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FIFTY YEARS OF THE ARCHIVES OF MECHANICS

Symmetric forms of the equations of heat transport in a rigid conductor of heat with internal state variables II. Alternative symmetric systems

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In Part I of this series, it has been shown that the field equations corresponding to the model of a rigid conductor of heat with (vector) internal state variable subject to the entropy inequality can be represented as the respective system of $N+1$ conservation equations for N unknowns, on which the "main dependency relation" (MDR) is imposed. In this paper (Part II), it is demonstrated how two families of symmetric systems corresponding to the consistent system of N conservation equations (family of symmetric systems for original unknowns and the family of $N+1$ symmetric conservative systems for transformed unknowns) can be directly derived with the aid of the MDR. The condition of equivalence of symmetric systems to the original system of conservation equations is analysed and alternatively formulated. For the considered model of a rigid conductor of heat, the conditions on free energy that assure symmetric hyperbolicity of symmetric systems are established, and it is shown that they are stronger than the conditions required for equivalence of symmetric systems to the original system of conservation equations. Two alternative symmetric conservative systems are derived for the considered model of a rigid conductor of heat and the conditions of symmetric hyperbolicity for those systems are established with the aid of the relation between convexity (concavity) of the respective generating potentials, and with the aid of the relation between symmetric hyperbolicity of the symmetric systems for original unknowns and symmetric conservative system for the transformed unknowns.

1. Introduction

In Part I of this paper [1], we have shown that the field equations corresponding to the model of a rigid conductor of heat with vector internal state variables, with the Clausius–Duhem entropy inequality taken into account, can be represented by an overdetermined system of conservation equations ($N+1$ equations for N unknowns)

$$(1.1)\qquad \begin{gathered} \partial_t g^{0\Lambda}(u^K) + \partial_\alpha g^{\alpha\Lambda}(u^K) = b^\Lambda(u^K, t, X^\gamma), \\ \Lambda = 1,2,\dots,N+1, \qquad K = 1,2,\dots,N, \qquad \alpha,\gamma = 1,2,\dots,m, \end{gathered}$$

where $N = 4$, $m = 3$ and

$$
\begin{aligned}
[u^K] &= [\theta, w_\gamma],\\
[g^{0A}(u^K)] &= [\varrho_0\widetilde{\varepsilon}(\theta, w_\beta), \tau w_\gamma, \varrho_0\widetilde{\eta}(\theta, w_\beta)],\\
[g^{\alpha A}(u^K)] &= \left[\widetilde{q}^\alpha(\theta, w_\beta), -f_1(\theta)\delta^\alpha{}_\gamma, \frac{1}{\theta}\widetilde{q}^\alpha(\theta, w_\beta)\right],\\
[b^A(u^K, t, X^\alpha)] &= [\varrho_0\widetilde{r}(\theta, w_\beta, t, X^\alpha), c_1 w_\gamma, \varrho_0\widetilde{\sigma}(\theta, w_\beta, t, X^\alpha)],\\
\gamma,\ \beta &= 1, 2, 3,
\end{aligned}
\tag{1.2}
$$

on which the "main dependency relation" (MDR) is imposed. In Eqs. (1.1), (1.2) $\theta(t, X^\alpha)$ represents the temperature field, $\mathbf{w}(t, X^\alpha) = [w_\gamma(t, X^\alpha)]$ is the field of internal state variables. In this model f_1, $\widetilde{\varepsilon}$, $\widetilde{q}^\alpha$, $\widetilde{r}$, and $\widetilde{\sigma}$ are postulated as the constitutive functions

$$
\begin{aligned}
&\widetilde{\varepsilon} = \widetilde{\varepsilon}(\theta, w_\gamma), \quad \widetilde{q}^\alpha = \widetilde{q}^\alpha(\theta, w_\gamma), \quad \widetilde{r} = \widetilde{r}(\theta, w_\gamma, t, X^\alpha),\\
&\widetilde{\eta} = \widetilde{\eta}(\theta, w_\gamma), \quad \widetilde{\sigma} = \widetilde{\sigma}(\theta, w_\gamma, t, X^\alpha), \quad f_1 = f_1(\theta).
\end{aligned}
\tag{1.3}
$$

The first equation of the system (1.1), (1.2) corresponds to balance of energy, the next three equations are the components of the evolution equation for internal state variable $\mathbf{w}$, and the last equation can be interpreted as the equation of balance of entropy.

In this paper, the possibilities of expressing the system (1.1), (1.2) in the form of symmetric systems of the first order partial differential equations are investigated. The emphasis is put on the employed method of symmetrization which directly leads to alternative symmetric systems. This method is based on direct application of the MDR which, as we have proved in Sec. 3.2 of Part I, can be used for derivation of thermodynamic restrictions. From the point of view of thermodynamics, the MDR formulated by FRIEDRICHS [2], can be regarded as a generalization of the "entropy principle" of extended thermodynamics of MÜLLER and LIU [3, 4, 5], in the sense that it assigns an analogue of Lagrange–Liu multiplier also to the balance of entropy. The advantage of employing the MDR instead of the "entropy principle" is that it enables one to derive equivalent (for classical solutions) alternative symmetric systems directly. In this approach, a family of symmetric systems (parameterized by real differentiable functions) with respect to original dependent variables as well as a family of $N+1$ symmetric conservative systems with respect to transformed dependent variables can be obtained, offering a possibility of selecting or constructing symmetric systems optimal from the point of view of the chosen numerical method and/or which are most suitable for the considered initial-boundary value problem. The symmetrization may be utilized in the design and analysis of numerical solutions. As it is mentioned by HARTEN [6], it offers the possibility of linearizing locally the equations in a way which preserves the hyperbolicity and conservation properties if the symmetric system is in the conservative form. Since the Cauchy problem is locally well-posed for the quasi-linear symmetric hyperbolic systems [7, 8], symmetrization enables the application of this result provided that the condition of symmetric hyperbolicity is satisfied.

In Secs. 2.1, 2.2 and 2.3 we recall restrictions on constitutive functions (1.3) and the family of solutions of the "main dependency relation" derived in Part I. Then, in Sec. 3.1, the general procedure of obtaining symmetric systems for the original unknowns is presented and a family of symmetric systems with respect to $[\theta, w_\gamma]$, parameterized by differentiable functions is given in Sec. 3.2.

The condition of equivalence of the symmetric systems to original system of conservation equations is discussed in Sec. 3.3. Two Observations, which provide alternative formulation of this condition are proved with the aid of three Lemmas given in the Appendix A. Then, in Sec. 3.4, the restrictions on the free energy $\widetilde{\Psi}_C(\theta, w_\gamma)$ that assure equivalence of the symmetric systems for $[\theta, w_\gamma]$ to the system (1.1), (1.2) are derived. Further conditions on the free energy $\widetilde{\Psi}_C(\theta, w_\gamma)$ that assure symmetric hyperbolicity of the symmetric systems for $[\theta, w_\gamma]$ are derived in Secs. 3.5.1 – 3.5.4. It is shown that the restriction on $\widetilde{\Psi}_C(\theta, w_\gamma)$ imposed by the condition of symmetric hyperbolicity is much stronger than the condition ensuring the equivalence of the symmetric systems and the original system of conservation equations or, in other words, that symmetric hyperbolicity implies equivalence to the original system of conservation equations (1.1).

In Secs. 4.1.1, 4.1.2, we present the general procedure of simultaneous derivation of $N + 1$ alternative symmetric conservative systems corresponding to the system (1.1) that satisfies the MDR. Taking into account Observations stated in Sec. 3.3, we show that the condition of equivalence of symmetric systems for original unknowns to (1.1), together with the assumption that (1.1) contains N independent equations (is determined), suffices for transformation of (1.1) into symmetric conservative form in $N + 1$ ways. In Sec. 4.1.4, we derive the general relations between alternative symmetric conservative systems that enable transformation of the given symmetric conservative system into the remaining N symmetric conservative systems, as well as to establish the relation between convexity (concavity) of the respective potentials (hence, symmetric hyprbolicity of those systems).

The procedure developed in Secs. 4.1.1, 4.1.2 is employed for derivation of two alternative symmetric conservative systems governing heat transport in a rigid heat conductor with (vector) internal state variable. The first one (Sec. 4.2.2) corresponds to the case of the equation of balance of energy treated as the additional conservation equation implied by the equation of balance of entropy and the equation of evolution for the internal state variable, while the second (Sec. 4.2.3) corresponds to the case of the equation of balance of entropy treated as the additional conservation equation implied by the equation of balance of energy and by the equation of evolution for the internal state variable. The possibility of interchanging ("switching") the role of the ("original") conserved quantity (like energy) and the role of the additional ("derived") conserved quantity (like entropy) in analysing systems of conservation equations admitting the additional conservation equation ("equations with convex extensions"), was for the first time discussed by FRIEDRICHS and LAX [9].

Finally, the conditions of symmetric hyperbolicity for both the symmetric conservative systems are established with the aid of the relation between the respective potentials generating those two systems (Sec. 4.3.1), and with the aid of the relation between symmetric hyperbolicity of the symmetric system for original dependent variables and the corresponding symmetric conservative system (Sec. 4.3.2). The proof that one of the potentials is covex (concave) if and only if the second one is concave(convex) is given in the Appendix B.

It has been shown in Part I that the model of a rigid conductor of heat with (vector) internal state variable considered here comprises, as special cases, various phenomenological models proposed in the literature. In particular, the equations (1.1), (1.2), (1.3), when supplemented by the respective involutive constraints, can be interpreted as corresponding to the special case of the model of a rigid conductor of heat with scalar internal state variable called "semi-empirical temperature". Symmetrization of the first-order system of equations corresponding to the more general version of the "semi-empirical temperature" model (five conservation equations for five unknowns) has been considered by DOMAŃSKI, JABŁOŃSKI and KOSIŃSKI [14] with the aid of the converse to the condition of Friedrichs and Lax, and the results obtained there are discussed by the author in a separate note [15].

2. Basic equations, restrictions on constitutive functions and solutions of the MDR

2.1. Basic equations in a matrix form

Performing the respective differentiation, we rewrite the system (1.1) in a matrix form

$$
\begin{gathered}
\mathcal{A}^{0\Lambda}{}_M(u^K)\partial_t u^M + \mathcal{A}^{\alpha\Lambda}{}_M(u^K)\partial_\alpha u^M = b^\Lambda(u^K,t,X^\alpha),\\
\mathcal{A}^{0\Lambda}{}_M(u^K) = \frac{\partial g^{0\Lambda}(u^K)}{\partial u^M}, \qquad \mathcal{A}^{\alpha\Lambda}{}_M(u^K) = \frac{\partial g^{\alpha\Lambda}(u^K)}{\partial u^M},
\end{gathered}
\tag{2.1}
$$

with the 5×4 matrices $[\mathcal{A}^{0\Lambda}{}_M]$, $[\mathcal{A}^{\alpha\Lambda}{}_M]$ of the form

(2.2)
$$\left[\mathcal{A}^{0\Lambda}{}_M\right] = \left[\frac{\partial g^{0\Lambda}}{\partial u^M}\right] = \begin{bmatrix} \varrho_0\dfrac{\partial\widetilde{\varepsilon}}{\partial\theta} & \varrho_0\dfrac{\partial\widetilde{\varepsilon}}{\partial w_1} & \varrho_0\dfrac{\partial\widetilde{\varepsilon}}{\partial w_2} & \varrho_0\dfrac{\partial\widetilde{\varepsilon}}{\partial w_3} \\ 0 & \tau & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & \tau \\ \varrho_0\dfrac{\partial\widetilde{\eta}}{\partial\theta} & \varrho_0\dfrac{\partial\widetilde{\eta}}{\partial w_1} & \varrho_0\dfrac{\partial\widetilde{\eta}}{\partial w_2} & \varrho_0\dfrac{\partial\widetilde{\eta}}{\partial w_3} \end{bmatrix},$$

$$\left[\mathcal{A}^{\alpha\Lambda}{}_M\right] = \left[\frac{\partial g^{\alpha\Lambda}}{\partial u^M}\right] = \begin{bmatrix} \dfrac{\partial\widetilde{q}^\alpha}{\partial\theta} & \dfrac{\partial\widetilde{q}^\alpha}{\partial w_1} & \dfrac{\partial\widetilde{q}^\alpha}{\partial w_2} & \dfrac{\partial\widetilde{q}^\alpha}{\partial w_3} \\ -\delta^\alpha{}_1 f_1'(\theta) & 0 & 0 & 0 \\ -\delta^\alpha{}_2 f_1'(\theta) & 0 & 0 & 0 \\ -\delta^\alpha{}_3 f_1'(\theta) & 0 & 0 & 0 \\ -\dfrac{1}{\theta^2}\widetilde{q}^\alpha + \dfrac{1}{\theta}\dfrac{\partial\widetilde{q}^\alpha}{\partial\theta} & \dfrac{1}{\theta}\dfrac{\partial\widetilde{q}^\alpha}{\partial w_1} & \dfrac{1}{\theta}\dfrac{\partial\widetilde{q}^\alpha}{\partial w_2} & \dfrac{1}{\theta}\dfrac{\partial\widetilde{q}^\alpha}{\partial w_3} \end{bmatrix}.$$

For completeness, we recall that, according to FRIEDRICHS [2], the MDR requires the existence of $N+1$ functions $y_\Lambda : \mathbb{R}^N \to \mathbb{R}$, $\Lambda = 1, 2, \ldots, N+1$, not all identically zero, such that (Property CI in [2])

(2.3)
$$y_\Lambda(u^K)\frac{\partial g^{0\Lambda}(u^K)}{\partial u^M}\partial_t u^M + y_\Lambda(u^K)\frac{\partial g^{\alpha\Lambda}(u^K)}{\partial u^M}\partial_\alpha u^M \equiv 0,$$
$$y_\Lambda(u^K) b^\Lambda(u^K) \equiv 0,$$

holds for all functions $u^K(t, X^\alpha)$, $K = 1, 2, \ldots, N$.

The identity $(2.3)_1$ is equivalent to the following system of identities (Property CI' in [2]):

(2.4)
$$y_\Lambda(u^K)\frac{\partial g^{0\Lambda}(u^K)}{\partial u^M} \equiv 0, \qquad y_\Lambda(u^K)\frac{\partial g^{\alpha\Lambda}(u^K)}{\partial u^M} \equiv 0.$$

The set of $N+1$ functions $y_\Lambda(u^K)$ is obtained as a solution of the overdetermined system of linear homogeneous equations $(2.3)_2$, (2.4) and therefore, if it exists, it is not unique.

2.2. Restrictions on constitutive functions

In Secs. 3.3, 3.4 of Part I, it has been found that the system (1.1) (1.2) satisfies the MDR if and only if constitutive functions $\widetilde{\varepsilon}$, $\widetilde{\eta}$ and $\widetilde{q}^\alpha$ satisfy the following relations:

(2.5)
$$\widetilde{q}^\gamma = -\frac{\varrho_0\theta f_1'(\theta)}{\tau}\left(\frac{\partial\widetilde{\varepsilon}}{\partial w_\gamma} - \theta\frac{\partial\widetilde{\eta}}{\partial w_\gamma}\right) = -\frac{\varrho_0\theta f_1'(\theta)}{\tau}\frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma},$$
$$\frac{\partial\widetilde{\eta}}{\partial\theta} = \frac{1}{\theta}\frac{\partial\widetilde{\varepsilon}}{\partial\theta},$$

and

$$\widetilde{\eta} = -\frac{\partial \widetilde{\Psi}_C}{\partial \theta}, \qquad \widetilde{\varepsilon} = \widetilde{\Psi}_C - \theta \frac{\partial \widetilde{\Psi}_C}{\partial \theta}, \tag{2.6}$$

where the free energy $\widetilde{\Psi}_C$ is introduced

$$\widetilde{\Psi}_C(\theta, w_\gamma) = \widetilde{\varepsilon}(\theta, w_\gamma) - \theta\widetilde{\eta}(\theta, w_\gamma). \tag{2.7}$$

The entropy inequality implies

$$\frac{c_1}{\theta^2 f_1'(\theta)}\widetilde{q}^{\gamma} w_\gamma = -\frac{\varrho_0}{\tau}\frac{c_1}{\theta}\left(\frac{\partial \widetilde{\varepsilon}}{\partial w_\gamma} - \theta\frac{\partial \widetilde{\eta}}{\partial w_\gamma}\right) w_\gamma = -\frac{\varrho_0}{\tau}\frac{c_1}{\theta}\frac{\partial \widetilde{\Psi}_C}{\partial w_\gamma} w_\gamma \geq 0. \tag{2.8}$$

The restrictions on constitutive functions (2.5), (2.6), (2.7) enable us to transform the systems (1.1), (1.2) into quasi-linear symmetric systems of 4 equations for 4 unknowns. We recall that, in Part I (Sec. 3.2) of this paper, we have assumed that $\dfrac{\partial \widetilde{\varepsilon}(\theta, w_\gamma)}{\partial \theta} \neq 0$ (or, equivalently $\dfrac{\partial^2 \widetilde{\Psi}_C(\theta, w_\gamma)}{\partial \theta^2} \neq 0$) for all θ, w_γ in order to make it possible to derive thermodynamic restrictions via the MDR.

2.3. Family of solutions of the MDR

In Sec. 3.4 of Part I, a family of solutions $\mathbf{y}^T = [y_A]$ of the MDR (2.3), (2.4) has been calculated. Introducing the notation for y_A, $[y_A] = [\lambda, z^\gamma, \mu]$ where by λ, z^γ, μ we mean functions $\lambda(\theta, w_\gamma)$, $z^\beta(\theta, w_\gamma)$ and $\mu(\theta, w_\gamma)$, we have obtained

$$\begin{aligned}
&\theta = -\frac{\mu}{\lambda} \\
&z^\gamma = -\lambda\frac{\varrho_0}{\tau}\left(\frac{\partial \widetilde{\varepsilon}}{\partial w_\gamma} - \theta\frac{\partial \widetilde{\eta}}{\partial w_\gamma}\right) = -\lambda\frac{\varrho_0}{\tau}\frac{\partial \widetilde{\Psi}_C}{\partial w_\gamma} \\
&\qquad = \mu\frac{\varrho_0}{\tau}\frac{1}{\theta}\left(\frac{\partial \widetilde{\varepsilon}}{\partial w_\gamma} - \theta\frac{\partial \widetilde{\eta}}{\partial w_\gamma}\right) = \mu\frac{\varrho_0}{\tau}\frac{1}{\theta}\frac{\partial \widetilde{\Psi}_C}{\partial w_\gamma} \\
&\qquad = \lambda\frac{1}{\theta f_1'(\theta)}\widetilde{q}^{\gamma} = -\mu\frac{1}{\theta^2 f_1'(\theta)}\widetilde{q}^{\gamma},
\end{aligned} \tag{2.9}$$

Thus, the family of solutions of the MDR can be written as

$$[y_A] = -\lambda\left[-1,\ \frac{\varrho_0}{\tau}\frac{\partial \widetilde{\Psi}_c}{\partial w_\gamma},\ \theta\right] = -\mu\left[\frac{1}{\theta},\ -\frac{1}{\theta}\frac{\varrho_0}{\tau}\frac{\partial \widetilde{\Psi}_c}{\partial w_\gamma},\ -1\right], \tag{2.10}$$

where λ and μ are arbitrary functions of $[\theta, w_\gamma]$, and $\lambda = -1$ and $[\widehat{y}_A] = \left[-1, \dfrac{\varrho_0}{\tau}\dfrac{\partial \widetilde{\Psi}_c}{\partial w_\gamma}, \theta\right]$ correspond to the case when the equation of balance of energy is treated as the additional conservation equation, while $\mu = -1$ and $[\breve{y}_A] = \left[\dfrac{1}{\theta}, -\dfrac{1}{\theta}\dfrac{\varrho_0}{\tau}\dfrac{\partial \widetilde{\Psi}_c}{\partial w_\gamma}, -1\right]$ correspond to the case when the equation of balance of entropy is treated as the additional conservation equation.

3. Symmetric systems for unknowns $[\theta, w_\gamma]$

3.1. General procedure

The procedure of symmetrization of consistent systems of $N+1$ conservation equations for N unknown fields, that is systems of the type (1.1) given by FREDRICHS [2], is employed to obtain the symmetric systems of the equations for $[\theta, w_\gamma]$. Differentiation of the identities (2.4) yields

$$\begin{aligned}\frac{\partial y_\Lambda}{\partial u^S}\frac{\partial g^{0\Lambda}}{\partial u^M} &= -y_\Lambda \frac{\partial^2 g^{0\Lambda}}{\partial u^S \partial u^M},\\ \frac{\partial y_\Lambda}{\partial u^S}\frac{\partial g^{\alpha\Lambda}}{\partial u^M} &= -y_\Lambda \frac{\partial^2 g^{\alpha\Lambda}}{\partial u^S \partial u^M}.\end{aligned} \tag{3.1}$$

It follows from (3.1) that the $N \times (N+1)$ matrix $[\mathcal{K}_{S\Lambda}(u^K)]$ with entries

$$\mathcal{K}_{S\Lambda}(u^K) = \frac{\partial y_\Lambda(u^K)}{\partial u^S} \tag{3.2}$$

is the left symmetrizer of the matrices $[\mathcal{A}^{0\Lambda}{}_M]$, $[\mathcal{A}^{\alpha\Lambda}{}_M]$. Left multiplication of the system (2.1) by the matrix $[\mathcal{K}_{S\Lambda}]$ gives therefore a symmetric system of N first order partial differential equations for N unknowns

$$A^0{}_{SM}(u^K)\partial_t u^M + A^\alpha{}_{SM}(u^K)\partial_t u^M = p_S(u^K, t, X^\alpha), \tag{3.3}$$

where

$$\begin{aligned}A^0{}_{SM}(u^K) &= \mathcal{K}_{S\Lambda}(u^K)\mathcal{A}^{0\Lambda}{}_M(u^K) = \frac{\partial y_\Lambda(u^K)}{\partial u^S}\frac{\partial g^{0\Lambda}(u^K)}{\partial u^M},\\ A^\alpha{}_{SM}(u^K) &= \mathcal{K}_{S\Lambda}(u^K)\mathcal{A}^{\alpha\Lambda}{}_M(u^K) = \frac{\partial y_\Lambda(u^K)}{\partial u^S}\frac{\partial g^{\alpha\Lambda}(u^K)}{\partial u^M},\\ p_S(u^K, t, X^\alpha) &= \mathcal{K}_{S\Lambda}(u^K)b^\Lambda(u^K, t, \alpha) = \frac{\partial y_\Lambda(u^K)}{\partial u^S}b^\Lambda(u^K, t, X^\alpha).\end{aligned} \tag{3.4}$$

The set of $N+1$ functions $y_\Lambda(u^K)$ which are solution of the MDR (2.3), (2.4), can be treated as a vector in $\mathbb{R}^{N+1}$. Since it is a solution of a homogeneous system of equations (2.4) and $(2.3)_2$, it is not unique. It can be easily verified that, in the case of the system (1.1) containing N independent equations, the solution set of the MDR takes the form of a family of collinear vectors. As a system (1.1) containing N independent equations, we understand here the system (1.1) for which at least one of the $(N+1)\times N$ matrices $\mathcal{A}^{0\Lambda}{}_M(u^K)$, $\mathcal{A}^{\alpha\Lambda}{}_M(u^K)$ is of rank N for all u^K. It follows from (2.2) that, in the case of the considered system (1.1), (1.2), rank $\mathcal{A}^{0\Lambda}{}_M(u^K) = N$ for all u^K if $\dfrac{\partial\widetilde{\varepsilon}(\theta, w_\gamma)}{\partial\theta} \neq 0$ for all θ, w_γ, what means that internal energy $\widetilde{\varepsilon}$ is a monotone function of the temperature θ for all w_γ. This condition coincides with the condition (3.14) of Part I, which has been assumed as necessary for application of the MDR for derivation of the restrictions on constitutive functions implied by the Clausius–Duhem entropy inequality. Therefore the considered system (1.1), (1.2) is assumed to contain N independent equations and if $\widehat{\mathbf{y}} = [\widehat{y}_\Lambda]$ is a particular solution of $(2.3)_2$ and (2.4), then any other solution $\check{\mathbf{y}} = [\check{y}_\Lambda]$ of $(2.3)_2$, (2.4) can be expressed as $\check{\mathbf{y}} = \xi\widehat{\mathbf{y}}, \check{y}_\Lambda(u^K) = \xi(u^K)\widehat{y}_\Lambda(u^K)$, $\Lambda = 1, 2, ..., N+1$ and $\xi(u^K)$ is a differentiable function of u^K, not identically zero. It immediately follows from (2.1), (2.3) and (2.4) that symmetric systems corresponding to $[\check{y}_\Lambda]$ and $[\widehat{y}_\Lambda]$, that is $\widehat{A}^0_{SM}(u^K)\partial_t u^M + \widehat{A}^\alpha_{SM}(u^K)\partial_\alpha u^M = \widehat{p}_S(u^K)$ and $\check{A}^0_{SM}(u^K)\partial_t u^M + \check{A}^\alpha_{SM}(u^K)\partial_\alpha u^M = \check{p}_S(u^K)$, respectively, differ by a scalar factor $\xi(u^K)$ since $\check{A}^0_{SM}(u^K) = \xi(u^K)\widehat{A}^0_{SM}(u^K)$, $\check{A}^\alpha_{SM}(u^K) = \xi(u^K)\widehat{A}^\alpha_{SM}(u^K)$ and $\check{p}_S(u^K) = \xi(u^K)\widehat{p}_S(u^K)$. Hence, a symmetric system (3.3) can be associated to each solution of the MDR and, therefore, we have one-parameter family of symmetric systems (3.3) parameterized by suitably differentiable functions $\xi(u^K)$.

3.2. Family of symmetric systems for $[\theta, w_\gamma]$

In order to symmetrize the system (2.1), (2.2), (2.5), (2.6), (2.8), we calculate the matrix

$$
(3.5)\qquad \left[\widehat{\mathcal{K}}_{S\Lambda}\right] = \left[\frac{\partial \widehat{y}_\Lambda}{\partial u^S}\right] = \begin{bmatrix}
0 & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial\theta\partial w_1} & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial\theta\partial w_2} & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial\theta\partial w_3} & 1 \\
0 & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_1^2} & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_1\partial w_2} & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_1\partial w_3} & 0 \\
0 & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_2\partial w_1} & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_2^2} & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_2\partial w_3} & 0 \\
0 & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_3\partial w_1} & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_3\partial w_2} & \dfrac{\varrho_0}{\tau}\dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_3^2} & 0
\end{bmatrix}
$$

for particular solution of the MDR $[\widehat{y}_\Lambda] = \left[-1, \dfrac{\varrho_0}{\tau}\dfrac{\partial\widetilde{\Psi}_c}{\partial w_\gamma}, \theta\right]$ and, taking into account (2.5), (2.6), express the matrices $\mathcal{A}^{0\Lambda}_M$ and $\mathcal{A}^{\alpha\Lambda}_M$ in terms of θ, $f_1(\theta)$ and derivatives of $\widetilde{\Psi}_C$

$$
[\mathcal{A}^{0\Lambda}_M] = \begin{bmatrix}
-\varrho_0\theta\dfrac{\partial^2\widetilde{\Psi}_C}{\partial\theta^2} & \varrho_0\left(\dfrac{\partial\widetilde{\Psi}_C}{\partial w_1} - \theta\dfrac{\partial^2\widetilde{\Psi}_C}{\partial\theta\partial w_1}\right) & \varrho_0\left(\dfrac{\partial\widetilde{\Psi}_C}{\partial w_2} - \theta\dfrac{\partial^2\widetilde{\Psi}_C}{\partial\theta\partial w_2}\right) & \varrho_0\left(\dfrac{\partial\widetilde{\Psi}_C}{\partial w_3} - \theta\dfrac{\partial^2\widetilde{\Psi}_C}{\partial\theta\partial w_3}\right) \\
0 & \tau & 0 & 0 \\
0 & 0 & \tau & 0 \\
0 & 0 & 0 & \tau \\
-\varrho_0\dfrac{\partial^2\widetilde{\Psi}_C}{\partial\theta^2} & -\varrho_0\dfrac{\partial\widetilde{\Psi}_C}{\partial\theta\partial w_1} & -\varrho_0\dfrac{\partial^2\widetilde{\Psi}_C}{\partial\theta\partial w_2} & -\varrho_0\dfrac{\partial^2\widetilde{\Psi}_C}{\partial\theta\partial w_3}
\end{bmatrix},
$$

$$[\mathcal{A}^{\alpha\Lambda}_M] = \begin{bmatrix} a^{\alpha}_{11} & -\frac{\varrho_0}{\tau}\theta f_1'(\theta)\frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha \partial w_1} & -\frac{\varrho_0}{\tau}\theta f_1'(\theta)\frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha \partial w_2} & -\frac{\varrho_0}{\tau}\theta f_1'(\theta)\frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha \partial w_3} \\ -f_1'(\theta)\delta^{\alpha}_1 & 0 & 0 & 0 \\ -f_1'(\theta)\delta^{\alpha}_2 & 0 & 0 & 0 \\ -f_1'(\theta)\delta^{\alpha}_3 & 0 & 0 & 0 \\ a^{\alpha}_{51} & -\frac{\varrho_0}{\tau} f_1'(\theta)\frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha \partial w_1} & -\frac{\varrho_0}{\tau} f_1'(\theta)\frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha \partial w_2} & -\frac{\varrho_0}{\tau} f_1'(\theta)\frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha \partial w_3} \end{bmatrix},$$

$$\begin{aligned} a^{\alpha}_{11} &= -\frac{\varrho_0}{\tau} f_1'(\theta)\frac{\partial\widetilde{\Psi}_C}{\partial w_\alpha} - \frac{\varrho_0}{\tau}\theta f_1''(\theta)\frac{\partial\widetilde{\Psi}_C}{\partial w_\alpha} - \frac{\varrho_0}{\tau}\theta f_1'(\theta)\frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha \partial\theta}, \\ a^{\alpha}_{51} &= -\frac{\varrho_0}{\tau} f_1''(\theta)\frac{\partial\widetilde{\Psi}_C}{\partial w_\alpha} - \frac{\varrho_0}{\tau} f_1'(\theta)\frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha \partial\theta}. \end{aligned} \tag{3.6}$$

Then, we obtain the matrices $\widehat{A}^0_{SM}$, $\widehat{A}^{\alpha}_{SM}$ and the production term $\widehat{p}_S$ of the symmetric system of four equations for four unknowns $[u^K] = [\theta, w_\gamma]$

$$\widehat{A}^0_{SM}(u^K)\partial_t u^M + \widehat{A}^{\alpha}_{SM}(u^K)\partial_\alpha u^M = \widehat{p}_S(u^K, t, X^\alpha),$$

$$[\widehat{A}^0_{SM}] = [\widehat{\mathcal{K}}_{S\Lambda}][\mathcal{A}^{0\Lambda}_M] = \varrho_0 \begin{bmatrix} -\frac{\partial^2\widetilde{\Psi}_C}{\partial\theta^2} & 0 & 0 & 0 \\ 0 & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_1^2} & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_1\partial w_2} & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_1\partial w_3} \\ 0 & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_2\partial w_1} & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_2^2} & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_2\partial w_3} \\ 0 & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_3\partial w_1} & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_3\partial w_2} & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_3^2} \end{bmatrix},$$

$$[\widehat{A}^{\alpha}_{SM}] = [\widehat{\mathcal{K}}_{S\Lambda}][\mathcal{A}^{\alpha\Lambda}_M] = -\frac{\varrho_0}{\tau} f_1'(\theta) \begin{bmatrix} \widehat{a}^{\alpha}_{11} & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha\partial w_1} & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha\partial w_2} & \frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha\partial w_3} \\ \frac{\partial^2\widetilde{\Psi}_C}{\partial w_1\partial w_\alpha} & 0 & 0 & 0 \\ \frac{\partial^2\widetilde{\Psi}_C}{\partial w_2\partial w_\alpha} & 0 & 0 & 0 \\ \frac{\partial^2\widetilde{\Psi}_C}{\partial w_3\partial w_\alpha} & 0 & 0 & 0 \end{bmatrix},$$

$$\widehat{a}^{\alpha}_{11} = 2\frac{\partial^2\widetilde{\Psi}_C}{\partial\theta\partial w_\alpha} + \frac{f_1''(\theta)}{f_1'(\theta)}\frac{\partial\widetilde{\Psi}_C}{\partial w_\alpha}, \tag{3.7}$$

(3.7) [cont.]

$$[\widehat{p}_S] = [\widehat{\mathcal{K}}_{S\Lambda}][b^\Lambda] = c_1 \frac{\varrho_0}{\tau}\left[\frac{\partial^2 \widetilde{\Psi}_C}{\partial\theta\partial w_\alpha} w_\alpha - \frac{1}{\theta}\frac{\partial \widetilde{\Psi}_C}{\partial w_\alpha} w_\alpha + \frac{\tau}{c_1}\widetilde{r}, \right.$$
$$\left. \frac{\partial^2 \widetilde{\Psi}_C}{\partial w_1 \partial w_\alpha} w_\alpha, \ \frac{\partial^2 \widetilde{\Psi}_C}{\partial w_2 \partial w_\alpha} w_\alpha, \ \frac{\partial^2 \widetilde{\Psi}_C}{\partial w_3 \partial w_\alpha} w_\alpha \right].$$

It should be noted that if we take the 4×4 minor $[\widehat{K}_{SM}]$ of the 4×5 matrix $[\widehat{\mathcal{K}}_{S\Lambda}]$, obtained by deleting the first column composed of zeros and take the 4×4 minors $[A^{0K}_M]$ and $[A^{\alpha K}_M]$ of the respective 5×4 matrices $[\mathcal{A}^{0\Lambda}_M]$ and $[\mathcal{A}^{\alpha\Lambda}_M]$ obtained by deleting the first row in each matrix, then the symmetric 4×4 matrices $[\widehat{A}^0_{SM}]$ and $[\widehat{A}^\alpha_{SM}]$ in the system (4.8) can be expressed as $[\widehat{A}^\alpha_{SM}] = [\widehat{K}_{SP}][A^{0P}_M]$ and $[\widehat{A}^\alpha_{SM}] = [\widehat{K}_{SP}][A^{\alpha P}_M]$. That is, $[\widehat{K}_{SP}]$ is the left symmetrizer of the square matrices $[A^{0P}_M]$, $[A^{\alpha P}_M]$.

Since the considered system (1.1), (1.2), (2.5), (2.6), (2.8) contains four independent equations, all symmetric systems with respect to the unknowns $[u^K] = [\theta, w_\gamma]$ can be obtained from (3.7) as the family of symmetric systems parametrized by differentiable functions $\xi(u^K)$,

$$\xi(u^K)\widehat{A}^0_{SM}(u^K)\partial_t u^M + \xi(u^K)\widehat{A}^\alpha_{SM}(u^K)\partial_\alpha u^M = \xi(u^K)\widehat{p}_S(u^K, t, X^\alpha), \tag{3.8}$$

where $\widehat{A}^0_{SM}$, $\widehat{A}^\alpha_{SM}$ and $\widehat{p}_S$ are given by (3.7). It follows from (2.10), (3.4), (3.7) that, for $\xi = 1$ we obtain the symmetric system corresponding to the case of balance of energy treated as the additional conservation equation and, for $\xi = -1/\theta$, we obtain the symmetric system corresponding to the case of balance of entropy treated as the additional conservation equation.

3.3. Equivalence of symmetric systems to original system of conservation equations

Sufficient condition. It has been proved by FRIEDRICHS [2] that nonsingularity of the $(N+1) \times (N+1)$ matrix composed of $y_\Lambda(u^K)$, $\dfrac{\partial y_\Lambda(u^K)}{\partial u^M}$

$$\begin{bmatrix} y_\Lambda(u^K) \\ \mathcal{K}_{M\Lambda}(u^K) \end{bmatrix} = \begin{bmatrix} y_\Lambda(u^K) \\ \dfrac{\partial y_\Lambda(u^K)}{\partial u^M} \end{bmatrix}, \tag{3.9}$$
$$\Lambda = 1, 2, \ldots, N+1, \qquad K, M = 1, 2, \ldots, N$$

for all u^K, for given solution $\mathbf{y}^T(u^K) = [y_\Lambda(u^K)]$ of the MDR, ensures the equivalence, for the classical (differentiable) solutions, of the symmetric system (3.3), (3.4) corresponding to $\mathbf{y}^T(u^K)$, to the original system of conservation equations (1.1). Equivalence is understood here in the sense that every classical (differentiable) solution of the symmetric system (3.8) satisfies the original system (1.1), and conversely. The matrices (3.9) have two interesting properties which can be formulated as the following Observations:

OBSERVATION 1. For the system (1.1) containing N independent equations, nonsingularity (singularity) of the matrix (3.9) for particular solution of the MDR implies nonsingularity (singularity) of all matrices of the type (3.9) corresponding to the family of solutions of the MDR.

OBSERVATION 2. Alternative sufficient condition of equivalence. The matrix (3.9) is nonsingular if and only if among $N+1$ functions

$$[y_1(u^K), y_2(u^K), \dots, y_{\Sigma-1}(u^K), y_\Sigma(u^K), y_{\Sigma+1}(u^K), \dots, y_N(u^K), y_{N+1}(u^K)]$$

there are N independent functions (say,

$$[y_1(u^K), y_2(u^K), \dots, y_{\Sigma-1}(u^K), y_{\Sigma+1}(u^K), \dots, y_N(u^K), y_{N+1}(u^K)]$$

are independent, what means that the transformation

$$[u^1, u^2, \dots, u^N] \to [y_1, y_2, \dots, y_{\Sigma-1}, y_{\Sigma+1}, \dots, y_N, y_{N+1}]$$

is invertible), and the remaining one (say, y_Σ) is not a homogeneous function of degree one of those independent functions (that is, the function $y_\Sigma(u^K(y_1, y_2, \dots, y_{\Sigma-1}, y_{\Sigma+1}, \dots, y_N, y_{N+1})) = \bar{y}_\Sigma(y_1, y_2, \dots, y_{\Sigma-1}, y_{\Sigma+1}, \dots, y_N, y_{N+1})$ is not a homogeneous function of degree one of the arguments $y_1, y_2, \dots, y_{\Sigma-1}, y_{\Sigma+1}, \dots, y_N, y_{N+1}$).

Observation 1 is a direct consequence of Lemma 1, and Observation 2 immediately follows from Lemma 2 and Lemma 3 proved in the Appendix A.

If rank $[\mathcal{K}_{S\Lambda}(u^K)] = N$ for all $u^K \in \mathcal{D}$ ($\mathcal{D}$ – convex domain in $\mathbb{R}^N$) and consequently, $y_1, \dots, y_{\Sigma-1}, y_{\Sigma+1}, \dots, y_{N+1}$ can be taken as new dependent variables, then the condition that $\bar{y}_\Sigma(y_1, \dots, y_{\Sigma-1}, y_{\Sigma+1}, \dots, y_{N+1})$ is not a homogeneous function of degree one with respect to all arguments expresses the fact that, roughly speaking, all equations of the system (1.1) are nontrivially involved in the MDR. It means that in the system (1.1) none of the equations is merely a linear combination with certain numerical factors of other equations or, in other words, any system of N equations selected from $N+1$ equations of (1.1) is the system of N independent equations. Equivalently, the solution set of the MDR takes the form of a family of collinear vectors in $\mathbb{R}^{N+1}$ in which there are no components y_Λ differing only by a numerical factor, and/or none of the components y_Λ identically vanishes ($\bar{y}_\Lambda(z_K) \equiv 0$ is a homogeneous function of degree one of $z_1, z_2, \dots z_N$).

3.4. Equivalence of the family of symmetric systems for $[\theta, w_\gamma]$

It follows from Observation 1 that in order to establish the sufficient condition of equivalence for the system (1.1), (1.2), (2.5), (2.6), (2.8), it suffices to consider the matrix of the type (3.9) for one particular solution of the MDR selected from the family (2.10). Hence, we select the solution $[\widehat{y}] = [-1, \widehat{z}_\gamma, \widehat{\mu}]$, $\widehat{z}_\gamma = \dfrac{\varrho_0}{\tau}\dfrac{\partial\widetilde{\psi}_C}{\partial w_\gamma}$, $\widehat{\mu} = \theta$, and take the following 5×5 matrix

$$(3.10)\qquad \left[\frac{\widehat{y}_\Lambda}{\widehat{\mathcal{K}}_{S\Lambda}}\right] = \begin{bmatrix} -1 & \widehat{z}_1 & \widehat{z}_2 & \widehat{z}_3 & \widehat{\mu} \\ 0 & \dfrac{\partial\widehat{z}_1}{\partial\theta} & \dfrac{\partial\widehat{z}_2}{\partial\theta} & \dfrac{\partial\widehat{z}_3}{\partial\theta} & \dfrac{\partial\widehat{\mu}}{\partial\theta} \\ 0 & \dfrac{\partial\widehat{z}_1}{\partial w_1} & \dfrac{\partial\widehat{z}_2}{\partial w_1} & \dfrac{\partial\widehat{z}_3}{\partial w_1} & \dfrac{\partial\widehat{\mu}}{\partial w_1} \\ 0 & \dfrac{\partial\widehat{z}_1}{\partial w_2} & \dfrac{\partial\widehat{z}_2}{\partial w_2} & \dfrac{\partial\widehat{z}_3}{\partial w_2} & \dfrac{\partial\widehat{\mu}}{\partial w_2} \\ 0 & \dfrac{\partial\widehat{z}_1}{\partial w_3} & \dfrac{\partial\widehat{z}_2}{\partial w_3} & \dfrac{\partial\widehat{z}_3}{\partial w_3} & \dfrac{\partial\widehat{\mu}}{\partial w_3} \end{bmatrix}$$

$$= \frac{\varrho_0}{\tau}\begin{bmatrix} -\dfrac{\tau}{\varrho_0} & \dfrac{\partial\widetilde{\Psi}_C}{\partial w_1} & \dfrac{\partial\widetilde{\Psi}_C}{\partial w_2} & \dfrac{\partial\widetilde{\Psi}_C}{\partial w_3} & \dfrac{\tau}{\varrho_0}\theta \\ 0 & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_1\partial\theta} & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_2\partial\theta} & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_3\partial\theta} & \dfrac{\tau}{\varrho_0} \\ 0 & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial^2 w_1} & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_1\partial w_2} & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_1\partial w_3} & 0 \\ 0 & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_1\partial w_2} & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial^2 w_2} & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_2\partial w_3} & 0 \\ 0 & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_1\partial w_3} & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial w_2\partial w_3} & \dfrac{\partial^2\widetilde{\Psi}_C}{\partial^2 w_3} & 0 \end{bmatrix}.$$

The matrix (3.10) is nonsingular and, as a consequence, the equivalence of the symmetric systems is ensured if the constitutive function $\widetilde{\Psi}_C(\theta, w_\gamma)$ satisfies the condition

$$(3.11)\qquad \det\left[\frac{\partial^2\widetilde{\Psi}_C}{\partial w_\alpha\partial w_\beta}\right] = \det\left[\frac{\partial^2\widetilde{\varepsilon}}{\partial w_\alpha\partial w_\beta} - \theta\frac{\partial^2\widetilde{\eta}}{\partial w_\alpha\partial w_\beta}\right] \neq 0 \quad \text{for all } [\theta, w_\gamma] \in D.$$

3.5. Symmetric hyperbolicity of symmetric systems for $[\theta, w_\gamma]$

3.5.1. Definition. According to the definition of FRIEDRICHS [2], a symmetric system (3.3) is symmetric hyperbolic (in time direction t) in an open convex domain $\mathcal{D} \subset \mathbb{R}^N$ if the matrix $[A^0_{SM}]$ is positive definite for all $u^K \in \mathcal{D}$.

3.5.2. Symmetric hyperbolicity of the system (3.7). It follows from $(3.7)_2$ that $[\widehat{A}^0_{SM}]$ is positive (negative) definite for $[\theta, w_1, w_2, w_3]$ from a convex domain $\mathcal{D} \subset \mathbb{R}^4$ if an only if $\widetilde{\Psi}_C(\theta, w_1, w_2, w_3)$ has the following properties:

$$\frac{\partial^2\widetilde{\Psi}_C}{\partial\theta^2} < 0 \qquad \left(\frac{\partial^2\widetilde{\Psi}_C}{\partial\theta^2} > 0\right)$$

($\widetilde{\Psi}_C$ is a concave (convex) function of θ while w_1, w_2, w_3 play the role of fixed parameters) and $\widetilde{\Psi}_C$ is a convex (concave) function of the three arguments w_1, w_2, w_3 while θ plays the role of a fixed parameter, for all $[\theta, w_1, w_2, w_3] \in \mathcal{D}$. In the case of $[\widehat{A}^0_{SM}]$ positive definite, the system $(3.7)_1$ is symmetric hyperbolic while, in the case of $[\widehat{A}^0_{SM}]$ negative definite, it can be transformed to the equivalent symmetric hyperbolic system by multiplying each equation by the factor (-1).

3.5.3. Symmetric hyperbolicity of the family (3.8). In choosing $\xi(u^K)$, the condition of symmetric hyperbolicity (positive definite $[\xi\widehat{A}^0_{SM}]$) should be taken into account. Since the conditions of positive (negative) definiteness of the matrix $[\widehat{A}^0_{SM}]$ are established above, the conditions on $\xi(u^K)$ ensuring positive definiteness of $[\xi\widehat{A}^0_{SM}]$ follow from the fact that, if $[\widehat{A}^0_{SM}(u^K)]$ is positive (negative) definite for $[u^K] \in \mathcal{D}$ then $[\xi(u^K)\widehat{A}^0_{SM}(u^K)]$ is positive (negative) for $\xi(u^K) > 0$, $[u^K] \in \mathcal{D}$ and negative (positive) definite for $\xi(u^K) < 0$, $[u^K] \in \mathcal{D}$.

3.5.4. Relation to the condition of equivalence to the original system of conservation equations (1.1). It should be noted that the condition rank $[\mathcal{K}_{S\Lambda}(u^K)] = N$ for all $u^K \in \mathcal{D}$ (which means that among $N+1$ functions $y(u^K)$ there are N independent functions of u^K, $u^K \in \mathcal{D}$) is necessary for symmetric hyperbolicity of the symmetric system (3.3), (3.4) (necessary for positive definiteness of $[A^0_{SM}] = [\mathcal{K}_{S\Lambda}][\mathcal{A}^0_{SM}]$) and necessary for nonsingularity of the matrix (3.9). It is the necessary and sufficient condition for transformation of dependent variables $u^K = u^K(y_1, \ldots, y_{\Sigma-1}, y_{\Sigma+1}, \ldots, y_{N+1})$ (where $y_1(u^K), \ldots, y_{\Sigma-1}(u^K)$, $y_{\Sigma+1}(u^K), \ldots, y_{N+1}(u^K)$ are independent functions of u^K) and such transformations will be employed in the following to obtain symmetric conservative systems. The restriction on $\widetilde{\Psi}_C(\theta, w_\gamma)$ imposed by the condition of symmetric hyperbolicity of the system (3.7) is much stronger (positive definiteness of $(\partial^2\widetilde{\Psi}_C)/(\partial w_\alpha \partial w_\beta)$ for all $[\theta, w_\gamma] \in \mathcal{D}$) than the condition (3.11) ensuring the equivalence of the symmetric systems and the original system of conservation equations or, in other words, symmetric hyperbolicity of (3.7) implies equivalence to original system of conservation equations (1.1).

4. Alternative symmetric conservative systems

4.1. Basic relations

4.1.1. Class of symmetric conservative systems. In [10], GODUNOV introduced the class of quasi-linear symmetric conservative systems of the following form

$$\partial_t\left(\frac{\partial\bar{\varphi}^0(l_M)}{\partial l_K}\right) + \partial_\alpha\left(\frac{\partial\bar{\varphi}^\alpha(l_M)}{\partial l_K}\right) = b^K(l_M), \tag{4.1}$$
$$\alpha = 1, 2, \ldots, m, \qquad I, K = 1, 2, \ldots, N.$$

The system (4.1) of N equations for N unknowns $l_K(t, X^\alpha)$ is specified completely by $m+1$ functions $\bar{\varphi}^0(l_M)$, $\bar{\varphi}^\alpha(l_M)$ and N functions $b^K(l_M)$. It implies the additional conservation equation which is obtained by multiplying (4.1) by the row vector $[l_1, l_2, \ldots, l_N]$

$$\partial_t \left(l_K \frac{\partial \bar{\varphi}^0(l_M)}{\partial l_K} - \bar{\varphi}^0 \right) + \partial_\alpha \left(l_K \frac{\partial \bar{\varphi}^\alpha(l_M)}{\partial l_K} - \bar{\varphi}^\alpha \right) = l_K b^K(l_M). \tag{4.2}$$

4.1.2. Direct derivation of $N+1$ alternative symmetric conservative systems. For derivation of alternative symmetric conservative systems, we assume that the system of $N+1$ conservation equations (1.1) contains N independent equations, satisfies the MDR (2.3), (2.4) (hence, solution set of the MDR takes the form of a family of collinear row vectors in $\mathbb{R}^{N+1}$) and the matrices (3.9) corresponding to the solutions of the MDR are nonsingular for all $u^K \in \mathcal{D}$. In order to derive alternative symmetric conservative systems, we define the potentials

$$\varphi^0 = y_\Lambda(u^K) g^{0\Lambda}(u^K), \qquad \varphi^\alpha = y_\Lambda(u^K) g^{\alpha\Lambda}(u^K). \tag{4.3}$$

According to Observation 1 and Observation 2, nonsingularity of the matrix (3.9) (for any solution of the MDR) for all $u^K \in \mathcal{D}$ implies that the family of solutions of the MDR contains $N+1$ vectors $\overset{(\Sigma)}{\mathbf{y}}$ such that Σ-th component is -1, namely, $\overset{(\Sigma)}{y} = \left[\overset{(\Sigma)}{y}_1, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, -1, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1} \right]$, $\Sigma = 1, 2, \ldots, N+1$, and each of $N+1$ transformations $[u^1, \ldots, u^N] \to \left[\overset{(\Sigma)}{y}_1 \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1} \right]$, is invertible. Hence, the components $[\overset{(\Sigma)}{y}_1, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1}]$ can be taken as new dependent variables and the respective potentials (4.3) can be expressed as functions of these new variables,

$$\begin{aligned}
\overset{(\Sigma)}{\varphi}{}^0 = \sum_{\substack{\Lambda=1 \\ \Lambda \neq \Sigma}}^{N+1} \overset{(\Sigma)}{y}_\Lambda g^{0\Lambda} \left(u^K \left(\overset{(\Sigma)}{y}_1, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1} \right) \right) \\
- g^{0\Sigma} \left(u^K \left(\overset{(\Sigma)}{y}_1, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1} \right) \right), \\
\overset{(\Sigma)}{\varphi}{}^\alpha = \sum_{\substack{\Lambda=1 \\ \Lambda \neq \Sigma}}^{N+1} \overset{(\Sigma)}{y}_\Lambda g^{\alpha\Lambda} \left(u^K \left(\overset{(\Sigma)}{y}_1, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1} \right) \right) \\
- g^{\alpha\Sigma} \left(u^K \left(\overset{(\Sigma)}{y}_1, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1} \right) \right).
\end{aligned} \tag{4.4}$$

Differentiating (4.4) with respect to $\overset{(\Sigma)}{y}_{\Delta}$, $\Delta = 1, 2, \ldots, \Sigma - 1, \Sigma + 1, \ldots, N + 1$, and taking into account the MDR (2.4), we obtain

$$
\begin{aligned}
\frac{\partial \overset{(\Sigma)}{\varphi^0}}{\partial \overset{(\Sigma)}{y_\Delta}} &= g^{0\Delta}\left(u^K\left(\overset{(\Sigma)}{y}_1, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1}\right)\right), \\
\frac{\partial \overset{(\Sigma)}{\varphi^\alpha}}{\partial \overset{(\Sigma)}{y_\Delta}} &= g^{\alpha\Delta}\left(u^K\left(\overset{(\Sigma)}{y}_1, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1}\right)\right),
\end{aligned}
\tag{4.5}
$$

and

$$
\begin{aligned}
\sum_{\substack{\Delta=1\\ \Delta\neq\Sigma}}^{N+1} \overset{(\Sigma)}{y}_{\Delta} \frac{\partial \overset{(\Sigma)}{\varphi^0}}{\partial \overset{(\Sigma)}{y_\Delta}} - \overset{(\Sigma)}{\varphi^0} &= g^{0\Sigma}\left(u^K\left(\overset{(\Sigma)}{y}_1, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1}\right)\right), \\
\sum_{\substack{\Delta=1\\ \Delta\neq\Sigma}}^{N+1} \overset{(\Sigma)}{y_\Delta} \frac{\partial \overset{(\Sigma)}{\varphi^\alpha}}{\partial \overset{(\Sigma)}{y_\Delta}} - \overset{(\Sigma)}{\varphi^\alpha} &= g^{\alpha\Sigma}\left(u^K\left(\overset{(\Sigma)}{y}_1, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1}\right)\right).
\end{aligned}
\tag{4.6}
$$

Thus, the introduced transformation of dependent variables enables us to express the equations $\partial_t g^{0\Delta}(u^K) + \partial_\alpha g^{\alpha\Delta}(u^K) = b^\Delta(u^K, t, X^\gamma)$, $\Delta = 1, 2, \ldots, \Sigma - 1, \Sigma + 1, \ldots, N + 1$ of the system (1.1) in symmetric conservative form (4.1)

$$
\partial_t \left(\frac{\partial \overset{(\Sigma)}{\varphi^0}}{\partial \overset{(\Sigma)}{y_\Delta}}\right) + \partial_\alpha \left(\frac{\partial \overset{(\Sigma)}{\varphi^\alpha}}{\partial \overset{(\Sigma)}{y_\Delta}}\right) = b^\Delta\left(u^K\left(\overset{(\Sigma)}{y_\Omega}\right), t, X^\gamma\right),
\tag{4.7}
$$
$$
\Delta, \Omega = 1, 2, \ldots, \Sigma - 1, \Sigma + 1, \ldots, N + 1,
$$

while the equation $\partial_t g^{0\Sigma}(u^K) + \partial_\alpha g^{\alpha\Sigma}(u^K) = b^\Sigma(u^K, t, X^\gamma)$ takes the form (4.2) and is interpreted as the additional conservation equation

$$
\begin{aligned}
\partial_t \left(\sum_{\substack{\Delta=1\\ \Delta\neq\Sigma}}^{N+1} \overset{(\Sigma)}{y}_{\Delta} \frac{\partial \overset{(\Sigma)}{\varphi^0}}{\partial \overset{(\Sigma)}{y_\Delta}} - \overset{(\Sigma)}{\varphi^0}\right) &+ \partial_\alpha \left(\sum_{\substack{\Delta=1\\ \Delta\neq\Sigma}}^{N+1} \overset{(\Sigma)}{y_\Delta} \frac{\partial \overset{(\Sigma)}{\varphi^\alpha}}{\partial \overset{(\Sigma)}{y_\Delta}} - \overset{(\Sigma)}{\varphi^\alpha}\right) \\
&= \sum_{\substack{\Delta=1\\ \Delta\neq\Sigma}}^{N+1} \overset{(\Sigma)}{y_\Delta}\, b^\Delta\left(u^K\left(\overset{(\Sigma)}{y_\Omega}\right), t, X^\gamma\right).
\end{aligned}
\tag{4.8}
$$

In this way, $N + 1$ alternative symmetric conservative systems corresponding to the system (1.1) can be obtained.

4.1.3. Symmetric hyperbolicity of symmetric conservative systems. According to the definition of symmetric hyperboliccity of symmetric systems (see Sec. 3.5.1), the symmetric conservative system (4.7) is symmetric hyperbolic if the Hessian matrix

$$\left[\frac{\partial \overset{(\Sigma)}{\varphi^0}}{\partial \overset{(\Sigma)}{y_\Delta} \, \partial \overset{(\Sigma)}{y_\Omega}}\right]$$

is positive definite, what is equivalent to convexity of the potential $\overset{(\Sigma)}{\varphi}{}^0$. In the case of concave $\overset{(\Sigma)}{\varphi}{}^0$ and

$$\left[\frac{\partial^2 \overset{(\Sigma)}{\varphi^0}}{\partial \overset{(\Sigma)}{y_\Delta} \, \partial \overset{(\Sigma)}{y_\Omega}}\right]$$

negative definite, the system (4.7) can be brought into equivalent symmetric conservative system simply by multiplying each equation by the factor (-1).

4.1.4. Relations between alternative symmetric conservative systems. In order to demonstrate the relations betwen different symmetric conservative systems corresponding to the same system of $N+1$ conservation equations (satisfying the assumptions mentioned in Sec. 4.1.2), we take two arbitrarily chosen solutions of the MDR

$$\overset{(\Sigma)}{\mathbf{y}} = \left[\overset{(\Sigma)}{y}_1, \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}, -1, \overset{(\Sigma)}{y}_{\Sigma+1}, \ldots, \overset{(\Sigma)}{y}_{N+1}\right]$$

and

$$\overset{(\Delta)}{\mathbf{y}} = \left[\overset{(\Delta)}{y}_1, \ldots, \overset{(\Delta)}{y}_{\Delta-1}, -1, \overset{(\Delta)}{y}_{\Delta+1}, \ldots, \overset{(\Delta)}{y}_{N+1}\right]$$

(without loss of generality, we may assume $1 \le \Sigma < \Delta \le N+1$). The symmetric conservative system can be assigned to each of them according to (4.4) – (4.8). The components of $\overset{(\Sigma)}{\mathbf{y}}$ and $\overset{(\Delta)}{\mathbf{y}}$ are mutualy related

$$\overset{(\Delta)}{y}_\Lambda = -\frac{\overset{(\Sigma)}{y}_\Lambda}{\overset{(\Sigma)}{y}_\Delta}, \qquad \overset{(\Sigma)}{y}_\Lambda = -\frac{\overset{(\Delta)}{y}_\Lambda}{\overset{(\Delta)}{y}_\Sigma}, \qquad \Lambda \neq \Delta, \qquad \Lambda \neq \Sigma, \tag{4.9}$$

$$\overset{(\Delta)}{y}_\Sigma \, \overset{(\Sigma)}{y}_\Delta = 1, \qquad \overset{(\Delta)}{y}_\Delta = \overset{(\Sigma)}{y}_\Sigma = -1.$$

According to (4.3), (4.4), the respective potentials $\left[\overset{(\Sigma)}{\varphi}{}^0, \overset{(\Sigma)}{\varphi}{}^\alpha\right]$ and $\left[\overset{(\Delta)}{\varphi}{}^0, \overset{(\Delta)}{\varphi}{}^\alpha\right]$, corresponding to $\overset{(\Sigma)}{\mathbf{y}}$ and $\overset{(\Delta)}{\mathbf{y}}$, can be obtained and, in view of (4.9), they satisfy the following relations:

$$
\begin{aligned}
&\overset{(\Delta)}{\varphi}{}^{0}\left(\overset{(\Delta)}{y}_{1},\ldots,\overset{(\Delta)}{y}_{\Sigma+1},\overset{(\Delta)}{y}_{\Sigma},\overset{(\Delta)}{y}_{\Sigma+1},\ldots,\overset{(\Delta)}{y}_{\Delta-1},\overset{(\Delta)}{y}_{\Delta+1},\ldots,\overset{(\Delta)}{y}_{N+1}\right)\\
&\qquad= -\overset{(\Delta)}{y}_{\Sigma}\overset{(\Sigma)}{\varphi}{}^{0}\left(-\frac{\overset{(\Delta)}{y}_{1}}{\overset{(\Delta)}{y}_{\Sigma}},\ldots,-\frac{\overset{(\Delta)}{y}_{\Sigma-1}}{\overset{(\Delta)}{y}_{\Sigma}},-\frac{\overset{(\Delta)}{y}_{\Sigma+1}}{\overset{(\Delta)}{y}_{\Sigma}},\ldots,\right.\\
&\qquad\qquad\left.-\frac{\overset{(\Delta)}{y}_{\Delta-1}}{\overset{(\Delta)}{y}_{\Sigma}},\frac{1}{\overset{(\Delta)}{y}_{\Sigma}},-\frac{\overset{(\Delta)}{y}_{\Delta+1}}{\overset{(\Delta)}{y}_{\Sigma}},\ldots,-\frac{\overset{(\Delta)}{y}_{N+1}}{\overset{(\Delta)}{y}_{\Sigma}}\right),\\
&\overset{(\Delta)}{\varphi}{}^{\alpha}\left(\overset{(\Delta)}{y}_{1},\ldots,\overset{(\Delta)}{y}_{\Sigma+1},\overset{(\Delta)}{y}_{\Sigma},\overset{(\Delta)}{y}_{\Sigma+1},\ldots,\overset{(\Delta)}{y}_{\Delta-1},\overset{(\Delta)}{y}_{\Delta+1},\ldots,\overset{(\Delta)}{y}_{N+1}\right)\\
&\qquad= -\overset{(\Delta)}{y}_{\Sigma}\overset{(\Sigma)}{\varphi}{}^{\alpha}\left(-\frac{\overset{(\Delta)}{y}_{1}}{\overset{(\Delta)}{y}_{\Sigma}},\ldots,-\frac{\overset{(\Delta)}{y}_{\Sigma-1}}{\overset{(\Delta)}{y}_{\Sigma}},-\frac{\overset{(\Delta)}{y}_{\Sigma+1}}{\overset{(\Delta)}{y}_{\Sigma}},\ldots,\right.\\
&\qquad\qquad\left.-\frac{\overset{(\Delta)}{y}_{\Delta-1}}{\overset{(\Delta)}{y}_{\Sigma}},\frac{1}{\overset{(\Delta)}{y}_{\Sigma}},-\frac{\overset{(\Delta)}{y}_{\Delta+1}}{\overset{(\Delta)}{y}_{\Sigma}},\ldots,-\frac{\overset{(\Delta)}{y}_{N+1}}{\overset{(\Delta)}{y}_{\Sigma}}\right),
\end{aligned}
\tag{4.10}
$$

and

$$
\begin{aligned}
&\overset{(\Sigma)}{\varphi}{}^{0}\left(\overset{(\Sigma)}{y}_{1},\ldots,\overset{(\Sigma)}{y}_{\Sigma-1},\overset{(\Sigma)}{y}_{\Sigma+1},\ldots,\overset{(\Sigma)}{y}_{\Delta-1},\overset{(\Sigma)}{y}_{\Delta},\overset{(\Sigma)}{y}_{\Delta+1},\ldots,\overset{(\Sigma)}{y}_{N+1}\right)\\
&\qquad= -\overset{(\Sigma)}{y}_{\Delta}\overset{(\Delta)}{\varphi}{}^{0}\left(-\frac{\overset{(\Sigma)}{y}_{1}}{\overset{(\Sigma)}{y}_{\Delta}},\ldots,-\frac{\overset{(\Sigma)}{y}_{\Sigma-1}}{\overset{(\Sigma)}{y}_{\Delta}},\frac{1}{\overset{(\Sigma)}{y}_{\Delta}},-\frac{\overset{(\Sigma)}{y}_{\Sigma+1}}{\overset{(\Sigma)}{y}_{\Delta}},\ldots,\right.\\
&\qquad\qquad\left.-\frac{\overset{(\Sigma)}{y}_{\Delta-1}}{\overset{(\Sigma)}{y}_{\Delta}},-\frac{\overset{(\Sigma)}{y}_{\Delta+1}}{\overset{(\Sigma)}{y}_{\Delta}},\ldots,-\frac{\overset{(\Sigma)}{y}_{N+1}}{\overset{(\Sigma)}{y}_{\Delta}}\right),\\
&\overset{(\Sigma)}{\varphi}{}^{\alpha}\left(\overset{(\Sigma)}{y}_{1},\ldots,\overset{(\Sigma)}{y}_{\Sigma+1},\overset{(\Sigma)}{y}_{\Sigma+1},\ldots,\overset{(\Sigma)}{y}_{\Delta-1},\overset{(\Sigma)}{y}_{\Delta},\overset{(\Sigma)}{y}_{\Delta+1},\ldots,\overset{(\Sigma)}{y}_{N+1}\right)\\
&\qquad= -\overset{(\Sigma)}{y}_{\Sigma}\overset{(\Delta)}{\varphi}{}^{\alpha}\left(-\frac{\overset{(\Sigma)}{y}_{1}}{\overset{(\Sigma)}{y}_{\Delta}},\ldots,-\frac{\overset{(\Sigma)}{y}_{\Sigma-1}}{\overset{(\Sigma)}{y}_{\Delta}},\frac{1}{\overset{(\Sigma)}{y}_{\Delta}},-\frac{\overset{(\Sigma)}{y}_{\Sigma+1}}{\overset{(\Sigma)}{y}_{\Delta}},\ldots,\right.\\
&\qquad\qquad\left.-\frac{\overset{(\Sigma)}{y}_{\Delta-1}}{\overset{(\Sigma)}{y}_{\Delta}},-\frac{\overset{(\Sigma)}{y}_{\Delta+1}}{\overset{(\Sigma)}{y}_{\Delta}},\ldots,-\frac{\overset{(\Sigma)}{y}_{N+1}}{\overset{(\Sigma)}{y}_{\Delta}}\right).
\end{aligned}
\tag{4.11}
$$

If the system (4.7), (4.8) corresponding to $\overset{(\Sigma)}{\mathbf{y}}$ is given, then, with the aid of the relations (4.9), (4.10), (4.11), the "new" system (4.7), (4.8) corresponding to $\overset{(\Delta)}{\mathbf{y}}$ can be obtained. Of course, in such transformations the "role" of the equations in the system (4.7), (4.8) changes, namely, the equation (4.8) of the system (4.7), (4.8) corresponding to $\overset{(\Sigma)}{\mathbf{y}}$ will be, according to (4.9), (4.10), (4.11), transformed to a "member" of the system (4.7) for $\overset{(\Delta)}{\mathbf{y}}$, while this member of the system (4.7) for $\overset{(\Sigma)}{\mathbf{y}}$ which corresponds to the conservation equation $\partial_t g^{0\Delta}(u^K)+\partial_\alpha g^{\alpha\Delta}(u^K)=b^{\Delta}(u^K,t,X^{\gamma})$ in (1.1) will be transformed to the equation (4.8) of the system (4.7), (4.8) for $\overset{(\Delta)}{\mathbf{y}}$. This transformation can be easily demonstrated by substituting (4.9) into (4.7), (4.8), performing the respective differentiations and taking into account (4.9), (4.10), (4.11), (4.4).

With the aid of $(4.10)_1$, $(4.11)_1$, the relation between convexity (concavity) of $\overset{(\Delta)}{\varphi}{}^0$ and concavity (convexity) of $\overset{(\Sigma)}{\varphi}{}^0$ can be established provided that either $\overset{(\Delta)}{y}{}_{\Sigma}(u^K)>0$ or $\overset{(\Delta)}{y}{}_{\Sigma}(u^K)<0$ for all u^K. Since the proof of this relation in a general case requires lengthy calculations, we restrict ourselvs to demonstrating this relation for the case of two symmetric conservative systems given in Secs. 4.2.2 and 4.2.3. Therefore, for those systems we derive the relations (4.9), (4.10), (4.11) in Sec. 4.3.1 and examine convexity (concavity) of the respective potentials in Appendix B. The reasoning given in Appendix B can be directly generalized for $(4.10)_1$, $(4.11)_1$. It should be noted that the problem of transformation of the given symmetric conservative system (4.1) in one spatial dimension into another symmetric conservative system has been considered by GODUNOV and SULTANGAZIAN [11, 12]. For such transformation, the relations differing by numerical factor (-1) from $(4.9)_{1,2}$ were postulated in [11, 12]. Hence, the procedure of derivation of "new" symmetric conservative system from the "old" one proposed by GODUNOV and SULTANGAZIN [11, 12] results in changing the sign of that conservation equation in "new" symmetric conservative system which is the additional conservation equation implied by the "old" symmetric conservative system and, as a consequence, the potential $\varphi^0(l_K)$ (see, (4.1)) for "new" symmetric conservative system and the potential $\varphi^0(l_K)$ for "old" symmetric conservative system are both convex (concave). In our approach to the derivation of alternative symmetric conservative systems, the signs of all $N+1$ conservation equations are preserved.

4.2. Two alternative symmetric conservative systems corresponding to (1.1), (1.2)

4.2.1. Family of potentials corresponding to (1.1), (1.2). According to the general procedure developed in Sec. 4.1.2, we derive in this paper two alternative symmetric conservative systems, one corresponding to the case of balance of energy treated as the additional conservation equation and described by $\mathbf{y} = \widehat{\mathbf{y}} = [-1, \widehat{z}^{\gamma}, \widehat{\mu}]$, and the second one corresponding to the case of balance of entropy treated as the additional conservation equation and described by $\mathbf{y} = \check{\mathbf{y}} = [\check{\lambda}, \check{z}^{\gamma}, -1]$.

In order to transform the system of field equations (1.1), (1.2) with constitutive functions satisfying the conditions (2.5), (2.8) into symmetric conservative systems with respect to the field variables $[\check{\lambda}, \check{z}^{\gamma}]$ or $[\check{z}^{\gamma}, \check{\mu}]$, we therefore introduce the following potentials (4.3):

$$\begin{aligned} \varphi_C^0 &= \lambda \varrho_0 \widetilde{\varepsilon} + z^{\gamma} \tau w_{\gamma} + \mu \varrho_0 \widetilde{\eta}, \\ \varphi_C^{\alpha} &= \lambda \widetilde{q}^{\alpha} - z^{\alpha} f_1(\theta) + \mu \frac{1}{\theta} \widetilde{q}^{\alpha}. \end{aligned} \tag{4.12}$$

In view of (2.9), (2.5), (2.6), potentials (4.12) can be expressed as

$$\begin{aligned} \varphi_C^0 &= \lambda \widetilde{\varphi}_C^0 = -\frac{\mu}{\theta} \widetilde{\varphi}_C^0, \\ \varphi_C^{\alpha} &= -z^{\alpha} f_1(\theta) = \lambda \widetilde{\varphi}_C^{\alpha} = -\frac{\mu}{\theta} \widetilde{\varphi}_C^{\alpha}, \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} \widetilde{\varphi}_C^0 &= \varrho_0 \widetilde{\varepsilon} + \frac{f_1(\theta)}{\theta f_1'(\theta)} \widetilde{q}^{\gamma} w_{\gamma} - \varrho_0 \theta \widetilde{\eta} = \varrho_0 \left[\widetilde{\Psi}_C - \frac{\partial \widetilde{\Psi}_C}{\partial w_{\gamma}} w_{\gamma} \right], \\ \widetilde{\varphi}_C^{\alpha} &= -\frac{f_1(\theta)}{\theta f_1'(\theta)} \widetilde{q}^{\alpha} = \frac{\varrho_0}{\tau} f_1(\theta) \frac{\partial \widetilde{\Psi}_C}{\partial w_{\alpha}}. \end{aligned} \tag{4.14}$$

4.2.2. Balance of energy treated as the additional conservation equation. When the first equation of the system (1.1), (1.2) corresponding to the balance of energy is treated as the additional conservation equation implied by the system of four remaining equations, we put $\lambda = -1$ in (2.10), (2.9), (4.13), and obtain

$$\begin{aligned} \mu &= \widehat{\mu} = \theta, \\ z^{\gamma} &= \widehat{z}^{\gamma} = \frac{\varrho_0}{\tau} \left(\frac{\partial \widetilde{\varepsilon}}{\partial w_{\gamma}} - \theta \frac{\partial \widetilde{\eta}}{\partial w_{\gamma}} \right) = \frac{\varrho_0}{\tau} \frac{\partial \widetilde{\Psi}_C}{\partial w_{\gamma}} = \frac{1}{\theta f_1'(\theta)} \widetilde{q}^{\gamma}, \\ \widehat{\varphi}_C^0 &= -\widetilde{\varphi}_C^0 = -\varrho_0 \widetilde{\varepsilon} - \frac{\tau}{\theta f_1'(\theta)} \widetilde{q}^{\alpha} w_{\gamma} + \varrho_0 \theta \widetilde{\eta} = -\varrho_0 \left[\widetilde{\Psi}_C - \frac{\partial \widetilde{\Psi}_C}{\partial w_{\gamma}} w_{\gamma} \right], \\ \widehat{\varphi}_C^{\alpha} &= -\widetilde{\varphi}_C^{\alpha} = \frac{f_1(\theta)}{\theta f_1'(\theta)} \widetilde{q}^{\alpha} = -\frac{\varrho_0}{\tau} f_1(\theta) \frac{\partial \widetilde{\Psi}_C}{\partial w_{\alpha}}. \end{aligned} \tag{4.15}$$

In $(4.15)_{1,2}$ $\widehat{\mu}$ and $\widehat{z}^{\gamma}$ are functions of θ and w_{γ}. The condition (3.11) that the matrix (3.10) is nonsingular coincides in this case with the condition that Jacobian of the mapping $[\theta, w_{\alpha}] \to [\widehat{z}_{\gamma}, \widehat{\mu}]$ is nonsingular. Hence, there exists the inverse

$[\widehat{z}^\gamma, \widehat{\mu}] \to [\theta, w_\alpha]$. Assuming that constitutive functions $\widehat{\varepsilon}(\theta, w_\gamma)$ and $\widehat{\eta}(\theta, w_\gamma)$ are chosen such that (3.11) is satisfied, we define functions $\theta = \widehat{\theta}(\widehat{\mu}) = \widehat{\mu}$ and $w_\gamma = \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha)$, and use them to express $\widehat{\varphi}^0_C$, $\widehat{\varphi}^\alpha_C$ as functions of $\widehat{\mu}$ and $\widehat{z}^\alpha$,

$$
\begin{aligned}
\widehat{\varphi}^0_C(\widehat{\mu}, \widehat{z}^\alpha) &= -\varrho_0 \widetilde{\varepsilon}(\widehat{\theta}(\widehat{\mu}), \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha)) + \varrho_0 \widehat{\mu} \widetilde{\eta}(\widehat{\theta}(\widehat{\mu}), \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha)) \\
&\qquad - \frac{\tau}{\widehat{\mu} f_1'(\widehat{\mu})} \widetilde{q}^\beta(\widehat{\theta}(\widehat{\mu}), \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha)) \widehat{w}_\beta(\widehat{\mu}, \widehat{z}^\alpha) \\
&= -\varrho_0 \widetilde{\varepsilon}(\widehat{\theta}(\widehat{\mu}), \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha)) + \varrho_0 \widehat{\mu} \widetilde{\eta}(\widehat{\theta}(\widehat{\mu}), \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha)) + \tau \widehat{z}^\gamma \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha), \\
\widehat{\varphi}^\alpha_C(\widehat{\mu}, \widehat{z}^\alpha) &= \frac{f_1(\widehat{\mu})}{\widehat{\mu} f_1'(\widehat{\mu})} \widetilde{q}^\alpha(\widehat{\theta}(\widehat{\mu}), \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha)) = f_1(\widehat{\mu}) \widehat{z}^\alpha,
\end{aligned}
\tag{4.16}
$$

since it follows from $(4.15)_{1,2}$ that

$$
\widetilde{q}^\gamma(\widehat{\theta}(\widehat{\mu}), \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha)) = -\widehat{\mu} f_1'(\widehat{\mu}) \widehat{z}^\gamma. \tag{4.17}
$$

Differentiating (4.16) with respect to $\widehat{z}^\alpha$ and $\widehat{\mu}$, and taking into account (2.5), (2.6), $(4.15)_{1,2}$, we obtain

$$
\begin{aligned}
\frac{\partial \widehat{\varphi}^0_C}{\partial \widehat{z}^\beta} &= \tau \widehat{w}_\beta(\widehat{\mu}, \widehat{z}^\alpha) = \tau w_\beta, \\
\frac{\partial \widehat{\varphi}^\alpha_C}{\partial \widehat{z}^\beta} &= -f_1(\widehat{\mu}) \delta^\alpha{}_\beta = -f_1(\theta) \delta^\alpha{}_\beta, \\
\frac{\partial \widehat{\varphi}^0_C}{\partial \widehat{\mu}} &= \varrho_0 \widetilde{\eta}(\widehat{\theta}(\widehat{\mu}), \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha)) = \varrho_0 \widetilde{\eta}, \\
\frac{\partial \widehat{\varphi}^\alpha_C}{\partial \widehat{\mu}} &= \frac{1}{\widehat{\mu}} \widetilde{q}^\alpha(\widehat{\theta}(\widehat{\mu}), \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha)) = \frac{1}{\theta} \widetilde{q}^\alpha.
\end{aligned}
\tag{4.18}
$$

Hence, the system of four equations of (1.1), $(1.2)_{2,3,4,5}$ can be written as an equivalent symmetric conservative system for four unknown fields $\widehat{z}^\alpha(t, X^\alpha)$, $\widehat{\mu}(t, X^\alpha)$,

$$
\begin{aligned}
\partial_t \left(\frac{\partial \widehat{\varphi}^0_C}{\partial \widehat{z}^\beta} \right) + \partial_\alpha \left(\frac{\partial \widehat{\varphi}^\alpha_C}{\partial \widehat{z}^\beta} \right) &= c_1 \widehat{w}_\beta(\widehat{\mu}, \widehat{z}^\alpha) = \frac{c_1}{\tau} \frac{\partial \widehat{\varphi}^0_C}{\partial \widehat{z}^\beta}, \\
\partial_t \left(\frac{\partial \widehat{\varphi}^0_C}{\partial \widehat{\mu}} \right) + \partial_\alpha \left(\frac{\partial \widehat{\varphi}^\alpha_C}{\partial \widehat{\mu}} \right) &= \varrho_0 \widetilde{\sigma}(\widehat{\theta}(\widehat{\mu}), \widehat{w}_\gamma(\widehat{\mu}, \widehat{z}^\alpha), t, X^\beta),
\end{aligned}
\tag{4.19}
$$

while the equation

$$
\begin{aligned}
\partial_t \left(\widehat{\mu} \frac{\partial \widehat{\varphi}^0_C}{\partial \widehat{\mu}} + \widehat{z}^\beta \frac{\partial \widehat{\varphi}^0_C}{\partial \widehat{z}^\beta} - \widehat{\varphi}^0_C \right) + \partial_\alpha \left(\widehat{\mu} \frac{\partial \widehat{\varphi}^\alpha_C}{\partial \widehat{\mu}} + \widehat{z}^\beta \frac{\partial \widehat{\varphi}^\alpha_C}{\partial \widehat{z}^\beta} - \widehat{\varphi}^\alpha_C \right) \\
= c_1 \widehat{z}^\beta \widehat{w}_\beta + \varrho_0 \widehat{\mu} \widetilde{\sigma} = \varrho_0 \widetilde{r},
\end{aligned}
\tag{4.20}
$$

which is a consequence of the system (4.19), corresponds to the first equation of the system (1.1), (1.2), since

$$
\begin{aligned}
&\widehat{\mu}\frac{\partial\widehat{\varphi}^0_C}{\partial\widehat{\mu}} + \widehat{z}^\beta\frac{\partial\widehat{\varphi}^0_C}{\partial\widehat{z}^\beta} - \widehat{\varphi}^0_c = \varrho_0\widetilde{\varepsilon}(\widehat{\theta}(\widehat{\mu}),\widehat{w}_\gamma(\widehat{\mu},\widehat{z}^\alpha)) = \varrho_0\widehat{\varepsilon},\\
&\widehat{\mu}\frac{\partial\widehat{\varphi}^\alpha_C}{\partial\widehat{\mu}} + \widehat{z}^\beta\frac{\partial\widehat{\varphi}^\alpha_C}{\partial\widehat{z}^\beta} - \widehat{\varphi}^\alpha_C = \widetilde{q}^\alpha(\widehat{\theta}(\widehat{\mu}),\widehat{w}_\gamma(\widehat{\mu},\widehat{z}^\beta)) = \widetilde{q}^\alpha.
\end{aligned}
\tag{4.21}
$$

4.2.3. Balance of entropy treated as the additional conservation equation. In order to obtain an alternative symmetric conservative system corresponding to (1.1), (1.2), in which the balance of entropy is treated as the additional conservation equation, we put $\mu = -1$ in (2.10), (2.9), (4.13), and obtain

$$
\begin{aligned}
&\lambda = \check{\lambda} = \frac{1}{\theta},\\
&z^\gamma = \check{z}^\gamma = -\frac{\varrho_0}{\tau}\frac{1}{\theta}\left(\frac{\partial\widetilde{\varepsilon}}{\partial w_\gamma} - \theta\frac{\partial\widetilde{\eta}}{\partial w_\gamma}\right) = -\frac{\varrho_0}{\tau}\frac{1}{\theta}\frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma} = \frac{1}{\theta^2 f_1'(\theta)}\widetilde{q}^\gamma,\\
&\check{\varphi}^0_C = \frac{1}{\theta}\widetilde{\varphi}^0_C = \varrho_0\frac{1}{\theta}\widetilde{\varepsilon} + \frac{\tau\widetilde{q}^\gamma w_\gamma}{\theta^2 f_1'(\theta)} - \varrho_0\widetilde{\eta} = \varrho_0\frac{1}{\theta}\left[\widetilde{\Psi}_C - \frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma}w_\gamma\right],\\
&\check{\varphi}^\alpha_C = \frac{1}{\theta}\widetilde{\varphi}^\alpha_C = -\frac{f_1(\theta)}{\theta^2 f_1'(\theta)}\widetilde{q}^\alpha = \frac{\varrho_0}{\tau}\frac{f_1(\theta)}{\theta}\frac{\partial\widetilde{\Psi}_C}{\partial w_\alpha},
\end{aligned}
\tag{4.22}
$$

where, $\check{\lambda}$ and $\check{z}^\gamma$ are functions of θ and w_γ. As it follows from Observation 1, for the system of conservation equations considered in this paper, nonsingularity of the matrix of the type (3.9) for one particular solution of the MDR implies nonsingularity of that matrix for all other solutions of the MDR. Hence, the condition (3.11) is necessary and sufficient for nonsingularity of the matrix (3.9) for $\check{\mathbf{y}} = [\check{\lambda}, \check{z}^\gamma, -1]$ and, as in the case of $\widehat{\mathbf{y}} = [-1, \widehat{z}^\gamma, \widehat{\mu}]$, it obviously coincides with the condition that Jacobian of the mapping $[\theta, w_\alpha] \to [\check{\lambda}, \check{z}^\gamma]$ is nonsingular. As in the previous case, assuming that constitutive functions $\widetilde{\varepsilon}(\theta, w_\gamma)$, $\widetilde{\eta}(\theta, w_\gamma)$ are chosen such that this condition holds, we define functions $\theta = \check{\theta}(\check{\lambda}) = \dfrac{1}{\check{\lambda}}$ and $w_\gamma = \check{w}_\gamma(\check{\lambda}, \check{z}^\alpha)$. Then, we express potentials $\check{\varphi}^0_C$, $\check{\varphi}^\alpha_C$ as functions of $\check{\lambda}$ and $\check{z}^\alpha$,

$$
\begin{aligned}
\check{\varphi}^0_C(\check{\lambda}, \check{z}^\alpha) &= \varrho_0\check{\lambda}\widetilde{\varepsilon}(\check{\theta}(\check{\lambda}), \check{w}_\gamma(\check{\lambda}, \check{z}^\alpha)) - \varrho_0\widetilde{\eta}(\check{\theta}(\check{\lambda}), \check{w}_\gamma(\check{\lambda}, \check{z}^\alpha))\\
&\quad + \tau\check{z}^\gamma\check{w}_\gamma(\check{\lambda}, \check{z}^\alpha),\\
\check{\varphi}^\alpha_C(\check{\lambda}, \check{z}^\alpha) &= -f_1\left(\frac{1}{\check{\lambda}}\right)\check{z}^\alpha,
\end{aligned}
\tag{4.23}
$$

taking into account that

$$
\widetilde{q}^\gamma = f_1'\left(\frac{1}{\check{\lambda}}\right)\frac{1}{\check{\lambda}^2}\check{z}^\gamma,
\tag{4.24}
$$

according to $(4.22)_{1,2}$. Differentiating (4.23) with respect to $\check{\lambda}$ and $\check{z}^\alpha$ and employing (2.5), (2.6), (4.22), we obtain

$$
\begin{aligned}
&\frac{\partial \check{\varphi}^0_C}{\partial \check{\lambda}} = \varrho_0 \widetilde{\varepsilon}(\check{\theta}(\check{\lambda}), \check{w}_\gamma(\check{\lambda}, \check{z}^\alpha)) = \varrho_0 \widetilde{\varepsilon},\\
&\frac{\partial \check{\varphi}^\alpha_C}{\partial \check{\lambda}} = f_1'\left(\frac{1}{\check{\lambda}}\right)\frac{\check{z}^\alpha}{\check{\lambda}^2} = \widetilde{q}^\alpha,\\
&\frac{\partial \check{\varphi}^0_C}{\partial \check{z}^\gamma} = \tau \check{w}_\gamma(\check{\lambda}, \check{z}^\alpha) = \tau w_\gamma,\\
&\frac{\partial \check{\varphi}^\alpha_C}{\partial \check{z}^\beta} = -f_1\left(\frac{1}{\check{\lambda}}\right)\delta^\alpha_\gamma = -f_1(\theta)\delta^\alpha_\gamma .
\end{aligned} \tag{4.25}
$$

The system of the first four equations of (1.1), (1.2) can therefore be written as a symmetric conservative system for four unknown fields $\check{\lambda}(t, X^\alpha)$, $\check{z}^\alpha(t, X^\alpha)$

$$
\begin{aligned}
&\partial_t\left(\frac{\partial \check{\varphi}^0_C}{\partial \check{\lambda}}\right) + \partial_\alpha\left(\frac{\partial \check{\varphi}^\alpha_C}{\partial \check{\lambda}}\right) = \varrho_0 \widetilde{r}(\check{\theta}(\check{\lambda}), \check{w}_\gamma(\check{\lambda}, \check{z}^\alpha), t, X^\beta),\\
&\partial_t\left(\frac{\partial \check{\varphi}^0_C}{\partial \check{z}^\gamma}\right) + \partial_\alpha\left(\frac{\partial \check{\varphi}^\alpha_C}{\partial \check{z}^\gamma}\right) = c_1 \check{w}_\gamma(\check{\lambda}, \check{z}^\alpha) = \frac{c_1}{\tau}\frac{\partial \check{\varphi}^0_C}{\partial \check{z}^\gamma} .
\end{aligned} \tag{4.26}
$$

The conservation equation implied by (4.26)

$$
\begin{aligned}
\partial_t\left(\check{\lambda}\frac{\partial \check{\varphi}^0_C}{\partial \check{\lambda}} + \check{z}^\gamma\frac{\partial \check{\varphi}^0_C}{\partial \check{z}^\gamma} - \check{\varphi}^0_C\right) &+ \partial_\alpha\left(\frac{\partial \check{\varphi}^\alpha_C}{\partial \check{\lambda}} + \check{z}^\gamma\frac{\partial \check{\varphi}^\alpha_C}{\partial \check{z}^\gamma} - \check{\varphi}^\alpha_C\right)\\
&= \varrho_0 \check{\lambda}\widetilde{r} + c_1 \check{z}^\gamma \check{w}^\gamma = \varrho_0 \widetilde{\sigma},
\end{aligned} \tag{4.27}
$$

corresponds to the equation of balance of entropy (the last equation in the system (1.1), (1.2)) since

$$
\begin{aligned}
&\check{\lambda}\frac{\partial \check{\varphi}^0_C}{\partial \check{\lambda}} + \check{z}^\gamma\frac{\partial \check{\varphi}^0_C}{\partial \check{z}^\gamma} - \check{\varphi}^0_C = \varrho_0 \widetilde{\eta}(\check{\theta}(\check{\lambda}), \check{w}_\gamma(\check{\lambda}, \check{z}^\alpha)) = \varrho_0 \widetilde{\eta},\\
&\check{\lambda}\frac{\partial \check{\varphi}^\alpha_C}{\partial \check{\lambda}} + \check{z}^\gamma\frac{\partial \check{\varphi}^\alpha_C}{\partial \check{z}^\gamma} - \check{\varphi}^\alpha_C = f_1'\left(\frac{1}{\check{\lambda}}\right)\frac{1}{\check{\lambda}}\check{z}^\alpha = \frac{1}{\theta}\widetilde{q}^\alpha.
\end{aligned} \tag{4.28}
$$

It should be noted that the potential $\check{\varphi}^0_C(\theta, w_\gamma)$ is not the Legendre transform of the entropy $\widetilde{\eta}$ since "multipliers" $\check{\lambda}(\theta, w_\gamma)$, $\check{z}^\alpha(\theta, w_\gamma)$ are not derivatives of the entropy with respect to primitive field variables θ, w_γ, respectively. This is because of the fact that internal energy $\widetilde{\varepsilon}$, which is an extensive variable, is not a primitive field variable of the theory. It is well known (see, for example, [13]) that, in thermodynamic theories formulated in terms of extensive quantities as primitive fields and balances of them, the Lagrange multipliers of the variational problem of maximization of entropy or, equivalently, Liu multipliers in the entropy principle of extended thermodynamics, which are intensive quantities, are the respective derivatives of the entropy with respect to extensive quantities. As a consequence, the potential, which gives extensive quantities as derivatives with respect to intensive quantities, is the Legendre transform of the entropy.

4.3. Symmetric hyperbolicity of symmmetric conservative systems (4.19), (4.26)

In Sec. 3.5, we have discussed the conditions upon which symmetric systems (3.8) for unknowns $[u^K(t, X^\alpha)]$ are symmetric hyperbolic. Now, the question of symmetric hyperbolicity of symmetric conservative systems (4.19) and (4.26) arises. There are various possibilities of establishing the conditions which ensure that systems (4.26), (4.19) are symmetric hyperbolic (or can be transformed to symmetric hyperbolic systems by multiplication by numerical factor (-1)), namely: a) to investigate directly either convexity (concavity) of $\widehat{\varphi}^0_C(\widehat{z}, \widehat{\mu})$ and $\check{\varphi}^0_C(\check{\lambda}, \check{z}^\gamma)$ or positive (negative) definiteness of the respective Hessians, b) to investigate convexity (concavity) of one of the potentials (either $\widehat{\varphi}^0_C(\widehat{z}, \widehat{\mu})$ or $\check{\varphi}^0_C(\check{\lambda}, \check{z}^\gamma)$) and employ the relations between $\widehat{\varphi}^0_C(\widehat{z}, \widehat{\mu})$ and $\check{\varphi}^0_C(\check{\lambda}, \check{z}^\gamma)$,, and c) to make use of the relation between symmetric hyperbolicity of the symmetric systems (3.8) and symmetric hyperbolicity of the symmetric conservative systems (4.7). Since, in this paper, the emphasis is put on various aspects of the employed method of symmetrization, we shall demonstrate here indirect examination of symmetric hyperbolicity of the systems (4.19) and (4.26) with the aid of the relation between potentials $\widehat{\varphi}^0_C(\widehat{z}, \widehat{\mu})$ and $\check{\varphi}^0_C(\check{\lambda}, \check{z}^\gamma)$, and with the aid of the relation between symmetric hyperbolicity of symmetric systems (3.8) and symmetric conservative systems (4.7).

4.3.1. Relation between potentials generating alternative symmetric conservative systems. It follows from (4.15), (4.16), (4.22), (4.23) that the potentials $\check{\varphi}^0_C(\widehat{\mu}, \widehat{z}^\alpha)$ and $\check{\varphi}^0_C(\check{\lambda}, \check{z}^\gamma)$ as well as the new field variables $\widehat{\mu}(t, X^\alpha)$, $\widehat{z}^\gamma(t, X^\alpha)$ and $\check{\lambda}(t, X^\alpha)$, $\check{z}^\gamma(t, X^\alpha)$ of the respective symmetric conservative systems (4.19), (4.26) are mutually related in accordance with the relations derived in Sec. 4.1.4.

$$\text{(4.29)}\qquad \begin{gathered} \check{\lambda} = \frac{1}{\widehat{\mu}}, \qquad \check{z}^\gamma = -\frac{\widehat{z}^\gamma}{\widehat{\mu}}, \\ \check{\varphi}^0_C(\check{\lambda}, \check{z}^\gamma) = -\check{\lambda}\widehat{\varphi}^0_C\left(\frac{1}{\check{\lambda}}, -\frac{\check{z}^\gamma}{\check{\lambda}}\right), \qquad \check{\varphi}^\alpha_C(\check{\lambda}, \check{z}^\gamma) = -\check{\lambda}\widehat{\varphi}^\alpha_C\left(\frac{1}{\check{\lambda}}, -\frac{\check{z}^\gamma}{\check{\lambda}}\right) \end{gathered}$$

and

$$\text{(4.30)}\qquad \begin{gathered} \widehat{\mu} = \frac{1}{\check{\lambda}}, \qquad \widehat{z}^\gamma = \frac{\check{z}^\gamma}{\check{\lambda}}, \\ \widehat{\varphi}^0_C(\widehat{\mu}, \widehat{z}^\gamma) = -\widehat{\mu}\check{\varphi}^0_C\left(\frac{1}{\widehat{\mu}}, -\frac{\widehat{z}^\gamma}{\widehat{\mu}}\right), \qquad \widehat{\varphi}^\alpha_C(\widehat{\mu}, \widehat{z}^\gamma) = -\widehat{\mu}\check{\varphi}^\alpha_C\left(\frac{1}{\widehat{\mu}}, -\frac{\widehat{z}^\gamma}{\widehat{\mu}}\right), \end{gathered}$$

where $\widehat{\mu} = \theta$ is positive. The relations (4.29), (4.30) lead to the following

PROPERTY. The potential $\widehat{\varphi}^0(\widehat{\mu}, \widehat{z}^\gamma)$ is convex (concave) if and only if the potential $\check{\varphi}^0(\check{\lambda}, \check{z}^\gamma)$ is concave (convex).

The proof of the Property is given in the Appendix B. Since we have established the relation between convexity (concavity) of $\widehat{\varphi}^0_C(\widehat{z},\widehat{\mu})$ and concavity (convexity) of $\check{\varphi}^0_C(\check{\lambda},\check{z}^\gamma)$, it now suffices to examine only the conditions of symmetric hyperbolicity for the system (4.19) (positive definiteness of Hessian matrix of $\widehat{\varphi}^0_C(\widehat{z},\widehat{\mu})$).

4.3.2. Symmetric hyperbolicity of the symmetric system for $[\theta, w_\gamma]$ and of the symmetric conservative system, both corresponding to the same solution of the MDR. We recall that the symmetric system (3.7) for unknowns $[u_K(t,X^\alpha)] = [\theta(t,X^\alpha), w_\gamma(t,X^\alpha)]$ and the symmetric conservative system (4.19) for unknowns $[\widehat{z}^\gamma(t,X^\alpha),\widehat{\mu}(t,X^\alpha)]$ correspond to the same solution of the MDR, namely,

$$\mathbf{y} = \widehat{\mathbf{y}} = [-1, \widehat{z}^\gamma(\theta,w_\gamma), \widehat{\mu}(\theta,w_\gamma)] = \left[-1, \frac{\varrho_0}{\tau}\frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma}, \theta\right].$$

It follows from (4.18) that the components of Hessian of $\widehat{\varphi}^0_C(\widehat{z}^\gamma,\widehat{\mu}))$ satisfy the relations

$$\begin{aligned}
&\frac{\partial^2\widehat{\varphi}^0_C}{\partial\widehat{z}^\alpha\partial\widehat{z}^\beta} = \tau\frac{\partial\widehat{w}_\alpha(\widehat{\mu},\widehat{z}^\gamma)}{\partial\widehat{z}^\beta} = \tau\frac{\partial\widehat{w}_\beta(\widehat{\mu},\widehat{z}^\gamma)}{\partial\widehat{z}^\beta},\\
&\frac{\partial^2\widehat{\varphi}^0_C}{\partial\widehat{\mu}\partial\widehat{z}^\alpha} = \tau\frac{\partial\widehat{w}_\alpha(\widehat{\mu},\widehat{z}^\gamma)}{\partial\widehat{\mu}} = \varrho_0\frac{\partial\widetilde{\eta}(\widehat{\theta}(\widehat{\mu}),\widehat{w}_\beta(\widehat{\mu},\widehat{z}^\lambda))}{\partial\widehat{w}_\gamma}\frac{\partial\widehat{w}_\gamma(\widehat{\mu},\widehat{z}^\lambda)}{\partial\widehat{z}^\alpha},\\
&\frac{\partial^2\widehat{\varphi}^0_C}{\partial\widehat{\mu}^2} = \varrho_0\frac{\partial\widetilde{\eta}(\widehat{\theta}(\widehat{\mu}),\widehat{w}_\beta(\widehat{\mu},\widehat{z}^\lambda))}{\partial\widehat{\theta}}\frac{\partial\widehat{\theta}}{\partial\widehat{\mu}} + \varrho_0\frac{\partial\widetilde{\eta}(\widehat{\theta}(\widehat{\mu}),\widehat{w}_\beta(\widehat{\mu},\widehat{z}^\lambda))}{\partial\widehat{w}_\gamma}\frac{\partial\widehat{w}_\gamma(\widehat{\mu},\widehat{z}^\lambda)}{\partial\widehat{\mu}},
\end{aligned}\tag{4.31}$$

which enable us to represent this Hessian as the following product:

$$[\widehat{B}^0_{PQ}] = \begin{bmatrix} \dfrac{\partial^2\widehat{\varphi}^0_C}{\partial\widehat{z}^\alpha\partial\widehat{z}^\beta} & \dfrac{\partial^2\widehat{\varphi}^0_C}{\partial\widehat{z}^\alpha\partial\widehat{\mu}} \\ \dfrac{\partial^2\widehat{\varphi}^0_C}{\partial\widehat{z}^\beta\partial\widehat{\mu}} & \dfrac{\partial^2\widehat{\varphi}^0_C}{\partial\widehat{\mu}^2}\end{bmatrix} \tag{4.32}$$

$$= \begin{bmatrix} 0 & \tau & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & \tau \\ \varrho_0\dfrac{\partial\widetilde{\eta}}{\partial\widehat{\theta}} & \varrho_0\dfrac{\partial\widetilde{\eta}}{\partial\widehat{w}_1} & \varrho_0\dfrac{\partial\widetilde{\eta}}{\partial\widehat{w}_2} & \varrho_0\dfrac{\partial\widetilde{\eta}}{\partial\widehat{w}_3}\end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \dfrac{\partial\widehat{\theta}}{\partial\widehat{\mu}} \\ \dfrac{\partial\widehat{w}_1}{\partial\widehat{z}^1} & \dfrac{\partial\widehat{w}_1}{\partial\widehat{z}^2} & \dfrac{\partial\widehat{w}_1}{\partial\widehat{z}^3} & \dfrac{\partial\widehat{w}_1}{\partial\widehat{\mu}} \\ \dfrac{\partial\widehat{w}_2}{\partial\widehat{z}^1} & \dfrac{\partial\widehat{w}_2}{\partial\widehat{z}^2} & \dfrac{\partial\widehat{w}_2}{\partial\widehat{z}^3} & \dfrac{\partial\widehat{w}_2}{\partial\widehat{\mu}} \\ \dfrac{\partial\widehat{w}_3}{\partial\widehat{z}^1} & \dfrac{\partial\widehat{w}_3}{\partial\widehat{z}^2} & \dfrac{\partial\widehat{w}_3}{\partial\widehat{z}^3} & \dfrac{\partial\widehat{w}_3}{\partial\widehat{\mu}}\end{bmatrix}.$$

The first term in the product (4.38) is the matrix $[A^{0M}{}_S]$ which, as we have mentioned in Sec. 3.2, is a 4×4 minor of the 5×4 matrix $[\mathcal{A}^{0\Lambda}{}_S]$, given by (2.2), obtained by deleting the first row in $[\mathcal{A}^{0\Lambda}{}_S]$. The second term is the Jacobi matrix of the transformation $[\widehat{z}^\gamma, \widehat{\mu}] \to [\theta, w_\gamma]$ defined as $\theta = \widehat{\theta}(\widehat{\mu}) = \widehat{\mu}$, $w_\gamma = \widehat{w}_\gamma(\widehat{z}^\gamma, \widehat{\mu})$ in (4.16). In Sec. 3.2, the 4×4 matrix $[\widehat{K}_{SM}]$ is introduced as a 4×4 minor of the 4×5 matrix $[\widehat{\mathcal{K}}_{S\Lambda}]$ given by (3.5), which is obtained by deleting the first column in $[\widehat{\mathcal{K}}_{S\Lambda}]$. This matrix can be obtained as a 4×4 minor of the 5×5 matrix (3.10) by deleting the first row and the first column. It follows from (3.10) that $[\widehat{K}_{S\Lambda}]$ corresponds to the transposed Jacobi matrix of the transformation $[\theta, w_\gamma] \to [\widehat{z}^\gamma, \widehat{\mu}]$. It will be convenient to employ the following notation for the matrices $\widehat{\mathbf{B}}^0 := [\widehat{B}^0_{PQ}]$, $\widehat{\mathbf{A}}^0 := [\widehat{A}^0_{PQ}]$, $\mathbf{A}^0 := [A^{0M}_R]$, $\widehat{\mathbf{K}} := [\widehat{K}_{SQ}]$ and rewrite (4.32) in the form

$$\widehat{\mathbf{B}}^0 = \mathbf{A}^0(\widehat{\mathbf{K}}^T)^{-1} = \widehat{\mathbf{B}}^{0T}, \tag{4.33}$$

since the transformation $[\widehat{z}^\gamma, \widehat{\mu}] \to [\theta, w_\gamma]$ is the inverse of the transformation $[\theta, w_\gamma] \to [\widehat{z}^\gamma, \widehat{\mu}]$ and $\widehat{\mathbf{B}}^0$ is symmetric. It is noted in Sec. 3.2 that $\widehat{\mathbf{K}}$ is the left symmetrizer of $\mathbf{A}^0$ and the symmetric matrix $\widehat{\mathbf{A}}^0$ of the symmetric system (3.7) can be obtained as

$$\widehat{\mathbf{A}}^0 = \widehat{\mathbf{K}}\mathbf{A}^0 = \widehat{\mathbf{A}}^{0T}. \tag{4.34}$$

The relations (4.33), (4.34) imply that the symmetric matrices $\widehat{\mathbf{A}}^0$ and $\widehat{\mathbf{B}}^0$ are mutually related by the following congruent transformation

$$\begin{aligned} \widehat{\mathbf{B}}^0 &= \widehat{\mathbf{K}}^{-1}\widehat{\mathbf{A}}^0(\widehat{\mathbf{K}}^{-1})^T, \\ \widehat{\mathbf{A}}^0 &= \widehat{\mathbf{K}}\widehat{\mathbf{B}}^0\widehat{\mathbf{K}}^T, \end{aligned} \tag{4.35}$$

where $\widehat{\mathbf{K}}$ is nonsingular. Positive (negative) definiteness of $\widehat{\mathbf{A}}^0$ means that $\mathbf{x}^T\widehat{\mathbf{A}}^0\mathbf{x} > 0$ (< 0) for all $\mathbf{x} \neq 0$ and therefore $\mathbf{x}^T\widehat{\mathbf{K}}\widehat{\mathbf{B}}^0\widehat{\mathbf{K}}^T\mathbf{x} > 0$ (< 0) for all $\mathbf{x} \neq \mathbf{0}$. Hence, $\mathbf{y}^T\widehat{\mathbf{B}}^0\mathbf{y} > 0$ (< 0) for all $\mathbf{y} \neq \mathbf{0}$ since $\mathbf{y} = \widehat{\mathbf{K}}^T\mathbf{x}$ represents all nonzero vectors for all $\mathbf{x} \neq \mathbf{0}$. This shows that positive (negative) definiteness of $\widehat{\mathbf{A}}^0$ implies positive (negative) definiteness of $\widehat{\mathbf{B}}^0$. The converse can be proved in the same way. Therefore, the restrictions on constitutive functions $\widetilde{\Psi}_C(\theta, w_\gamma)$ (or $\widetilde{\varepsilon}(\theta, w_\gamma)$ and $\widetilde{\eta}(\theta, w_\gamma)$) established in Sec. 3.5.2, which ensure symmetric hyperbolicity of the system (3.7) (positive definiteness of $[\widehat{A}^0_{SM}]$), also ensure symmetric hyperbolicity of the symmetric conservative system (4.19) (convexity of $\widehat{\varphi}^0_C(\widehat{z}^\gamma, \widehat{\mu})$ or, equivalently, positive definiteness of the Hessian matrix of $\widehat{\varphi}^0_C(\widehat{z}^\gamma, \widehat{\mu})$).

The reasoning similar to that presented here for particular systems (3.7) and (4.19) can be directly applied to each of the $N+1$ symmetric conservative systems derived in Sec. 4.1.2 and the corresponding symmetric systems for original unknowns u^K (derived in Sec. 3.1), thus leading to the following observation.

OBSERVATION 3. Symmetric conservative system (4.7) for unknowns

$$\overset{(\Sigma)}{y}_1(t, X^\alpha), \ldots, \overset{(\Sigma)}{y}_{\Sigma-1}(t, X^\alpha), \overset{(\Sigma)}{y}_{\Sigma+1}(t, X^\alpha), \ldots, \overset{(\Sigma)}{y}_{N+1}(t, X^\alpha)$$

is symmetric hyperbolic if and only if the symmetric system (3.3), (3.4) for unknowns $u^1(t, X^\alpha), \ldots, u^N(t, X^\alpha)$ corresponding to the solution

$$\mathbf{y} = \overset{(\Sigma)}{\mathbf{y}} = \left[\overset{(\Sigma)}{y}_1 (u^K), \ldots, \overset{(\Sigma)}{y}_{\Sigma-1} (u^K), -1, \overset{(\Sigma)}{y}_{\Sigma+1} (u^K), \ldots, \overset{(\Sigma)}{y}_{N+1} (u^K)\right]$$

of the MDR is symmetric hyperbolic.

Now, it follows from the relation between convexity (concavity) of $\widehat{\varphi}^0_C(\widehat{z}^\gamma, \widehat{\mu})$ and concavity (convexity) of $\check{\varphi}^0_C(\check{\lambda}, \check{z}^\gamma)$, that the systems (3.7) and (4.19) are symmetric hyperbolic if and only if the system (4.26) can be brought to symmetric hyperbolic form by multiplication by numerical factor (-1) and, conversely, the systems (3.7) and (4.19) can be brought to symmetric hyperbolic form by multiplication by numerical factor (-1) if and only if the system (4.26) is symmetric hyperbolic.

As it is emphasized by GODUNOV and SULTANGAZIAN [11, 12], alternative symmetric conservative systems corresponding to the same consistent system of $N+1$ conservation equations for N unknowns are not equivalent when weak (discontinuous) solutions are considered. This nonequivalence manifests itself in that the different equations are to be replaced by inequalities for different symmetric conservative systems. Namely, for the symmetric conservative system (4.19) corresponding to the equation of balance of energy treated as the additional conservation equation (energy being a "derived" conserved quantity), Eqs. (4.19) are now satisfied in the weak sense while the additional conservation equation (4.20) should be replaced by the inequality (≥ 0), also in the weak sense. When the symmetric conservative system (4.26) corresponding to the equation of balance of entropy in the role of the additional conservation equation (entropy being a "derived" conserved quantity) is considered for weak solutions, the equations (4.26) are now satisfied in the weak sense but the equation (4.27) should be replaced by the inequality (≥ 0) in a weak sense. A physical interpretation of those inequalities is mentioned in [9]. In the case of the energy taken as a "derived" conserved quantity, we have an increase of energy across the surface of discontinuity. This may be interpreted as energy production on the shock. Hence, in the processes, in which the energy is conserved, it must be taken as one of the "original" conserved quantities. If the entropy is taken as a "derived" conserved quantity, the inequality for discontinuous solutions means that entropy increases on the shock.

Appendix A

LEMMA 1. If the solution set of the MDR (2.3) (2.4) for the system (1.1) has a form of a family of collinear $N+1$ component row vectors $\mathbf{y}^T(u^K) = [y_\Lambda(u^K)]$ parametrized by differentiable functions $\alpha(u^K)$ (equivalently, if the system (1.1) contains N independent equations), then every matrix (3.9) corresponding to the solution of the MDR has the same kernel.

P r o o f. Let $\mathbf{y}^{*T}(\mathbf{u})$ and $\mathbf{y}^{**T}(\mathbf{u})$ be the two arbitrary different solutions of the "main dependency relation" (2.3), (2.4) for the system (1.1). It is assumed that there exists a differentiable function $\alpha(u^K)$, $\alpha(u^K) \not\equiv 0$, such that $\mathbf{y}^{**T}(u^K) = \alpha(u^K)\mathbf{y}^{*T}(u^K)$. Let $\mathcal{M}^*(u^K)$ and $\mathcal{M}^{**}(u^K)$ be the $(N+1) \times (N+1)$ matrices (3.9) corresponding to $\mathbf{y}^{*T}(u^K)$ and $\mathbf{y}^{**T}(u^K)$, respectively. Those matrices can be written in the following form:

$$\text{(A.1)} \qquad \mathcal{M}^*(u^K) := \begin{bmatrix} \mathbf{y}^{*T}(u^K) \\ (\nabla_{\mathbf{u}}\mathbf{y}^*(u^K))^T \end{bmatrix},$$

$$\text{(A.2)} \qquad \mathcal{M}^{**}(u^K) := \begin{bmatrix} \mathbf{y}^{**T}(u^K) \\ (\nabla_{\mathbf{u}}\mathbf{y}^{**}(u^K))^T \end{bmatrix} = \begin{bmatrix} \alpha(u^K)\mathbf{y}^{*T}(u^K) \\ \alpha(u^K)(\nabla_{\mathbf{u}}\mathbf{y}^*(u^K))^T + [\nabla_{\mathbf{u}}\alpha(u^K)] \otimes \mathbf{y}^{*T}(u^K) \end{bmatrix},$$

where $\nabla_{\mathbf{u}}$ denotes differentiation with respect to u^K. Let $\mathbf{z}^*(u^K) \in \operatorname{Ker} \mathcal{M}^*(u^K)$. It follws from (A.1) that

$$\text{(A.3)} \qquad \mathbf{y}^{*T}(u^K)\mathbf{z}^*(u^K) \equiv 0, \qquad [\nabla_{\mathbf{u}}\mathbf{y}^*(u^K)]^T\mathbf{z}^*(u^K) \equiv \mathbf{0}$$

and (A.3), (A.2) imply $\mathcal{M}^{**}(u^K)\mathbf{z}^*(u^K) \equiv \mathbf{0}$. Therefore, $\mathbf{z}^*(u^K) \in \operatorname{Ker} \mathcal{M}^{**}(u^K)$. In the same way it can be proved that if $\mathbf{z}^{**}(u^K) \in \operatorname{Ker} \mathcal{M}^{**}(u^K)$ then $\mathbf{z}^{**}(u^K) \in \operatorname{Ker} \mathcal{M}^*(u^K)$. Hence $\operatorname{Ker} \mathcal{M}^*(u^K) = \operatorname{Ker} \mathcal{M}^{**}(u^K)$.

LEMMA 2. The necessary condition for the $(N+1) \times (N+1)$ matrix (3.9) to be nonsingular for all u^K is that rank $\mathcal{K}(u^K) = N$ for all u^K, where

$$\mathcal{K}(u^K) = [\mathcal{K}_{M\Lambda}(u^K)] = \left[\frac{\partial y_\Lambda(u^K)}{\partial u^M}\right] = [\nabla_{\mathbf{u}}\mathbf{y}(u^K)]^T$$

is a $N \times (N+1)$ matrix.

P r o o f. According to (3.9),

$$\text{(A.4)} \qquad \det \mathcal{M}(u^K) = \det \begin{bmatrix} \mathbf{y}^T(u^K) \\ \mathcal{K}(u^K) \end{bmatrix} = \det \begin{bmatrix} y_\Lambda(u^K) \\ \mathcal{K}_{M\Lambda}(u^K) \end{bmatrix} = \sum_{\Gamma=1}^{N+1} (-1)^{\Gamma+1} y_\Gamma(u^K) \det \overset{(\Gamma)}{\mathbf{H}}(u^K),$$

where $\overset{(\Gamma)}{\mathbf{H}}(u^K)$ are $N \times N$ minors of $\mathcal{K}(u^K)$. Suppose that rank $\mathcal{K}(u^K) < N$. Then, all $\overset{(\Gamma)}{\mathbf{H}}(u^K)$ are singular and therefore (3.9) is singular according to (A.4).

LEMMA 3. If among $N+1$ components $y_\Lambda(u^K)$ of the solution $\mathbf{y}^T(u^K)$ of the MDR there are N independent functions of u^K, that is if rank $\mathcal{K}(u^K) = N$ for all u^K, then the matrix (3.9) is singular for all u^K if and only if the remaining (dependent) component of $\mathbf{y}^T(u^K)$ is a homogeneous function of degree one of the independent components of $\mathbf{y}^T(u^K)$ (if $y_1(u^K), y_2(u^K), \dots, y_{\Sigma-1}(u^K), y_{\Sigma+1}(u^K), \dots, y_{N+1}(u^K)$ are independent components of $\mathbf{y}^T(u^K)$, then the remaining component $y_\Sigma(u^K)$ expressed as $y_\Sigma = \bar{y}_\Sigma(y_1, \dots, y_{\Sigma-1}, y_{\sigma+1}, \dots, y_{N+1})$ is a homogeneous function of degree one with respect to all arguments).

P r o o f. Let $y_1(u^K), \dots, y_{\Sigma-1}(u^K), y_{\Sigma+1}(u^K) \dots, y_{N+1}(u^K)$ be N independent functions of u^K. Then the $N \times N$ nonsingular minor $\overset{(\Sigma)}{\mathbf{H}}(u^K)$ of $\mathcal{K}(u^K)$ corresponding to deletion of the column Σ is a Jacobian of the invertible mapping $(u^1, \dots, u^2) \to (y_1, \dots, y_{\Sigma-1}, y_{\Sigma+1} \dots, y_{N+1})$. Employing the inverse $u^K = \bar{u}^K(y_1, \dots, y_{\Sigma-1}, y_{\Sigma+1}, \dots, y_{N+1})$, $K = 1, 2, \dots, N$, we may express y_Σ in a form of the following composed function:

$$
\text{(A.5)} \qquad \begin{aligned} y_\Sigma = y_\Sigma(\bar{u}^K(y_1, \dots, y_{\Sigma-1}, y_{\Sigma+1}, \dots, y_{N+1})) \\ = \widetilde{y}_\Sigma(y_1, \dots, y_{\Sigma-1}, y_{\Sigma+1} \dots, y_{N+1}) \\ = \widetilde{y}_\Sigma(y_1(u^K), \dots, y_{\Sigma-1}(u^K), y_{\Sigma+1}(u^K) \dots, y_{N+1}(u^K)). \end{aligned}
$$

According to the chain rule,

$$
\text{(A.6)} \qquad \frac{\partial \widetilde{y}_\Sigma}{\partial u^K} = \sum_{\substack{\Delta=1, \\ \Delta \neq \Sigma, \\ \Lambda \neq \Sigma}}^{N+1} \frac{\partial \widetilde{y}_\Sigma(y_\Lambda(u^K))}{\partial y_\Delta} \frac{\partial y_\Delta(u^K)}{\partial u^K}.
$$

Assume that the matrix (3.9), denoted as $\mathcal{M}(u^K)$, is singular for all u^K. Then, there exists a nonzero solution $\mathbf{z}(u^K) = [z^0(u^K), z^1(u^K), \dots, z^N(u^K)]$ of the equation

$$
\text{(A.7)} \qquad \mathcal{M}(u^K)\mathbf{z}(u^K) = \mathbf{0} \qquad \text{for all} \quad u^K,
$$

which corresponds to the following system of $N+1$ linear homogeneous equations:

$$
\text{(A.8)} \qquad \begin{gathered} z^0(u^K) y_\Lambda(u^K) + \sum_{R=1}^{N} z^R(u^K) \frac{\partial y_\Lambda(u^K)}{\partial u^R} = 0, \\ \Lambda = 1, 2, \dots, \Sigma-1, \Sigma+1, \dots, N+1, \end{gathered}
$$

$$
\text{(A.9)} \qquad z^0(u^K)\widetilde{y}_\Sigma(y_\Lambda(u^K)) + \sum_{R=1}^{N} z^R(u^K) \frac{\partial \widetilde{y}(y_\Delta(u^K))}{\partial u^R} = 0, \qquad \Delta \neq \Sigma.
$$

Substituting (A.6) into (A.9) and taking into account (A.8), we obtain

$$\text{(A.10)}\qquad z^0\widetilde{y}_\Sigma + \sum_{R=1}^{N} z^R\left[\sum_{\substack{\Delta=1,\\ \Delta\neq\Sigma}}^{N+1} \frac{\partial\widetilde{y}_\Sigma}{\partial y_\Delta}\frac{\partial y_\Delta}{\partial u^R}\right] = z^0\widetilde{y}_\Sigma + \sum_{\substack{\Delta=1,\\ \Delta\neq\Sigma}}^{N+1}\left[\sum_{R=1}^{N} z^R\frac{\partial y_\Delta}{\partial u^R}\right]\frac{\partial\widetilde{y}_\Sigma}{\partial y_\Delta}$$
$$= z_0\left(\widetilde{y}_\Sigma - \sum_{\substack{\Delta=1,\\ \Delta\neq\Sigma}}^{N+1} y_\Delta\frac{\partial\widetilde{y}_\Sigma}{\partial y_\Delta}\right) = 0.$$

Suppose that $z^0(u^K)\equiv 0$. Since $\mathbf{z}(u^K)$ is a nonzero vector, among the components $z^1(u^K),\dots,z^N(u^K)$ at least one cannot identically vanish. It then follows from (A.8) that, in this case, the $N\times N$ minor $\overset{(\Sigma)}{\mathbf{H}}(u^K)$ must be singular. Therefore, the condition $z^0(u^K)\equiv 0$ contradicts the assumptions and (A.10) implies that $\widetilde{y}_\Sigma(y_1,\dots,y_{\Sigma-1},y_{\Sigma+1},\dots,y_{N+1})$ is a homogeneous function of degree one with respect to all arguments. Assume that $\widetilde{y}_\Sigma(y_1,\dots,y_{\Sigma-1},y_{\Sigma+1},\dots,y_{N+1})$ is a homogeneous function of degree one with respect to all arguments. Then

$$\text{(A.11)}\qquad \widetilde{y}_\Sigma = \frac{\partial\widetilde{y}_\Sigma}{\partial y_1}y_1 + \ldots + \frac{\partial\widetilde{y}_\Sigma}{\partial y_{\Sigma-1}}y_{\Sigma-1} + \frac{\partial\widetilde{y}_\Sigma}{\partial y_{\Sigma+1}}y_{\Sigma+1} + \ldots + \frac{\partial\widetilde{y}_\Sigma}{\partial y_{N+1}}y_{N+1}$$

and, according to (A.6),

$$\text{(A.12)}\qquad \frac{\partial\widetilde{y}_\Sigma}{\partial u^K} = \frac{\partial\widetilde{y}_\Sigma}{\partial y_1}\frac{\partial y_1}{\partial u^K} + \ldots + \frac{\partial\widetilde{y}_\Sigma}{\partial y_{\Sigma-1}}\frac{\partial y_{\Sigma-1}}{\partial u^K}$$
$$+ \frac{\partial\widetilde{y}_\Sigma}{\partial y_{\Sigma+1}}\frac{\partial y_{\Sigma+1}}{\partial u^K} + \ldots + \frac{\partial\widetilde{y}_\Sigma}{\partial y_{N+1}}\frac{\partial y_{N+1}}{\partial u^K}, \qquad K=1,2,\dots,N.$$

It follows from (A.11), (A.12) that the column Σ of the matrix (4.14) is a linear combination of the remaining N columns, so the matrix is singular.

Appendix B

For the proof we employ the reasoning analogous to that of GODUNOV and SULTANGAZIN [11, 12]. Convexity of $\widehat{\varphi}^0_C(\widehat{\mu},\widehat{z}^\gamma)$ means that the inequality

$$\text{(B.1)}\qquad \alpha_1\widehat{\varphi}^0_C(\widehat{\mu}^*,\widehat{z}^{*\gamma}) + \alpha_2\widehat{\varphi}^0_C(\widehat{\mu}^{**},\widehat{z}^{**\gamma}) > \widehat{\varphi}^0_C(\alpha_1\widehat{\mu}^* + \alpha_2\widehat{\mu}^{**}, \alpha_1\widehat{z}^{*\gamma} + \alpha_2\widehat{z}^{**\gamma}),$$

holds for all $\alpha_2\geq 0$, $\alpha_2\geq 0$ such that $\alpha_1+\alpha_2=1$, and for all $\widehat{\mu}^*$, $\widehat{\mu}^{**}$, $\widehat{z}^{*\gamma}$, $\widehat{z}^{**\gamma}$ from a convex domain. Inequality (B.1) can be rearranged to the equivalent form

$$\text{(B.2)}\qquad \frac{\alpha_1\widehat{\mu}^*}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}\frac{1}{\widehat{\mu}^*}\widehat{\varphi}^0_C(\widehat{\mu}^*,\widehat{z}^{*\gamma}) + \frac{\alpha_2\widehat{\mu}^{**}}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}\frac{1}{\widehat{\mu}^{**}}\widehat{\varphi}^0_C(\widehat{\mu}^{**},\widehat{z}^{**\gamma})$$
$$> \frac{1}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}\widehat{\varphi}^0_C(\alpha_1\widehat{\mu}^* + \alpha_2\widehat{\mu}^{**}, \alpha_1\widehat{z}^{*\gamma} + \alpha_2\widehat{z}^{**\gamma}).$$

The right-hand side of (B.2) can be expressed as

$$\text{(B.3)}\qquad \left[\frac{1}{\widehat{\mu}^*}\frac{\alpha_1\widehat{\mu}^*}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}+\frac{1}{\widehat{\mu}^{**}}\frac{\alpha_2\widehat{\mu}^{**}}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}\right] \cdot\widehat{\varphi}^0_C\left(\frac{1}{\dfrac{1}{\widehat{\mu}^*}\dfrac{\alpha_1\widehat{\mu}^*}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}+\dfrac{1}{\widehat{\mu}^{**}}\dfrac{\alpha_2\widehat{\mu}^{**}}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}},\ \frac{\dfrac{\widehat{z}^{*\gamma}}{\widehat{\mu}^*}\dfrac{\alpha_2\widehat{\mu}^*}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}+\dfrac{\widehat{z}^{**\gamma}}{\widehat{\mu}^{**}}\dfrac{\alpha_2\widehat{\mu}^{**}}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}}{\dfrac{1}{\widehat{\mu}^*}\dfrac{\alpha_1\widehat{\mu}^*}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}+\dfrac{1}{\widehat{\mu}^{**}}\dfrac{\alpha_2\widehat{\mu}^{**}}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}}}\right).$$

Taking into account (B.3), (4.29), (4.30), we obtain from (B.2) the following inequality

$$\text{(B.4)}\qquad \beta_1\check{\lambda}^*\widehat{\varphi}^0_C\left(\frac{1}{\check{\lambda}^*},-\frac{\check{z}^{*\gamma}}{\check{\lambda}^*}\right)+\beta_2\check{\lambda}^{**}\widehat{\varphi}^0_C\left(\frac{1}{\check{\lambda}^{**}},-\frac{\check{z}^{**\gamma}}{\check{\lambda}^{**}}\right) > (\beta_1\check{\lambda}^*+\check{\beta}_2\check{\lambda}^{**})\widehat{\varphi}^0_C\left(\frac{1}{\beta_1\check{\lambda}^*+\beta_2\check{\lambda}^{**}},\ -\frac{\beta_1\check{z}^{*\gamma}+\beta_2\check{z}^{**\gamma}}{\beta_1\check{\lambda}^*+\beta_2\check{\lambda}^{**}}\right),$$

where

$$\text{(B.5)}\qquad \begin{gathered}\beta_1=\frac{\alpha_1\widehat{\mu}^*}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}},\qquad \beta_2=\frac{\alpha_2\widehat{\mu}^{**}}{\alpha_1\widehat{\mu}^*+\alpha_2\widehat{\mu}^{**}},\\ \check{\lambda}^*=\frac{1}{\widehat{\mu}^*},\quad \check{\lambda}^{**}=\frac{1}{\widehat{\mu}^{**}},\quad \check{z}^{*\gamma}=-\frac{\widehat{z}^{*\gamma}}{\widehat{\mu}^*},\quad \check{z}^{**\gamma}=-\frac{\widehat{z}^{**\gamma}}{\widehat{\mu}^{**}}\end{gathered}$$

and $\beta_1\geq 0$, $\beta_2\geq 0$, $\beta_1+\beta_2=1$ since $\widehat{\mu}^*=\theta^*>0$, $\widehat{\mu}^{**}=\theta^{**}>0$. From (5.4), (B.5) and (4.29), (4.30), we finally obtain the inequality

$$\text{(B.6)}\qquad \beta_1\check{\varphi}^0_C(\check{\lambda}^*,\check{z}^{*\gamma})+\check{\beta}_2\check{\varphi}^0_C(\check{\lambda}^{**},\check{z}^{**\gamma})<\check{\varphi}^0_C(\beta_1\check{\lambda}^*+\beta_2\check{\lambda}^{**},\beta_1\check{z}^{*\gamma}+\beta_2\check{z}^{**\gamma}),$$

which implies that $\check{\varphi}^0_C$ is concave since β_1 and β_2 take all admissible values when α_1 and α_2 take all admissible values from the region $\alpha_1\geq 0$, $\alpha_2\geq 0$, $\alpha_1+\alpha_2=1$. In the same way, it can be proved that concavity of $\check{\varphi}^0_C(\check{\lambda},\check{z}^\gamma)$ implies convexity of $\widehat{\varphi}^0_C(\widehat{\mu},\widehat{z}^\gamma)$ as well as the opposite situation when $\widehat{\varphi}^0_C(\widehat{\mu},\widehat{z}^\gamma)$ is concave and $\check{\varphi}^0_C(\check{\lambda},\check{z}^\gamma)$ is convex.

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