), \boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{u}})\cdot\mathbf{Wn}, [\mathbf{n}, \mathbf{Wn}, \boldsymbol{\rho}_{2m-1}(\overset{\circ}{\mathbf{u}})];$

$\overset{\circ}{\mathbf{u}}\cdot\mathbf{A}\overset{\circ}{\mathbf{u}}, [\mathbf{n}, \overset{\circ}{\mathbf{u}}, \mathbf{A}\overset{\circ}{\mathbf{u}}], \operatorname{tr}\boldsymbol{\phi}_m(\overset{\circ}{\mathbf{u}})\mathbf{A}, \operatorname{tr}\boldsymbol{\phi}_m(\overset{\circ}{\mathbf{u}})\mathbf{AN}, (\mathbf{u}\cdot\mathbf{n})\alpha_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n}), (\mathbf{u}\cdot\mathbf{n})\beta_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n}),$

$(\mathbf{u}\cdot\mathbf{n})\alpha_m(\mathbf{q}(\mathbf{A})), (\mathbf{u}\cdot\mathbf{n})\beta_m(\mathbf{q}(\mathbf{A}));$

$\mathbf{n}\cdot\mathbf{WHn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{Hn}];$

$\mathbf{n}\cdot\mathbf{WAn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{An}], \mathbf{n}\cdot\mathbf{WAWn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{WAn}];$

$\operatorname{tr}\mathbf{AB}, \operatorname{tr}\mathbf{ABN}, \mathbf{n}\cdot\overset{\circ}{\mathbf{A}}\mathbf{B}\overset{\circ}{\mathbf{A}}\mathbf{n}, [\mathbf{n}, \overset{\circ}{\mathbf{A}}\mathbf{n}, \mathbf{B}\overset{\circ}{\mathbf{A}}\mathbf{n}];$

and

$V_m^1(\mathbf{u}), V_m^1(\mathbf{W}), V_m^1(\mathbf{A});$

and

$\mathrm{Skw}_m^1(\mathbf{u}), \mathrm{Skw}^0(\mathbf{W}), \mathrm{Skw}^0(\mathbf{A});$

and

$\mathrm{Sym}_m^1(\mathbf{u}), \mathrm{Sym}^0(\mathbf{W}), \mathrm{Sym}^0(\mathbf{A});$

where $\mathbf{u}, \mathbf{v} = \mathbf{u}_1, \cdots, \mathbf{u}_a$; $\mathbf{W}, \mathbf{H} = \mathbf{W}_1, \cdots, \mathbf{W}_b$; $\mathbf{A}, \mathbf{B} = \mathbf{A}_1, \cdots, \mathbf{A}_c$; $\mathbf{u} \neq \mathbf{v}$, $\mathbf{W} \neq \mathbf{H}$, $\mathbf{A} \neq \mathbf{B}$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X \in \mathcal{D}$ *under the group* $S_{4m}(\mathbf{n})$ *for each integer* $m \geq 1$, *respectively.*

6. The classes C_{2m+1h}

The classes C_{2m+1h} include the hexagonal crystal class C_{3h} as the particular case when $m = 1$.

Following the scheme designed in Sec. 2, in what follows we discuss various cases for a single variable and two variables.

CASE 1. A single vector variable $\mathbf{u}$

By means of the criterion (1.1) and (1.2)-(1.4) as well as the facts

$$g(\mathbf{u}) \cap C_{2m+1h}(\mathbf{n}) = \begin{cases} C_{2m+1h}(\mathbf{n}), & \mathbf{u} = \mathbf{0}, \\ C_{2m+1}(\mathbf{n}), & \overset{\circ}{\mathbf{u}} = \mathbf{0}, \quad \mathbf{u}\cdot\mathbf{n} \neq 0, \\ C_{1h}(\mathbf{n}), & \mathbf{u}\cdot\mathbf{n} = 0, \quad \overset{\circ}{\mathbf{u}} \neq \mathbf{0}, \\ C_1, & (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{u}} \neq \mathbf{0}, \end{cases} \tag{6.1}$$

we infer that the sets $V^0(\mathbf{u})$, $\mathrm{Skw}^0(\mathbf{u})$ and $\mathrm{Sym}^0(\mathbf{u})$ listed in the table for Case 1 in Sec. 4.1 are also generating sets for vector-valued and skewsymmetric and symmetric tensor-valued anisotropic functions of the vector variable $\mathbf{u}$ under $C_{2m+1h}(\mathbf{n})$. In the following, we show that the set given below is an irreducible functional basis of the vector variable $\mathbf{u}$ under $C_{2m+1h}(\mathbf{n})$.

$$I_m^2(\mathbf{u}) = \{(\mathbf{u}\cdot\mathbf{n})^2, \alpha_{2m+1}(\overset{\circ}{\mathbf{u}}), \beta_{2m+1}(\overset{\circ}{\mathbf{u}})\}. \tag{6.2}$$

In fact, we have

$$I_m^2(\overline{\mathbf{u}}) = I_m^2(\mathbf{u}) \Longrightarrow \begin{cases} \overline{\mathbf{u}}\cdot\mathbf{n} = \delta\mathbf{u}\cdot\mathbf{n}, \ |\overset{\circ}{\overline{\mathbf{u}}}| = |\overset{\circ}{\mathbf{u}}|, \\ \cos(2m+1)\overline{\theta} = \cos(2m+1)\theta, \ \sin(2m+1)\overline{\theta} = \sin(2m+1)\theta, \end{cases}$$

$$\Longrightarrow \overline{\mathbf{u}}\cdot\mathbf{n} = \delta\mathbf{u}\cdot\mathbf{n}, |\overset{\circ}{\overline{\mathbf{u}}}| = |\overset{\circ}{\mathbf{u}}|, \overline{\theta} = \frac{2k\pi}{2m+1} + \theta, k = 0, \pm 1, \cdots$$

$$\Longrightarrow \overline{\mathbf{u}} = \mathbf{Q}\mathbf{u},$$

where $\delta^2 = 1$, $\overline{\theta} =< \overset{\circ}{\overline{\mathbf{u}}}, \mathbf{e} >$, $\theta =< \overset{\circ}{\mathbf{u}}, \mathbf{e} >$, and $\mathbf{Q} \in C_{2m+1h}(\mathbf{n})$ depends on δ, given by

$$\mathbf{Q} = \begin{cases} \mathbf{R}_{\mathbf{n}}^{2k\pi/2m+1}, & \delta = 1, \\ -\mathbf{R}_{\mathbf{n}}^{\pi}\mathbf{R}_{\mathbf{n}}^{2k\pi/2m+1}, & \delta = -1. \end{cases}$$

Thus, we infer that the set $I_m^2(\mathbf{u})$ obeys the criterion (1.5) and hence that it is a functional basis of $\mathbf{u}$ under $C_{2m+1h}(\mathbf{n})$. It is readily shown that this basis is irreducible.

From the above, we conclude that for the case at issue, the derived results are obtained by taking $\delta_{1m} = 0$ and replacing the basis $I_m^0(\mathbf{u})$ with the basis $I_m^2(\mathbf{u})$ in the Table for Case 1 in Sec. 4.1.

CASE 2. A single second order tensor variable

Since a scalar-valued (resp. second order tensor-valued) anisotropic function of a second order tensor variable $\mathbf{x}$ under $C_{2m+1h}(\mathbf{n})$ is equivalent to a scalar-valued (resp. second order tensor-valued) anisotropic functions of $\mathbf{x}$ under $C_{4m+2h}(\mathbf{n})$, we know that irreducible functional basis or generating sets for scalar-valued and skewsymmetric and symmetric tensor-valued anisotropic functions of the variable $\mathbf{x} = \mathbf{W} \in \mathrm{Skw}$ (resp. $\mathbf{x} = \mathbf{A} \in \mathrm{Sym}$) under $C_{2m+1h}(\mathbf{n})$, as well as the invariants formed by means of the inner product between each presented second order tensor generator and a generic second order tensor variable, are given by the corresponding ones listed in the Tables for Cases 2—3 in Sec. 4.1 with m replaced by $2m+1$.

In view of the above facts, in what follows we need only to derive a generating set for the vector-valued anisotropic functions of the variable $\mathbf{W} \in \mathrm{Skw}$ or $\mathbf{A} \in \mathrm{Sym}$ under $C_{2m+1h}(\mathbf{n})$, and meanwhile provide the invariants formed by the inner product between each presented vector generator and the generic vector variable $\mathbf{r}$. The desired generating sets are obtainable from those for vector-valued isotropic functions of the extended variables $(\mathbf{W}, \boldsymbol{\rho}_{2m}(\mathbf{Wn}), \mathbf{N})$ or $(\mathbf{A}, \boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{A}\mathbf{n}}), \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), \mathbf{N})$ (see XIAO [35, 39]).

Applying the above facts, we construct the following tables for the single variables $\mathbf{W}$ and $\mathbf{A}$, respectively.

V	$\{\boldsymbol{\rho}_{2m}(\mathbf{Wn}), \mathbf{n}\times\boldsymbol{\rho}_{2m}(\mathbf{Wn}), \alpha_{2m+1}(\mathbf{Wn})\mathbf{n}, \beta_{2m+1}(\mathbf{Wn})\mathbf{n}\}\ (= V_m^2(\mathbf{W}))$,
R	$\mathbf{r}\cdot\boldsymbol{\rho}_{2m}(\mathbf{Wn}), [\mathbf{n}, \mathbf{r}, \boldsymbol{\rho}_{2m}(\mathbf{Wn})], (\mathbf{r}\cdot\mathbf{n})\alpha_{2m+1}(\mathbf{Wn}), (\mathbf{r}\cdot\mathbf{n})\beta_{2m+1}(\mathbf{Wn})$.

and

$$
\begin{aligned}
V \quad & \{\boldsymbol{\rho}_{2m}(\mathring{\mathbf{A}}\mathbf{n}), \mathbf{n} \times \boldsymbol{\rho}_{2m}(\mathring{\mathbf{A}}\mathbf{n}), |\mathbf{q}(\mathbf{A})|^{m+1}\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})) + \alpha_{2m+1}(\mathring{\mathbf{A}}\mathbf{n})\mathbf{n}, \\
& |\mathbf{q}(\mathbf{A})|^{m+1}\mathbf{n} \times \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})) + \beta_{2m+1}(\mathring{\mathbf{A}}\mathbf{n})\mathbf{n}\} \; (= V_m^2(\mathbf{A})), \\
R \quad & \mathbf{r}\cdot\boldsymbol{\rho}_{2m}(\mathring{\mathbf{A}}\mathbf{n}), [\mathbf{n}, \mathbf{r}, \boldsymbol{\rho}_{2m}(\mathring{\mathbf{A}}\mathbf{n})], \alpha_{2m+1}(\mathring{\mathbf{A}}\mathbf{n})(\mathbf{r}\cdot\mathbf{n}), \beta_{2m+1}(\mathring{\mathbf{A}}\mathbf{n})(\mathbf{r}\cdot\mathbf{n}).
\end{aligned}
$$

We need to prove that the two sets $V_m^2(\mathbf{W})$ and $V_m^2(\mathbf{A})$ given in the above tables obey the criterion (1.1). First, let $\mathbf{z}$ be a vector in the $\mathbf{n}$-plane. Then we have

$$\boldsymbol{\rho}_m(\mathbf{z}) = \mathbf{0} \iff \mathbf{z} = \mathbf{0}. \tag{6.3}$$

From this and

$$g(\mathbf{W}) \cap C_{2m+1h}(\mathbf{n}) = \begin{cases} C_{2m+1h}(\mathbf{n}), & \mathbf{W}\mathbf{n} = \mathbf{0}, \\ C_1, & \mathbf{W}\mathbf{n} \neq \mathbf{0}, \end{cases} \tag{6.4}$$

$$g(\mathbf{A}) \cap C_{2m+1h}(\mathbf{n}) = \begin{cases} C_{2m+1h}(\mathbf{n}), & \mathring{\mathbf{A}}\mathbf{n} = \mathbf{q}(\mathbf{A}) = \mathbf{0}, \\ C_{1h}(\mathbf{n}), & \mathbf{q}(\mathbf{A}) \neq \mathbf{0}, \quad \mathring{\mathbf{A}}\mathbf{n} = \mathbf{0}, \\ C_1, & \mathring{\mathbf{A}}\mathbf{n} \neq \mathbf{0}, \end{cases} \tag{6.5}$$

as well as (1.2), we conclude that $V_m^2(\mathbf{W})$ and $V_m^2(\mathbf{A})$ obey the criterion (1.1) separately.

Furthermore, let $\mathbf{W}_i = \mathbf{n} \wedge \mathbf{e}_i, \mathbf{A}_i = \mathbf{n} \vee \mathbf{e}_i, i = 1, 2$. Then for $i, j = 1, 2$,

$$\sin(2m+1) < \mathbf{W}_1\mathbf{n}, \mathbf{e} >= 0, \qquad \cos(2m+1) < \mathbf{W}_2\mathbf{n}, \mathbf{e} >= 0,$$
$$g(\mathbf{W}_i) \cap C_{2m+1h}(\mathbf{n}) = C_1;$$

$$\sin(2m+1) < \mathring{\mathbf{A}}_1\mathbf{n}, \mathbf{e} >= 0, \qquad \cos(2m+1) < \mathring{\mathbf{A}}_2\mathbf{n}, \mathbf{e} >= 0,$$
$$g(\mathbf{A}_i) \cap C_{2m+1h}(\mathbf{n}) = C_1.$$

Thus, we infer that either of the two generating sets $V_m^2(\mathbf{W})$ and $V_m^2(\mathbf{A})$ is irreducible.

CASE 3. Two vector variables $(\mathbf{u}, \mathbf{v})$

From the condition (4.12), where $C_{2mh}(\mathbf{n})$ is replaced by $C_{2m+1h}(\mathbf{n})$, we infer that the vector variables $(\mathbf{u}, \mathbf{v})$ are specified by (4.13). Hence, we construct the following table (the first case in (4.13) is considered).

V	$\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}} \times \mathbf{n}, (\mathbf{v}\cdot\mathbf{n})\mathbf{n}$
Skw	$\mathbf{N}, \mathbf{u} \wedge \mathbf{v}, (\mathbf{u} \times \mathbf{n}) \wedge \mathbf{v} + (\mathbf{v} \times \mathbf{n}) \wedge \mathbf{u}$
Sym	$\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \overset{\circ}{\mathbf{u}} \otimes \overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}} \vee (\overset{\circ}{\mathbf{u}} \times \mathbf{n}); \mathbf{u} \vee \mathbf{v}, (\mathbf{u} \times \mathbf{n}) \vee \mathbf{v} + (\mathbf{v} \times \mathbf{n}) \vee \mathbf{u}$
R	$\mathbf{u}\cdot\mathbf{H}\mathbf{v}, [\mathbf{n}, \mathbf{u}, \mathbf{H}\mathbf{v}] + [\mathbf{n}, \mathbf{v}, \mathbf{H}\mathbf{u}];$ $\mathbf{u}\cdot\mathbf{B}\mathbf{v}, [\mathbf{n}, \mathbf{u}, \mathbf{B}\mathbf{v}] + [\mathbf{n}, \mathbf{v}, \mathbf{B}\mathbf{u}];$ $< (\mathbf{v}\cdot\mathbf{n})^2, \alpha_{2m+1}(\overset{\circ}{\mathbf{u}}), \beta_{2m+1}(\overset{\circ}{\mathbf{u}}) > .$

The results listed in the above table can easily be verified by means of the condition (4.13).

CASE 4. A vector variable $\mathbf{u}$ and a skewsymmetric tensor variable $\mathbf{W}$

From (6.1) and (6.4) we infer that there is $\mathbf{x} \in \{\mathbf{u}, \mathbf{W}\}$ such that

$$g(\mathbf{x}) \cap C_{2m+1h}(\mathbf{n}) = g(\mathbf{u}, \mathbf{W}) \cap C_{2m+1h}(\mathbf{n}).$$

Thus, the case at issue can be reduced to the cases for the single variables $\mathbf{u}$ and $\mathbf{W}$.

CASE 5. A vector variable $\mathbf{u}$ and a symmetric tensor variable $\mathbf{A}$

The condition (2.6) yields

$$g(\mathbf{u}, \mathbf{A}) \cap C_{2m+1h}(\mathbf{n}) \neq g(\mathbf{x}) \cap C_{2m+1h}(\mathbf{n}), \qquad \mathbf{x} = \mathbf{u}, \mathbf{A}.$$

From (6.1) and (6.5) and the above condition we derive (cf. §3.3 in XIAO [34] for detail)

$$\begin{aligned} &\mathbf{u} = a\mathbf{n}, \qquad a \neq 0, \\ &\mathbf{A} = x(\mathbf{e} \otimes \mathbf{e} - \mathbf{e}' \otimes \mathbf{e}') + y\mathbf{e} \vee \mathbf{e}' + c\mathbf{I} + d\mathbf{n} \otimes \mathbf{n}, \qquad x^2 + y^2 \neq 0. \end{aligned} \tag{6.6}$$

By means of the criterion (1.5), it can easily be verified that for the variables $(\mathbf{u}, \mathbf{A})$ specified above, the union $\{(\mathbf{u}\cdot\mathbf{n})^2\} \cup I_m^2(\mathbf{A})$ supplies an irreducible functonal basis of the variables $(\mathbf{u}, \mathbf{A})$ specified above under $C_{2m+1h}(\mathbf{n})$. On the other hand, generating sets for vector-valued and second order tensor-valued anisotropic functions of the variables $(\mathbf{u}, \mathbf{A})$ under $C_{2m+1h}(\mathbf{n})$ can be derived from vector-valued and second order tensor-valued isotropic functions of the extended variables $(\mathbf{u}, \mathbf{A}, \boldsymbol{\rho}_m(\mathbf{q}(A)), \mathbf{N})$ (see XIAO [35, 39]). Thus, we construct the following table for irreducible generating sets.

V	$(\mathbf{u}\cdot\mathbf{n})\mathbf{n}, \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})) + \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\mathbf{n})\mathbf{n}, \mathbf{n} \times \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})) + \beta_{2m+1}(\overset{\circ}{\mathbf{A}}\mathbf{n})\mathbf{n}$
Skw	$\mathbf{N}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \wedge \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \wedge (\mathbf{n} \times \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})))$
Sym	$\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \mathbf{A}, \mathbf{A}\mathbf{N} - \mathbf{N}\mathbf{A}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \vee \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \vee (\mathbf{n} \times \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))).$

The above results can easily be verified by means of the condition (6.6). We mentioned that in the above table, the generic variables $\mathbf{r}, \mathbf{H}, \mathbf{B} \in X \in \mathcal{D}$ are treated as being subject to the conditions: $\mathbf{r} \times \mathbf{n} = \mathbf{0}$, $\mathbf{H}\mathbf{n} = \mathbf{0}$ and $\overset{\circ}{\mathbf{A}}\mathbf{n} = \mathbf{0}$

respectively (hence no new invariants appear), since any set $X \in \mathcal{D}$ violating the just-stated conditions has been covered by one of the preceding four cases.

Combining the above cases, we arrive at the main result of this section as follows.

THEOREM 6.1. *The four sets given by*

$I_m^2(\mathbf{u}), I_{2m+1}^0(\mathbf{W}), I_{2m+1}^0(\mathbf{A})$;

$\mathring{\mathbf{u}}\cdot\mathring{\mathbf{v}}, (\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n}), [\mathbf{n}, \mathring{\mathbf{u}}, \mathring{\mathbf{v}}]$;

$\mathbf{n}\cdot\mathbf{WHn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{Hn}]$;

$\operatorname{tr}\mathbf{AB}, \operatorname{tr}\mathbf{ABN}, \mathbf{n}\cdot\mathring{\mathbf{A}}\mathbf{B}\mathring{\mathbf{A}}\mathbf{n}, [\mathbf{n}, \mathring{\mathbf{A}}\mathbf{n}, \mathbf{B}\mathring{\mathbf{A}}\mathbf{n}]$;

$\mathbf{n}\cdot\mathbf{WAn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{An}], \mathbf{n}\cdot\mathbf{WAWn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{WAn}]$;

$(\mathbf{u}\cdot\mathbf{n})\mathring{\mathbf{u}}\cdot\mathbf{Wn}, (\mathbf{u}\cdot\mathbf{n})[\mathbf{n}, \mathring{\mathbf{u}}, \mathbf{Wn}], \mathbf{u}\cdot\boldsymbol{\rho}_{2m}(\mathbf{Wn}), [\mathbf{n}, \mathbf{u}, \boldsymbol{\rho}_{2m}(\mathbf{Wn})]$,

$(\mathbf{u}\cdot\mathbf{n})\alpha_{2m+1}(\mathbf{Wn}), (\mathbf{u}\cdot\mathbf{n})\beta_{2m+1}(\mathbf{Wn})$;

$\mathring{\mathbf{u}}\cdot\mathbf{A}\mathring{\mathbf{u}}, [\mathbf{n}, \mathring{\mathbf{u}}, \mathbf{A}\mathring{\mathbf{u}}], (\mathbf{u}\cdot\mathbf{n})\mathring{\mathbf{u}}\cdot\mathbf{An}, (\mathbf{u}\cdot\mathbf{n})[\mathbf{n}, \mathring{\mathbf{u}}, \mathbf{An}]$,

$\mathbf{u}\cdot\boldsymbol{\rho}_{2m}(\mathring{\mathbf{A}}\mathbf{n}), [\mathbf{n}, \mathbf{u}, \boldsymbol{\rho}_{2m}(\mathring{\mathbf{A}}\mathbf{n})], \alpha_{2m+1}(\mathring{\mathbf{A}}\mathbf{n})(\mathbf{u}\cdot\mathbf{n}), \beta_{2m+1}(\mathring{\mathbf{A}}\mathbf{n})(\mathbf{u}\cdot\mathbf{n})$;

$\mathbf{u}\cdot\mathbf{Wv}, [\mathbf{n}, \mathbf{u}, \mathbf{Wv}] + [\mathbf{n}, \mathbf{v}, \mathbf{Wu}]$;

$\mathbf{u}\cdot\mathbf{Av}, [\mathbf{n}, \mathbf{u}, \mathbf{Av}] + [\mathbf{n}, \mathbf{v}, \mathbf{Au}]$;

and

$V^0(\mathbf{u}), V_m^2(\mathbf{W}), V_m^2(\mathbf{A})$;

and

$\operatorname{Skw}^0(\mathbf{u}), \operatorname{Skw}^0(\mathbf{W}), \operatorname{Skw}^0(\mathbf{A})$;

$\mathbf{u}\wedge\mathbf{v}, (\mathbf{u}\times\mathbf{n})\wedge\mathbf{v} + (\mathbf{v}\times\mathbf{n})\wedge\mathbf{u}$;

$(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\wedge(\mathbf{n}\times\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})))$;

and

$\operatorname{Sym}^0(\mathbf{u}), \operatorname{Sym}^0(\mathbf{W}), \operatorname{Sym}^0(\mathbf{A})$;

$\mathbf{u}\vee\mathbf{v}, (\mathbf{u}\times\mathbf{n})\vee\mathbf{v} + (\mathbf{v}\times\mathbf{n})\vee\mathbf{u}$;

$(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), (\mathbf{u}\cdot\mathbf{n})\mathbf{n}\vee(\mathbf{n}\times\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})))$;

where $\mathbf{u}, \mathbf{v} = \mathbf{u}_1, \cdots, \mathbf{u}_a$; $\mathbf{W}, \mathbf{H} = \mathbf{W}_1, \cdots, \mathbf{W}_b$; $\mathbf{A}, \mathbf{B} = \mathbf{A}_1, \cdots, \mathbf{A}_c$; $\mathbf{u} \neq \mathbf{v}$, $\mathbf{W} \neq \mathbf{H}$, $\mathbf{A} \neq \mathbf{B}$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X \in \mathcal{D}$ *under the group* $C_{2m+1h}(\mathbf{n})$ *for each integer* $m \geq 1$, *respectively.*

7. The classes S_{4m+2} and C_{2m+1}

The classes at issue include the trigonal crystal classes S_6 and C_3 as the particular case when $m = 1$.

7.1. The classes S_{4m+2}

With the aid of the fact

$$g(\mathbf{r}) \cap S_{4m+2}(\mathbf{n}) = \begin{cases} S_{4m+2}(\mathbf{n}), & \mathbf{r} = \mathbf{0}, \\ C_{2m+1}(\mathbf{n}), & \overset{\circ}{\mathbf{r}} = \mathbf{0}, \quad \mathbf{r}\cdot\mathbf{n} \neq 0, \\ C_1, & \overset{\circ}{\mathbf{r}} \neq \mathbf{0}, \end{cases} \tag{7.1}$$

for any vector $\mathbf{r} \in V$, we infer that for any two vectors $(\mathbf{u}, \mathbf{v})$, there exists $\mathbf{z} \in \{\mathbf{u}, \mathbf{v}\}$ such that

$$g(\mathbf{u}, \mathbf{v}) \cap S_{4m+2}(\mathbf{n}) = g(\mathbf{z}) \cap S_{4m+2}\,.$$

This indicates that the case for two vector variables can be reduced to the case for a single vector variable. Accordingly, following the scheme outlined in Sec. 2, five classes are discussed as follows.

CASE 1. A single vector variable $\mathbf{u}$

A scalar-valued (resp. second order tensor-valued) anisotropic function of the vector variable $\mathbf{u}$ under $S_{4m+2}(\mathbf{n})$ is equivalent to a scalar-valued (resp. second order tensor-valued) anisotropic function of the symmetric tensor variable $\mathbf{u} \otimes \mathbf{u}$ under $S_{4m+2}(\mathbf{n})$. As a result, generating sets for the former can be derived by taking $\mathbf{A} = \mathbf{u} \otimes \mathbf{u}$ in the corresponding generating sets listed in the table for Case 3 below. Moreover, according to XIAO [35, 39], generating sets for vector-valued anisotropic functions of $\mathbf{u}$ under $S_{4m+2}(\mathbf{n})$ can be obtained from those for vector-valued isotropic functions of the extended variables $(\mathbf{u}, \mathbf{E}\boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}}), \mathbf{N})$, where $\boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}})$ is given by (1.11).

Basing on the above facts, we construct the following table.

V	$\{\mathbf{u}, \overset{\circ}{\mathbf{u}} \times \mathbf{n}, \alpha_{2m+1}(\overset{\circ}{\mathbf{u}})\mathbf{n}, \beta_{2m+1}(\overset{\circ}{\mathbf{u}})\mathbf{n}\}$ $(= V_m^3(\mathbf{u}))$
Skw	$\{\mathbf{N}, \mathbf{n} \wedge \boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}}), \mathbf{n} \wedge (\mathbf{n} \times \boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}}))\}$ $(= \mathrm{Skw}_m^3(\mathbf{u}))$
Sym	$\{\mathbf{I}, \mathbf{n} \otimes \mathbf{n}, \overset{\circ}{\mathbf{u}} \otimes \overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}} \vee (\mathbf{n} \times \overset{\circ}{\mathbf{u}}), \mathbf{n} \vee \boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}}), \mathbf{n} \vee (\mathbf{n} \times \boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}}))\}$ $(= \mathrm{Sym}_m^3(\mathbf{u}))$
R	$\mathbf{u}\cdot\mathbf{r}, [\mathbf{n}, \overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{r}}], (\mathbf{r}\cdot\mathbf{n})\alpha_{2m+1}(\overset{\circ}{\mathbf{u}}), (\mathbf{r}\cdot\mathbf{n})\beta_{2m+1}(\overset{\circ}{\mathbf{u}})$; $\mathrm{tr}\,\mathbf{HN}, \boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}})\cdot\mathbf{Hn}, [\mathbf{n}, \mathbf{Hn}, \boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}})]$; $\mathrm{tr}\,\mathbf{B}, \mathbf{n}\cdot\mathbf{Bn}, \overset{\circ}{\mathbf{u}}\cdot\mathbf{B}\overset{\circ}{\mathbf{u}}, [\mathbf{n}, \overset{\circ}{\mathbf{u}}, \mathbf{B}\overset{\circ}{\mathbf{u}}], \boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}})\cdot\mathbf{Bn}, [\mathbf{n}, \mathbf{Bn}, \boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}})]$; $< (\mathbf{u}\cdot\mathbf{n})\alpha_{2m+1}(\overset{\circ}{\mathbf{u}}), (\mathbf{u}\cdot\mathbf{n})\beta_{2m+1}(\overset{\circ}{\mathbf{u}}), \alpha_{4m+2}(\overset{\circ}{\mathbf{u}}), \beta_{4m+2}(\overset{\circ}{\mathbf{u}}) >$ $(= I_m^3(\mathbf{u}))$.

With the aid of (7.1) and (1.2) – (1.4), it can be verified that the first three sets given above, denoted by $V_m^3(\mathbf{u})$, $\mathrm{Skw}_m^3(\mathbf{u})$ and $\mathrm{Sym}_m^3(\mathbf{u})$ henceforth, obey the criterion (1.1) and hence they are the desired generating sets.

CASE 2. A single skewsymmetric tensor variable $\mathbf{W}$

Every vector-valued anisotropic function of $\mathbf{W}$ under the group S_{4m+2} vanishes. Anisotropic functional bases of the variable $\mathbf{W} \in \mathrm{Skw}$ under $S_{4m+2}(\mathbf{n})$ can be obtained from isotropic functional bases of the extended variables

$(\mathbf{W}, \mathbf{E}\boldsymbol{\rho}_{2m}(\mathbf{Wn}), \mathbf{N})$ (see Xiao [35, 39]). Moreover, the generating sets $\mathrm{Skw}^0(\mathbf{W})$ and $\mathrm{Sym}^0(\mathbf{W})$ under $C_{2mh}(\mathbf{n})$ for each $m \geq 2$ (cf. Case 2 in Sec. 4.1) also provide irreducible generating sets for skewsymmetric and symmetric tensor-valued anisotropic functions of the variable $\mathbf{W}$ under $S_{4m+2}(\mathbf{n})$.

An irreducible functional basis of $\mathbf{W}$ under $S_{4m+2}(\mathbf{n})$ is given by

$$I^3_m(\mathbf{W}) = \{\mathrm{tr}\,\mathbf{WN},\ \alpha_{2m+1}(\mathbf{Wn}),\ \beta_{2m+1}(\mathbf{Wn})\}. \tag{7.2}$$

This result can be verified by means of the procedure used at Case 1 in Sec. 6.

Case 3. A single symmetric tensor variable $\mathbf{A}$

Every vector-valued anisotropic function of the variable $\mathbf{A} \in \mathrm{Sym}$ under $S_{4m+2}(\mathbf{n})$ vanishes. Irreducible representations for scalar-valued and second order tensor-valued anisotropic functions of the single variable $\mathbf{A} \in \mathrm{Sym}$ under $S_{4m+2}(\mathbf{n})$ can be obtained from those for scalar-valued and second order tensor-valued isotropic functions of the extended variables

$$(\mathbf{A}, \mathbf{E}\boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{A}}\mathbf{n}), \mathbf{E}\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), \mathbf{N})$$

(see Xiao [35, 39]).

Based on this fact, we construct the following table.

Skw $\{\mathbf{N}, \mathbf{n}\wedge\overset{\circ}{\mathbf{A}}\mathbf{n}, \mathbf{n}\wedge(\mathbf{n}\times\overset{\circ}{\mathbf{A}}\mathbf{n}), \mathbf{n}\wedge\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), \mathbf{n}\wedge(\mathbf{n}\times\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})))\}$ $(=\mathrm{Skw}^3_m(\mathbf{A}))$

Sym $\{\mathbf{I}, \mathbf{n}\otimes\mathbf{n}, \mathbf{A}, \mathbf{AN}-\mathbf{NA}, \overset{\circ}{\mathbf{A}}\mathbf{n}\otimes\overset{\circ}{\mathbf{A}}\mathbf{n}, \overset{\circ}{\mathbf{A}}\mathbf{n}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{A}}\mathbf{n}),$
$\mathbf{n}\vee\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), \mathbf{n}\vee(\mathbf{n}\times\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})))\}$ $(=\mathrm{Sym}^3_m(\mathbf{A}))$

R $\mathrm{tr}\,\mathbf{HN}; \mathrm{tr}\,\mathbf{B}, \mathbf{n}\cdot\mathbf{Bn}, \mathrm{tr}\,\mathbf{BA}, \mathrm{tr}\,\mathbf{BAN},$
$\mathbf{n}\cdot\overset{\circ}{\mathbf{A}}\mathbf{B}\overset{\circ}{\mathbf{A}}\mathbf{n}, [\mathbf{n}, \overset{\circ}{\mathbf{A}}\mathbf{n}, \mathbf{B}\overset{\circ}{\mathbf{A}}\mathbf{n}], \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))\cdot\mathbf{Bn}, [\mathbf{n}, \mathbf{Bn}, \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))];$
$< I^3_m(\mathbf{A}) >$

In the above, the set $I^3_m(\mathbf{A})$ is an irreducible functional basis of a symmetric tensor $\mathbf{A}$ under the group S_{4m+2}, given by (see Xiao [38])

$$I^3_m(\mathbf{A}) = \{\mathbf{n}\cdot\mathbf{An}, \mathrm{tr}\,\mathbf{A}, \mathrm{tr}\,\mathbf{A}^3, [\mathbf{n}, \mathbf{An}, \mathbf{A}^2\mathbf{n}], \alpha_{2m+1}(\overset{\circ}{\mathbf{A}}\mathbf{n}), \beta_{2m+1}(\overset{\circ}{\mathbf{A}}\mathbf{n}), \alpha_{2m+1}(\mathbf{q}(\mathbf{A})), \beta_{2m+1}(\mathbf{q}(\mathbf{A}))\}. \tag{7.3}$$

By means of the criterion (1.1) and the facts

$$g(\mathbf{A}) \cap S_{4m+2}(\mathbf{n}) = \begin{cases} S_{4m+2}(\mathbf{n}), & \mathbf{A} = x\mathbf{I} + y\mathbf{n}\otimes\mathbf{n}, \\ S_2, & |\mathbf{q}(\mathbf{A})|^2 + |\overset{\circ}{\mathbf{A}}\mathbf{n}|^2 \neq 0, \end{cases} \tag{7.4}$$

it can be verified that the first two sets in the above table, denoted by $\mathrm{Skw}^3_m(\mathbf{A})$ and $\mathrm{Sym}^3_m(\mathbf{A})$ henceforth, are generating sets for skewsymmetric and symmetric tensor-valued anisotropic functions of the variable $\mathbf{A} \in \mathrm{Sym}$ under $S_{4m+2}(\mathbf{n})$ and moreover, that the two sets are irreducible.

It should be pointed out that in the above table, the generic skewsymmetric tensor variable $\mathbf{H}$ has been treated as being subject to the condition that $\mathbf{Hn} = \mathbf{0}$, since the case when $\mathbf{Wn} \neq \mathbf{0}$ has been covered by the last case. As a result, of the five invariants obtained by forming the inner product between $\mathbf{H}$ and each of the five skewsymmetric tensor generators presented, only one, i.e. $\mathrm{tr}\,\mathbf{HN}$, does not vanish.

REMARK. The generating set $\mathrm{Sym}^3_m(\mathbf{A})$ consists of eight generators. For $m = 1$, i.e. the trigonal crystal class S_6, a minimal generating set of six generators is available (see XIAO [37]). For the sake of consistency, here for all $m \geq 1$ we employ the set $\mathrm{Sym}^3_m(\mathbf{A})$, which supplies a unified form of all generating sets in question.

CASE 4. A vector variable $\mathbf{u}$ and a skewsymmetric tensor variable $\mathbf{W}$

The condition (2.6) yields

$$g(\mathbf{u}, \mathbf{W}) \cap S_{4m+2}(\mathbf{n}) \neq g(\mathbf{x}) \cap S_{4m+2}(\mathbf{n}), \qquad \mathbf{x} = \mathbf{u}, \mathbf{W}.$$

From (7.1) and the above condition we derive

$$\begin{aligned} &\mathbf{u} = a\mathbf{n}, \quad a \neq 0\,, \\ &\mathbf{Wn} \neq \mathbf{0}, \quad \text{i.e. } g(\mathbf{W}) \cap S_{4m+2}(\mathbf{n}) = S_2\,. \end{aligned} \tag{7.5}$$

As a result, $\mathrm{Skw}^0(\mathbf{W})$ and $\mathrm{Sym}^0(\mathbf{W})$ supply the desired irreducible generating sets for second order tensor-valued functions. It is evident that the union $I^3_m(\mathbf{W}) \cup \{(\mathbf{u}\cdot\mathbf{n})^2\}$ provides an irreducible functional basis of the variables $(\mathbf{u}, \mathbf{W})$ specified by (7.5) under $S_{4m+2}(\mathbf{n})$. Moreover, the generic vector variable $\mathbf{r}$ is subject to the condition that $\mathring{\mathbf{r}} = \mathbf{0}$ (the other case has been covered at Case 1). Thus, we need only to provide an irreducible generating set for vector-valued functions of the variables $(\mathbf{u}, \mathbf{W})$ specified by (7.5) under $S_{4m+2}(\mathbf{n})$, which is given as follows.

$$V^3_m(\mathbf{u}, \mathbf{W}) = \{\mathbf{u}, (\mathbf{u}\cdot\mathbf{n})\boldsymbol{\rho}_{2m}(\mathbf{Wn}), (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \times \boldsymbol{\rho}_{2m}(\mathbf{Wn})\}\,. \tag{7.6}$$

CASE 5. A vector variable $\mathbf{u}$ and a symmetric tensor variable $\mathbf{A}$

The condition (2.6) yields

$$g(\mathbf{u}, \mathbf{A}) \cap S_{4m+2}(\mathbf{n}) \neq g(\mathbf{x}) \cap S_{4m+2}(\mathbf{n}), \qquad \mathbf{x} = \mathbf{u}, \mathbf{A}.$$

From (7.1) and (7.4) and the above condition we derive

$$\begin{aligned} &\mathbf{u} = a\mathbf{n}, \quad a \neq 0\,, \\ &g(\mathbf{A}) \cap S_{4m+2}(\mathbf{n}) = S_2\,. \end{aligned} \tag{7.7}$$

As a result, $\mathrm{Skw}_m^3(\mathbf{A})$ and $\mathrm{Sym}_m^3(\mathbf{A})$ supply the desired irreducible generating sets for second order tensor-valued functions. It is evident that the union $I_m^3(\mathbf{A}) \cup \{(\mathbf{u}\cdot\mathbf{n})^2\}$ provides an irreducible functional basis of the variables $(\mathbf{u}, \mathbf{A})$ specified by (7.7) under $S_{4m+2}(\mathbf{n})$. Moreover, the generic vector variable $\mathbf{r}$ is subject to the condition that $\overset{\circ}{\mathbf{r}} = \mathbf{0}$ (the other case has been covered at Case 1). Thus, we need only to provide an irreducible generating set for vector-valued functions of the variables $(\mathbf{u}, \mathbf{A})$ specified by (7.7) under $S_{4m+2}(\mathbf{n})$, which is given by

$$V_m^3(\mathbf{u}, \mathbf{A}) = \{\mathbf{u}, (\mathbf{u}\cdot\mathbf{n})\overset{\circ}{\mathbf{A}}\mathbf{n}, (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \times \overset{\circ}{\mathbf{A}}\mathbf{n}, (\mathbf{u}\cdot\mathbf{n})\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), (\mathbf{u}\cdot\mathbf{n})\mathbf{n} \times \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}))\}. \tag{7.8}$$

Combining the above cases, we arrive at the main result of this subsection as follows.

THEOREM 7.1. *The four sets given by*

$I_m^3(\mathbf{u}), I_m^3(\mathbf{W}), I_m^3(\mathbf{A})$;

$\mathbf{u}\cdot\mathbf{v}, [\mathbf{n}, \overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{v}}], (\mathbf{u}\cdot\mathbf{n})\alpha_{2m+1}(\overset{\circ}{\mathbf{v}}), (\mathbf{u}\cdot\mathbf{n})\beta_{2m+1}(\overset{\circ}{\mathbf{v}})$;

$\mathbf{n}\cdot\mathbf{WHn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{Hn}]$;

$\mathbf{n}\cdot\mathbf{WAn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{An}], \mathbf{n}\cdot\mathbf{WAWn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{AWn}]$;

$\mathrm{tr}\,\mathbf{AB}, \mathrm{tr}\,\mathbf{ABN}, \mathbf{n}\cdot\overset{\circ}{\mathbf{A}}\mathbf{B}\overset{\circ}{\mathbf{A}}\mathbf{n}, [\mathbf{n}, \overset{\circ}{\mathbf{A}}\mathbf{n}, \mathbf{B}\overset{\circ}{\mathbf{A}}\mathbf{n}], \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{B}))\cdot\mathbf{An}, [\mathbf{n}, \mathbf{An}, \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{B}))]$;

$\boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}})\cdot\mathbf{Wn}, [\mathbf{n}, \mathbf{Wn}, \boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}})]$;

$\overset{\circ}{\mathbf{u}}\cdot\mathbf{A}\overset{\circ}{\mathbf{u}}, [\mathbf{n}, \overset{\circ}{\mathbf{u}}, \mathbf{A}\overset{\circ}{\mathbf{u}}], \boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}})\cdot\mathbf{An}, [\mathbf{n}, \mathbf{An}, \boldsymbol{\rho}_{2m}(\overset{\circ}{\mathbf{u}})]$;

and

$V_m^3(\mathbf{u}),\ V_m^3(\mathbf{u}, \mathbf{W}),\ V_m^3(\mathbf{u}, \mathbf{A})$;

and

$\mathrm{Skw}_m^3(\mathbf{u}),\ \mathrm{Skw}^0(\mathbf{W}),\ \mathrm{Skw}_m^3(\mathbf{A})$;

and

$\mathrm{Sym}_m^3(\mathbf{u}),\ \mathrm{Sym}^0(\mathbf{W}),\ \mathrm{Sym}_m^3(\mathbf{A})$;

where $\mathbf{u}, \mathbf{v} = \mathbf{u}_1, \cdots, \mathbf{u}_a$; $\mathbf{W}, \mathbf{H} = \mathbf{W}_1, \cdots, \mathbf{W}_b$; $\mathbf{A}, \mathbf{B} = \mathbf{A}_1, \cdots, \mathbf{A}_c$; $\mathbf{u} \neq \mathbf{v}$, $\mathbf{W} \neq \mathbf{H}$, $\mathbf{A} \neq \mathbf{B}$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X \in \mathcal{D}$ *under the group* $S_{4m+2}(\mathbf{n})$ *for each integer* $m \geq 1$, *respectively.*

7.2. The classes C_{2m+1}

Applying the argument given in Sec. 4.2 and Theorem 7.1, we derive the following result.

THEOREM 7.2. *The four sets given by*

$I_m^3(\mathbf{W}), I_m^3(\mathbf{A}), \mathbf{u}\cdot\mathbf{n}, \alpha_{2m+1}(\overset{\circ}{\mathbf{u}}), \beta_{2m+1}(\overset{\circ}{\mathbf{u}}); \mathbf{u}\cdot\mathbf{v}, [\mathbf{u}, \mathbf{v}, \mathbf{n}]$;

$\mathbf{n}\cdot\mathbf{WHn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{Hn}]$;

$\mathbf{n}\cdot\mathbf{WAn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{An}], \mathbf{n}\cdot\mathbf{WAWn}, [\mathbf{n}, \mathbf{Wn}, \mathbf{AWn}]$;

$\mathrm{tr}\,\mathbf{AB}, \mathrm{tr}\,\mathbf{ABN}, \mathbf{n}\cdot\overset{\circ}{\mathbf{A}}\mathbf{B}\overset{\circ}{\mathbf{A}}\mathbf{n}, [\mathbf{n}, \overset{\circ}{\mathbf{A}}\mathbf{n}, \mathbf{B}\overset{\circ}{\mathbf{A}}\mathbf{n}], \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{B}))\cdot\mathbf{An}, [\mathbf{n}, \mathbf{An}, \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{B}))]$;

$\mathbf{u}\cdot\mathbf{Wn}, [\mathbf{n}, \mathbf{u}, \mathbf{Wn}]$;

$\mathbf{u}\cdot\mathbf{A}\mathbf{n}, [\mathbf{n},\mathbf{u},\mathbf{A}\mathbf{n}], \overset{\circ}{\mathbf{u}}\cdot\mathbf{A}\overset{\circ}{\mathbf{u}}, [\mathbf{n},\mathbf{u},\mathbf{A}\mathbf{u}];$

and

$\mathbf{n}, \overset{\circ}{\mathbf{u}}, \mathbf{n}\times\overset{\circ}{\mathbf{u}}, \mathbf{W}\mathbf{n}, \mathbf{n}\times\mathbf{W}\mathbf{n}, \overset{\circ}{\mathbf{A}}\mathbf{n}, \mathbf{n}\times\overset{\circ}{\mathbf{A}}\mathbf{n}, \boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A})), \mathbf{n}\times\boldsymbol{\rho}_m(\mathbf{q}(\mathbf{A}));$

and

$\mathrm{Skw}^0(\mathbf{W}), \mathrm{Skw}^3_m(\mathbf{A}), \mathbf{n}\wedge\overset{\circ}{\mathbf{u}}, \mathbf{n}\wedge(\mathbf{n}\times\overset{\circ}{\mathbf{u}});$

and

$\mathrm{Sym}^0(\mathbf{W}), \mathrm{Sym}^3_m(\mathbf{A}), \overset{\circ}{\mathbf{u}}\otimes\overset{\circ}{\mathbf{u}}, \overset{\circ}{\mathbf{u}}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{u}}), \mathbf{n}\vee\overset{\circ}{\mathbf{u}}, \mathbf{n}\vee(\mathbf{n}\times\overset{\circ}{\mathbf{u}});$

where $\mathbf{u},\mathbf{v}=\mathbf{u}_1,\cdots,\mathbf{u}_a$; $\mathbf{W},\mathbf{H}=\mathbf{W}_1,\cdots,\mathbf{W}_b$; $\mathbf{A},\mathbf{B}=\mathbf{A}_1,\cdots,\mathbf{A}_c$; $\mathbf{u}\neq\mathbf{v}$, $\mathbf{W}\neq\mathbf{H}$, $\mathbf{A}\neq\mathbf{B}$, *provide irreducible representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of the variables* $X\in\mathcal{D}$ *under the group* $C_{2m+1}(\mathbf{n})$ *for each integer* $m\geq 1$, *respectively.*

8. Remarks

In the previous sections, complete nonpolynomial representations for scalar-, vector-, skewsymmetric and symmetric tensor-valued anisotropic functions of any finite number of vector and second order tensor variables under all kinds of subgroups of the transverse isotropy group $C_{\infty h}$ are derived, each of which consists of polynomial invariants or polynomial generators. The presented results offer unified forms of general representations for infinitely many subgroup classes concerned, respectively.

It can be seen that infinitely many different types of vector-valued or second order-tensor-valued anisotropic functions may have a common generating set. Indeed, the set $\mathrm{Skw}^0(\mathbf{W})$ is a common generating set which applies to all subgroups of $C_{\infty h}$ except the triclinic groups. Other examples are the sets $\mathrm{Skw}^0(\mathbf{u})$, $\mathrm{Skw}^0(\mathbf{A})$, $\mathrm{Sym}^0(\mathbf{W})$, $\mathrm{Sym}^0(\mathbf{A})$, etc.

The presented results for generating sets are irreducible. It has been shown that each presented invariant with a single variable is irreducible. Irreducibility of each presented invariant with two or three variables will be proved elsewhere.

The unified scheme described in Sec. 2 and the method for isotropic extension of anisotropic functions may be used to derive irreducible representations for other types of anisotropic functions. The results will be reported elsewhere.

Acknowledgment

This reasearch was completed during the author's visit to Institute of Mechanics, Ruhr-University Bochum under the support of a fellowship from Alexander von Humboldt-Stiftung. This support and the support from National Natural Science Foundation of China are greatly appreciated. In particular, the author would like to express his heartfelt thanks to Prof. Dr. OTTO BRUHNS and Dr. ALBERT MEYERS for their hospitality and kind help.

References

1. J.E. Adkins, *Dynamic properties of resilient materials: constitutive equations*, Phil. Trans. Roc. Soc. Lond., **A 250**, 519–534, 1958.
2. J.E. Adkins, *Symmetry relations for orthotropic and transversely isotropic materials*, Arch. Rat. Mech. Anal., **4**, 193–213, 1960.
3. J.E. Adkins, *Further symmetry relations for transversely isotropic materials*, Arch. Rat. Mech. Anal., **5**, 263–274, 1960.
4. J. Betten, *Recent advances in applications of tensor functions in solid mechanics*, Advances in Mech., **14**, 1, 79–109, 1991.
5. J.P. Boehler, *On irreducible representations for isotropic scalar functions*, Zeits. Angew. Math. Mech, **57**, 323–327, 1977.
6. J.P. Boehler, *A simple derivation of representations for non-polynomial constitutive equations in some cases of anisotropy*, Zeits. Angew. Math. Mech., **59**, 157–167, 1979.
7. J.P. Boehler [Ed.], *Applications of tensor functions in solid mechanics*, CISM Courses and Lectures No. 292, Springer, Berlin, New York, Wien 1987.
8. S. Jemioło and J.J. Telega, *Non-polynomial representations of orthotropic tensor functions in the three-dimensional case: an alternative approach*, Arch. Mech., **49**, 233–239, 1997.
9. E. Kiral and G.F. Smith, *On the constitutive relations for anisotropic materials: triclinic, monoclinic, rhombic, tetragonal and hexagonal crystal systems*, Int. J. Engng. Sci., **12**, pp. 471–490, 1974.
10. E. Kiral and A.C. Eringen, *Constitutive equations of nonlinear electromagnetic-elastic crystals*, Springer-Verlag, New York 1990.
11. I.S. Liu, *On representations of anisotropic invariants*, Int. J. Engng. Sci., **19**, 1099–1109, 1982.
12. V.V. Lokhin and L.I. Sedov, *Nonlinear tensor functions of several tensor arguments*, Prikl. Mat. Mekh., **29**, pp. 393–417, 1963.
13. S. Pennisi and M. Trovato, *On irreducibility of Professor G.F. Smith's representations for isotropic functions*, Int. J. Engng. Sci., **25**, 1059–1065, 1987.
14. A.C. Pipkin and R.S. Rivlin, *The formulation of constitutive equations in continuum physics I*, Arch. Rat. Mech. Anal., **4**, 129–144, 1959.
15. A.C. Pipkin and A.S. Wineman, *Material symmetry restrictions on nonpolynomial constitutive equations*, Arch. Rat. Mech. Anal., **12**, 420–426, 1963.
16. J. Rychlewski, *Symmetry of causes and effects*, SIAMM Research Report No. 8706, Shanghai Institute of Appl. Math. and Mech., Shanghai, 1988; also: Wydawnictwo Naukowe PWN, Warsaw 1991.
17. J. Rychlewski and J.M. Zhang, *On representations of tensor functions: a review*, Advances in Mech., **14**, 4, pp. 75–94, 1991.
18. G.F. Smith, *Further results on the strain-energy function for anisotropic elastic materials*, Arch. Rat. Mech. Anal., **10**, 108–118, 1962.
19. G.F. Smith, *On isotropic functions of symmetric tensors, skewsymmetric tensors and vectors*, Int. J. Engng. Sci., **9**, pp. 899–916, 1971.
20. G.F. Smith, *On transversely isotropic functions of vectors, symmetric second order tensors and skewsymmetric second order tensors*, Quart. Appl. Math., **39**, pp. 509–516, 1982.
21. G.F. Smith, *Constitutive equations for anisotropic and isotropic materials*, Elsevier, New York 1994.

22. G.F. SMITH and E. KIRAL, *Integrity bases for N symmetric second order tensors – the crystal classes*, Rend. Circ. Mat. Palermo. II, Ser. 18, pp. 5–22, 1969.

23. G.F. SMITH and R.S. RIVLIN, *The strain-energy functions for anisotropic elastic materials*, Trans. Am. Math. Soc., **88**, 175–193, 1958.

24. G.F. SMITH and R.S. RIVLIN, *Integrity bases for vectors – the crystal classes*, Arch. Rat. Mech. Anal., **15**, pp. 169–221, 1964.

25. G.F. SMITH, M.M. SMITH and R.S. RIVLIN, *Integrity bases for a symmetric tensor and a vector – the crystal classes*, Arch. Rat. Mech. Anal., **12**, 93–133, 1963.

26. A.J.M. SPENCER, *Theory of invariants*, [in:] Continuum Physics, Vol. I, A.C. ERINGEN [Ed.], Academic Press, New York 1971.

27. C. TRUESDELL and W. NOLL, *The nonlinear field theories of mechanics*, Handbuch der Physik III/3, S. FLÜGGE [Ed.], Springer-Verlag, Berlin, Heidelberg, New York 1965.

28. B.L. VAINSHTEIN, *Modern crystallography 1: Fundamentals of crystals*, Springer-Verlag, Berlin, Heidelberg, New York 1994.

29. C.C. WANG, *A new representation theorem for isotropic functions. Part I and II*, Arch. Rat. Mech. Anal., **36**, pp. 166–223, 1970; Corrigendum, ibid, **43**, pp. 392–395, 1971.

30. A.S. WINEMAN and A.C. PIPKIN, *Material symmetry restrictions on constitutive equations*, Arch. Rat. Mech. Anal., **17**, 184–214, 1964.

31. H. XIAO and Z.H. GUO, *A general representation theorem for anisotropic tensor functions*, [in:] Proc. of the 2nd International Conference on Nonlinear Mechanics, W.Z. CHIEN, Z.H. GUO and Y.Z. GUO [Eds.], pp. 206–210, Peking University Press, Beijing 1993.

32. H. XIAO, *General irreducible representations for constitutive equations of elastic crystals and transversely isotropic elastic solids*, J. Elasticity, **39**, pp. 47–73, 1995.

33. H. XIAO, *Two general representation theorems for arbitrary-order tensor-valued isotropic and anisotropic tensor functons of vectors and second order tensors*, Zeits. Angew. Math. Mech., **76**, pp. 151–162, 1996.

34. H. XIAO, *Further results on general representation theorems for arbitrary-order tensor-valued isotropic and anisotropic tensor functions of vectors and second order tensors*, Zeits. Angew. Math. Mech [to appear].

35. H. XIAO, *On isotropic extension of anisotropic tensor functions*, Zeits. Angew. Math. Mech., **76**, pp. 205–214, 1996.

36. H. XIAO, *On anisotropic scalar functions of a single symmetric tensor*, Proc. Roy. Soc. Lond., **A452**, 1545–1561, 1996.

37. H. XIAO, *On minimal representations for constitutive equations of anisotropic elastic materials*, J. Elasticity, **45**, pp. 13–32, 1996.

38. H. XIAO, *On anisotropic invariants of a single symmetric tensor: crystal classes, quasicrystal classes and others*, Proc. Roy. Soc. London, Ser. A, 1217–1240, 1998.

39. H. XIAO, *A unified theory of representations for scalar-, vector-, and second order tensor-valued anisotropic functions of vectors and second order tensors*, Arch. Mech., **49**, 995–1039, 1997.

40. J.M. ZHANG and J. RYCHLEWSKI, *On structural tensors for anisotropic solids*, Arch. Mech., **42**, 267–277, 1990.

41. Q.S. ZHENG, *On transversely isotropic, orthotropic and relatively isotropic functions of symmetric tensors, skewsymmetric tensors and vectors. Part I-V*, Int. J. Engng. Sci., **31**, pp. 1399–1453, 1993.

42. Q.S. ZHENG, *Two-dimensional tensor function representations for all kinds of material symmetries*, Proc. Roy. Soc. London, **A 443**, pp. 127–138, 1993.

43. Q.S. ZHENG, *Theory of representations for tensor functions: A unified invariant approach to constitutive equations*, Appl. Mech. Rev., **47**, pp. 545–587, 1994.

44. Q.S. ZHENG and A.J.M. SPENCER, *Tensors which characterize anisotropies*, Int. J. Engng. Sci., **31**, 679–693, 1993.

INSTITUTE OF MECHANICS
RUHR-UNIVERSITY BOCHUM
D-44780 BOCHUM, GERMANY

and

DEPARTMENT OF MATHEMATICS
PEKING UNIVERSITY
BEIJING 100871, CHINA.

Received July 9, 1997.