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FIFTY YEARS OF THE ARCHIVES OF MECHANICS

Symmetric forms of the equations of heat transport in a rigid conductor of heat with internal state variables I. Analysis of the model and thermodynamic restrictions via the "main dependency relation"

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THE OBJECTIVE of this series of two papers is twofold: to analyse the phenomenological model of a rigid conductor of heat with (vector) internal state variable, and to promote the application of the "main dependency relation" (MDR) as a tool for derivation of the restrictions on constitutive functions implied by the entropy inequality as well as a tool for direct derivation of alternative symmetric systems of field equations. In this paper (Part I), the analysis of the model of a rigid conductor of heat with (vector) internal state variable is focused on two aspects, namely, on the form of the respective field equations and on the relation to other phenomenological models proposed in the literature, with the emphasis put on those models which have been succesfully adjusted to experimental data on heat transport at finite speeds. The relation to the model of a rigid conductor of heat with scalar internal state variable, called "semi-empirical temperature" is demonstrated. It is proved that, for the system of N conservation equations, consistency with the entropy inequality (in the form of first-order unilateral differential constrains) is equivalent to the requirement that the corresponding system of $N+1$ conservation equations satisfies the "main depency relation" (MDR). For the model of a rigid conductor of heat with conservative evolution equation for internal state variable, the procedure of derivation of thermodynamic restrictions via the MDR is demonstrated.

1. Introduction

THE OBJECTIVE of this series of two papers is twofold: to analyse the phenomenological model of a rigid conductor of heat with (vector) internal state variable from the point of view of relations (consistency and/or possible generalization) to some other proposed (and applied to fit the experimental data, like Maxwell – Cattaneo – Vernotte equation [1, 2, 3], or the model proposed by MORRO and RUGGERI [4, 5, 6]) phenomenological models of hyperbolic heat transport as well as from the point of view of symmetrizability (and consequently, symmmetric hyperbolicity) of the resulting field equations; and to promote the application the "main dependency relation" (MDR) introduced by FRIEDRICHS [7] which, in fact, is a natural generalization of the "entropy principle" of Extended Thermodynamics [8, 9, 10] as a tool for derivation of thermodynamic restrictions on constitutive functions. Therefore, Part I in this series is focused on the analysis of the phenomenological model of a rigid conductor of heat with (vector) internal state variable, on the proof of the equivalence of the "main dependency relation" (MDR) and the entropy inequality, and on application of the MDR for derivation of the restrictions on constitutive functions for the specific model of a rigid conductor of heat with (vector) internal state variable that admits conservative form of field equations.

Various types of heat conduction equations leading to finite speed of propagation of thermal waves were postulated for rigid and deformable heat conductors in the last four decades. Different formulations of continuum thermodynamics (for example, Rational Thermodynamics, Extended Thermodynamics, Extended Irreversible Thermodynamics (EIT)) proved useful in deriving the heat transport equations leading to finite wave speeds. The literature on the subject is too extensive to be quoted here and therefore we refer the interested readers to the review papers by JOU and CASSAZ–VASQUEZ [11, 12] and to the general overeview by JOSEPH and PREZIOSI [13, 14].

Some phenolmenological modeles of heat transport with finite speeds are motivated by or even directly related ([15, 16, 5, 6]) to the second sound in solids. The second sound was detected in crystalline He^4, NaF and Bi in heat pulse experiments at very low temperatures but very few quantitative data concerning second sound measurements were published in early 70-ties (extensive list of references can be found in [17]). Besides the results of the second sound measurements, there is other published interesting experimental evidence (for example, [18, 19, 20]) of the wave nature of heat propagation (or dominance of wave behaviour over diffussion) at moderate temperature ranges in materials of practical applications in technology and engineering. KAMIŃSKI [18] determined the constant τ of the Maxwell–Cattoneo–Vernotte equation for various materials with "nonhomogeneous inner structure" ("complex systems made up of solid, liquid and gas, e.g., porous-capillary bodies, cellular systems, suspensions, etc.") with the aid of the original experimental method. MITRA, KUMAR, VEDAVARZ and MOALLEMI [19] presented an experimental evidence of the wave-type heat transport in processed meat and showed that Maxwell–Cattaneo–Vernotte equation provides an accurate description, on a macroscopic level, of the heat conduction process in such biological materials. In [18, 19], the investigated materials had refined complex inner structure and, as a consequence, the observed heat transport was due to cummulative effect of different transfer mechanisms, for instance, particle-to-particle contact, free convection in closed space, radiation, etc. Even in the absence of such inner structure, the effect of finite speed of heat transport has been confirmed experimentally, namely, TZOU [20] concluded the wave nature of heat conduction from the comparison of the wave solution of the Maxwell–Cattaneo–Vernotte equation for the temperature rise induced by a propagating crack tip with the experimental results obtained by ZEHNDER and ROSAKIS [21] for 4340 steel. In [18, 19, 20], the wave features of heat transport were concluded by means of adjusting the experimental data to the simplest phenomenological model of hyperbolic heat conduction, i.e.

Maxwell – Cattaneo – Vernotte equation in which the single numerical constant τ is the only material parameter responsible for wave features. It can be supposed that this simplest model will be insufficient to describe properly the material thermal response in wide range of thermal and mechanical conditions, especially in the case of materials with inner structure where different transport mechanisms are simultaneously involved with relative intensities dependent on thermal and mechanical conditions. The phenomenological model of hyperbolic heat conduction that can be proposed for practical use should involve relatively simple but sufficiently general constitutive relations, should enable one to analyse waves of weak and of strong discontinouity and should lead to the system of field equations for which the Cauchy problem is well-posed and numerical methods are easily applicable. One of the purely phenomenological models of heat transport with finite speed that seems to meet those requirements is the model of a rigid conductor of heat with (vector) internal state variable.

Basic constitutive assumptions for modelling heat transport at finite speed by means of internal state variable, formulated by KOSIŃSKI [22], are recalled and the general form of the model of a rigid conductor of heat with (vector) internal state variable is introduced in Sec. 2.2.1. Then, two types of the additional constitutive assumptions are analysed subsequently in Secs. 2.2.2 and 2.2.3, namely, assumptions concerning the form of the evolution equation for the vector internal state variable and the assumptions concerning dependence of the heat flux vector and the internal energy on the vector internal state variable and on the temperature.

In Secs. 2.2.4, 2.2.5, 2.2.6, further constitutive assumptions corresponding to specific forms of free energy and specific forms of the source term in the evolution equation for internal state variable are investigated in the case of the model of a rigid conductor of heat with conservative form of the evolution equation for internal state variable. It is shown that this model can be interpreted as a representation of the class of phenomenological models comprising both the Maxwell – Cattaneo – Vernotte equation and the models proposed in [4, 5, 6, 23]. Within this class, various particular generalizations or corrections to those models can be easily introduced.

Recently, a phenomenological model of rigid conductor of heat with scalar internal state variable called "semi-empirical temperature" has been proposed [24 – 29]. This model is discussed in Sec. 2.3, and it is shown that the same system of field equations as suggested in Sec. 2.2.6 for the model of a rigid conductor of heat with (vector) internal state variable also describes the model with "semi-empirical temperature", if supplemented by additional involutive constraints.

In order to promote the "main dependency relation" [7] as a tool for derivation of thermodynamic restrictions, its equivalence to the entropy inequality is established. Namely, we prove in Sec. 3.2 that if the system of N conservation equations has nonsingular $N \times N$ matrix multiplying time derivatives of the unknows, then every Lipschitz continuous solution of this system satisfies the entropy inequality (in the form of first-order unilateral differential constraints) if and only if the MDR is satisfied by the respective system of $N+1$ conservation equations (system of N conservation equations supplemented by the conservation equation corresponding to the entropy inequality). For the proof, the results established by Kosiński [30, 31] for systems of conservation equations in normal (Cauchy) form are employed. From the point of view of thermodynamics, the MDR can be regarded as a generalization of the "entropy principle" of extended thermodynamics of MÜLLER and LIU [8, 9, 10] in the sense that it assigns an analogue of Lagrange–Liu multiplier also to the balance of entropy. It reduces to the "entropy principle" for the value of that additional multiplier equal to -1. The advantage of employing the MDR instead the "entropy principle" is that it enables one to derive equivalent (for classical solutions) alternative symmetric systems directly (see, Part II).

For the model of a rigid conductor of heat with conservative evolution equation for internal state variable, the procedure of derivation of thermodynamic restrictions on constitutive functions via the MDR is demonstrated in Sec. 3.3. According to this procedure, a family of solutions of the MDR is determined in Sec. 3.4 and the restrictions on constitutive functions are derived in Sec. 3.5.

2. Rigid conductor of heat with internal state vriables

2.1. Rigid heat conductor

By a rigid heat conductor we mean the undeformable (rigid) continuous material body which can conduct the heat. It is assumed that for an inertial observer, the body remains at rest and the materal points (particles) are identified with points of the tree-dimensional Euclidean point space E^3. For a rigid heat conductor, all balance laws of continuum mechanics are satisfied trivially except the energy balance, which assumes the following local form:

$$\varrho_0\dot{\varepsilon} + \operatorname{div}\mathbf{q} = \varrho_0 r\,, \tag{2.1}$$

where ϱ_0 is a constant mass density, ε denotes energy density referred to the unit mass, $\mathbf{q} = [q^\alpha]$, $\alpha = 1, 2, 3$ is a heat flux and r is a heat source.

In the order to formulate the model of a rigid conductor of heat in the framework of Rational Mechanics, the Clausius–Duhem entropy inequality

$$\varrho_0\sigma := \varrho_0\dot{\eta} + \operatorname{div}\left(\frac{1}{\theta}\mathbf{q}\right) \geq \varrho_0\frac{1}{\theta}r\,, \tag{2.2}$$

should be taken into account, where η is the entropy density referred to the unit mass, θ denotes the temperature and σ is the entropy production.

In this paper, the usual summation convention over repeated lower and upper indices is employed and the notation $\partial_t := \frac{\partial}{\partial t}$, $\partial_\alpha := \frac{\partial}{\partial X^\alpha}$ is used for time and spatial derivatives, respectively. Spatial coordinates X^α correspond here to the fixed Cartesian frame, for simplicity. Dot over a letter denotes material time derivative which, in our case (rigid, unmoving body), corresponds to the partial time derivative ∂_t.

2.2. Model of a rigid conductor of heat with (vector) internal variable

2.2.1. General form of the model. The concept of constitutive modelling (in the framework of Rational Mechanics) with the aid of internal state variables is due to COLEMAN and CURTIN [32]. Constitutive modelling of heat propagation at finite speeds (thermal waves of weak and strong discontinuity) by means of internal state variables was proposed by KOSIŃSKI [22] ([1]) for the general case of deformable (inelastic) body. It can be easily reduced to the model of a rigid conductor of heat, simply by neglecting the dependence on deformation.

The following constitutive assumptions were intrduced in [22]:

• response of the material in particle X^α at time t depends on the values of the temperature $\theta(t, X^\alpha)$ and the internal state variables $\mathbf{w}(t, X^\alpha) = [w_\sigma(t, X^\alpha)]$;

• the evolution of the internal state variables $\mathbf{w}(t, X^\alpha) = [w_\sigma(t, X^\alpha)]$ during the thermodynamic processes is governed by a vector differential equation of the first order, dependent on the temperature gradient as the additional variable.

Taking into account those assumptions and conforming to the equipresence principle of Rational Mechanics, we postulate, according to [22], the constitutive equations

$$\varepsilon = \tilde{\varepsilon}(\theta, \mathbf{w}), \quad \mathbf{q} = \tilde{q}(\theta, \mathbf{w}), \quad r = \tilde{r}(\theta, \mathbf{w}, t, X^\alpha), \quad \eta = \tilde{\eta}(\theta, \mathbf{w}) \tag{2.3}$$

and the following evolution equation for internal state variable $\mathbf{w}$

$$\dot{\mathbf{w}}(t, X^\alpha) = \mathbf{g}(\theta(t, X^\alpha), \operatorname{grad}\theta(t, X^\alpha), \mathbf{w}(t, X^\alpha)) \tag{2.4}$$

with the initial-value problem

$$\mathbf{w}(t_0, X^\alpha) = \mathbf{w}_0(X^\alpha). \tag{2.5}$$

It is assumed that the initial-value problem has a unique solution and this implies that the function $\mathbf{g}$ is Lipshitz continuous with respect to $\mathbf{w}$ and continuous with respect to the other remaining arguments.

([1]) In this paper, we restrict considerations to the models of a rigid conductor of heat developed in the framework of Rational Mechanics, and therefore we do not discuss those phenomenological models employing internal state variables which violate the axioms of Rational Mechanics, like, for example, the model proposed by BAMPI, MORRO and JOU [33], where the entropy flux vector is assumed to be different from $(1/\theta)\mathbf{q}$.

Substitution of the constitutive functions $(2.3)_{1,2,3}$ into the balance of energy (2.1) together with the evolution equation (2.4), results in the nonlinear (for general $\mathbf{g}(\theta, \operatorname{grad}\theta, \mathbf{w})$) system of the four first-order partial differential equations for the unknowns θ, w_γ

$$(2.6)\qquad \begin{aligned} &\varrho_0\partial_t\tilde{\varepsilon}(\theta, w_\gamma) + \partial_\alpha\tilde{q}^\alpha(\theta, w_\gamma) = \varrho_0\tilde{r}(\theta, w_\gamma, t, X^\alpha), \\ &\partial_t w_\beta = g_\beta(\theta, \partial_\alpha\theta, w_\gamma), \qquad \alpha, \beta, r = 1, 2, 3 \end{aligned}$$

subject to unilateral first-order differential constraints

$$(2.7)\qquad \varrho_0\sigma := \varrho_0\partial_t\tilde{\eta}(\theta, w_\gamma) + \partial_\alpha\left[\frac{1}{\theta}\tilde{q}^\alpha(\theta, w_\gamma)\right] \geq \varrho_0\frac{1}{\theta}\tilde{r}(\theta, \mathbf{w}, t, X^\alpha),$$

obtained by substituting the constitutive functions $(2.3)_{2,4}$ into the entropy inequality (2.2).

With the aid of the free energy function $\tilde{\Psi}_C(\theta, w_\gamma)$

$$(2.8)\qquad \tilde{\Psi}_C(\theta, \mathbf{w}) = \tilde{\varepsilon}(\theta, \mathbf{w}) - \theta\tilde{\eta}(\theta, \mathbf{w}),$$

it has been derived in [22] that the entropy inequality (2.7) imposed on the system (2.6) implies

$$(2.9)\qquad \eta(\theta, w_\gamma) = -\frac{\partial\tilde{\Psi}_C(\theta, w)}{\partial\theta},$$

and the following inequality holds

$$(2.10)\qquad -\frac{\partial\tilde{\Psi}(\theta, w)}{\partial w_\alpha}\partial_t w_\alpha - \frac{1}{\varrho_0}\tilde{q}^\alpha(\theta, w_\gamma)\partial_\alpha\theta \geq 0.$$

2.2.2. Assumption leading to the quasi-linear system of field equations. The additional general assumption that the function $\mathbf{g}(\theta, \operatorname{grad}\theta, \mathbf{w})$ is linear in $\operatorname{grad}\theta$, namely

$$(2.11)\qquad \mathbf{g}(\theta, \operatorname{grad}\theta, \mathbf{w}) = \mathbf{M}(\theta, \mathbf{w})\operatorname{grad}\theta + \mathbf{b}(\theta, \mathbf{w}),$$

where $\mathbf{M}(\theta, \mathbf{w})$ is a tensor function of θ and $\mathbf{w}$, makes the system (2.6) quasi-linear but not in a conservative form.

Assuming that, besides (2.9), (2.10), no other restrictions are imposed, it is apparently possible to find specific restrictions on the components ${M_\beta}^\alpha(\theta, w_\gamma)$ of the matrix representation of $\mathbf{M}(\theta, \mathbf{w})$ and on the functions $\tilde{\varepsilon}(\theta, w_\gamma)$, $\tilde{q}^\alpha(\theta, w_\gamma)$ such that the system (2.6), (2.11) can be transformed into equivalent quasi-linear symmetric hyperbolic system by premultiplication (left multiplication) by a nonsingular 4×4 matrix $S^{IJ}(\theta, w_\gamma)$ $I, J = 1, 2, 3, 4$. It can be done simply by rewriting the system (2.6), (2.11) in a matrix form and requiring the resulting matrices to have common nonsingular left symmetrizer $S^{IJ}(\theta, w_\gamma)$ such that symmetrized matrix multiplying $[\partial_t\theta, \partial_t w_\gamma]^T$ is positive definite. In this context, the question arises wheather the restrictions on $M^{\alpha\beta}(\theta, w_\gamma)$, $\widetilde{\varepsilon}(\theta, w_\gamma)$ and $\widetilde{q}^\alpha(\theta, w_\gamma)$ imposed by this procedure are related to the restrictions imposed by the inequality (2.7). LEFLOCH [34, 35] introduced the concept of the additional conservation equation ("balance of entropy") for quasi-linear systems in non-conservative form in one spatial dimension and related the consistency of the non-conservative quasi-linear systems with the additional conservation equation to the existence of the left symmetrizer of this quasi-linear system in the form of matrix representation of transposed gradient of a vector function of the unknowns (gradient with respect to these unknowns). Therefore, for the system (2.6), (2.11) with assumed dependence of the unknowns θ, w_γ on only one spatial coordinate, the answer can be expected to be affirmative provided that it will be proved, for a non-conservative first-order quasi-linear system (at least in the one-dimensional case), that the restrictions imposed by consistency with the additional conservation equation ("balance of entropy") are equivalent to the restrictions imposed by the corresponding first-order differential unilateral constraints ("entropy inequality", like (2.7)). Hence, the problem is still open even in the one-dimensional case.

2.2.3. Conservative form of field equations. The system of field equations (2.6), (2.11) takes a conservative form if the next additional assumption that $\mathbf{M}(\theta, \mathbf{w})$ is an isotropic tensor function of the temperature is introduced since, without loss of generality, we may write

$$\mathbf{M}(\theta, \mathbf{w}) = -f'(\theta)\mathbf{I}, \qquad f'(\theta) = \frac{df(\theta)}{d\theta}, \qquad f'(\theta) \neq 0 \tag{2.12}$$

and, consequently, rearrange the evolution equation for $\mathbf{w}$ to the form

$$\dot{\mathbf{w}} = -\mathrm{div}(f(\theta)\mathbf{I}) + \mathbf{b}(\theta, \mathbf{w}). \tag{2.13}$$

The evolution equation for internal state variable with the source linear in $\mathbf{w}$

$$\mathbf{b}(\theta, \mathbf{w}) = -\mathbf{N}(\theta)\mathbf{w}, \tag{2.14}$$

and $\mathbf{N}(\theta)$ positive definite, was assumed for the model of a rigid conductor of heat proposed by MORRO and RUGGERI [4] and, therefore, it can be considered as a special case of the model developed previously in [22]. It has been shown in [4] that, in the case of evolution equation of the type (2.13), the entropy inequality (2.7) implies

$$\widetilde{q}^\alpha(\theta, w_\gamma) = \varrho_0 \theta f'(\theta) \frac{\partial \widetilde{\Psi}_C(\theta, w_\gamma)}{\partial w_\alpha}, \tag{2.15}$$

and

$$-\varrho_0 \frac{\partial \widetilde{\Psi}_C(\theta, w_\gamma)}{\partial w_\alpha} \widetilde{b}^\alpha(\theta, w_\gamma) \geq 0. \tag{2.16}$$

To this end, we note that (2.8), (2.9) imply

$$\widetilde{\varepsilon}(\theta, w_\gamma) = \widetilde{\Psi}_C(\theta, w_\gamma) - \theta\frac{\partial\widetilde{\Psi}_C(\theta, w_\gamma)}{\partial\theta} \tag{2.17}$$

and therefore, the considered model of a rigid conductor of heat is entirely determined by the free energy $\widetilde{\Psi}(\theta, \mathbf{w})$, scalar function of the temperature $f(\theta)$ and the source terms $\widetilde{r}(\theta, \mathbf{w}, X^\alpha, t)$, $\mathbf{b}(\theta, \mathbf{w})$.

2.2.4. Further constitutive assumptions

A. $\widetilde{\mathbf{q}}(\theta, \mathbf{w})$ *linear in* $\mathbf{w}$

According to (2.15), the additional constitutive assumption that $\widetilde{\mathbf{q}}(\theta, \mathbf{w})$ is a linear function of $\mathbf{w}$

$$\widetilde{\mathbf{q}}(\theta, \mathbf{w}) = \mathbf{S}(\theta)\mathbf{w}, \tag{2.18}$$

with $\mathbf{S}(\theta)$ symmetric nonsingular tensor function of $\theta > 0$, is equivalent to the postulate that free energy takes the following special form:

$$\widetilde{\Psi}_C(\theta, \mathbf{w}) = \widetilde{\Psi}_{C0}(\theta) + \frac{1}{2\theta f'(\theta)\varrho_0}\mathbf{w}\cdot\mathbf{S}(\theta)\mathbf{w}. \tag{2.19}$$

As a consequence of this assumption, we obtain from (2.8), (2.9)

$$\begin{aligned}
\widetilde{\eta}(\theta, \mathbf{w}) &= \widetilde{\eta}_0(\theta) - \frac{1}{2\varrho_0}\mathbf{w}\cdot\left[\frac{d}{d\theta}\left(\frac{1}{\theta f'(\theta)}\mathbf{S}(\theta)\right)\right]\mathbf{w},\\
\widetilde{\varepsilon}(\theta, \mathbf{w}) &= \widetilde{\varepsilon}_0(\theta) + \frac{1}{2\varrho_0}\mathbf{w}\cdot\left[\frac{1}{\theta f'(\theta)}\mathbf{S}(\theta) - \theta\frac{d}{d\theta}\left(\frac{1}{\theta f'(\theta)}\mathbf{S}(\theta)\right)\right]\mathbf{w},\\
\widetilde{\eta}_0(\theta) &= -\widetilde{\Psi}'_{C0}(\theta), \qquad \widetilde{\varepsilon}_0(\theta) = \widetilde{\Psi}_{C0}(\theta) - \theta\widetilde{\Psi}'_{C0}(\theta).
\end{aligned} \tag{2.20}$$

In this case, the inequality (2.16) implies

$$-\varrho_0\frac{1}{\theta f'(\theta)}\mathbf{w}\cdot\mathbf{S}(\theta)\widetilde{\mathbf{b}}(\theta, \mathbf{w}) \geq 0, \tag{2.21}$$

and, for the source term linear in $\mathbf{w}$ (2.14), it gives the following condition:

$$\varrho_0\frac{1}{\theta f'(\theta)}\mathbf{w}\cdot\mathbf{S}(\theta)\mathbf{N}(\theta)\mathbf{w} \geq 0, \tag{2.22}$$

which requires $\mathbf{S}(\theta)\mathbf{N}(\theta)$ to be positive semi-definite for $f'(\theta) > 0$ or negative semi-definite for $f'(\theta) < 0$. This additional constitutive assumption (2.18), together with the assumption (2.14), was introduced in [4] and motivated by the requirement that, in stationary conditions defined as $\dot{\mathbf{q}} = \mathbf{0}$ and $\dot{\theta} = 0$, the evolution equation (2.13), (2.14) should coincide with Fourier's law assumed in the following form:

$$\mathbf{q} = -\mathbf{K}\,\mathrm{grad}\,\theta, \tag{2.23}$$

where $\mathbf{K}$ is positive definite heat conductivity tensor which may be generalized to be temperature-dependent $\mathbf{K} = \mathbf{K}(\theta)$. In fact, taking the evolution equation (2.13), (2.14) for $\dot{\mathbf{w}} = \mathbf{0}$ (in view of (2.18), this corresponds to $\dot{\mathbf{q}} = \mathbf{0}$, $\dot{\theta} = 0$) and comparing with (2.23), one obtains

$$\mathbf{q}(\theta, \mathbf{w}) = \mathbf{S}(\theta)\mathbf{w}, \qquad \mathbf{S}(\theta) = \frac{1}{f'(\theta)}\mathbf{K}(\theta)\mathbf{N}(\theta). \tag{2.24}$$

With the additional assumptions (2.18), (2.14), the model considered is completely determined by the constitutive functions dependent only on the temperature $f(\theta)$, $\widetilde{\Psi}_{C0}(\theta)$, $\mathbf{S}(\theta)$, $\mathbf{N}(\theta)$, except the source term in the balance of energy. The corresponding system of equations, in view of (2.18), (2.20), (2.14), assumes the following form:

$$\begin{aligned} &\varrho_0\partial_t\left\{\widetilde{\Psi}_{C0}(\theta) - \theta\widetilde{\Psi}'_{C0}(\theta) + \frac{1}{2}\mathbf{w}\cdot\left[\frac{1}{f'(\theta)}\mathbf{S}(\theta) - \theta\frac{d}{d\theta}\left(\frac{1}{\theta f'(\theta)}\mathbf{S}(\theta)\right)\right]\cdot\mathbf{w}\right\} \\ &\qquad\qquad + \operatorname{div}(\mathbf{S}(\theta)\mathbf{w}) = \varrho_0\widetilde{r}(\theta, \mathbf{w}), \\ &\partial_t\mathbf{w} + \operatorname{div}(f(\theta)\mathbf{I}) = -\mathbf{N}(\theta)\mathbf{w}, \end{aligned} \tag{2.25}$$

and is consistent with the entropy inequality

$$\begin{aligned} &\varrho_0\partial_t\left\{-\widetilde{\Psi}'_{C0}(\theta) - \frac{1}{2}\mathbf{w}\cdot\left[\frac{d}{d\theta}\left(\frac{1}{\theta f'(\theta)}\mathbf{S}(\theta)\right)\right]\mathbf{w}\right\} \\ &\qquad\qquad + \operatorname{div}\left[\frac{1}{\theta}(\mathbf{S}(\theta)\mathbf{w}\right] \geq \varrho_0\frac{1}{\theta}\widetilde{r}(\theta, \mathbf{w}, t, X^\alpha) \end{aligned} \tag{2.26}$$

provided that the inequality (2.22) holds for all $\mathbf{w}$ and $\theta > 0$.

In the case of "thermally" isotropic body, the relations (2.18) – (2.20) simplify according to the well known representation theorems for tensor functions

$$\mathbf{S}(\theta) = \alpha(\theta)\mathbf{I}, \qquad \mathbf{N}(\theta) = \nu(\theta)\mathbf{I}, \qquad \mathbf{K}(\theta) = \kappa(\theta)\mathbf{I}. \tag{2.27}$$

B. $\widetilde{\varepsilon}(\theta, \mathbf{w}) = \widetilde{\varepsilon}_0(\theta)$

The next simplifying assumption introduced in [4] together with the assumptions (2.14), (2.18) is that internal energy $\widetilde{\varepsilon}(\theta, \mathbf{w})$ does not depend on the internal state variable $\mathbf{w}$

$$\widetilde{\varepsilon}(\theta, \mathbf{w}) = \widetilde{\varepsilon}_0(\theta). \tag{2.28}$$

According to $(2.20)_2$, the following equation for $\mathbf{S}(\theta)$ results as a consequence of this assumption:

$$\mathbf{S}(\theta) = \theta^2 f(\theta)\frac{d}{d\theta}\left[\frac{1}{\theta f'(\theta)}\mathbf{S}(\theta)\right]. \tag{2.29}$$

In [4], the following solution of (2.29) was found

$$\mathbf{S}(\theta) = \theta^2 f(\theta)\mathbf{B}, \tag{2.30}$$

where $\mathbf{B}$ is a positive definite constant tensor, and, according to (2.24), it relates the source factor $\mathbf{N}(\theta)$ to the temperature-dependent heat conductivity $\mathbf{K}(\theta)$

$$\mathbf{N}(\theta) = \theta^2[f'(\theta)]^2\mathbf{K}^{-1}(\theta)\mathbf{B}. \tag{2.31}$$

It follows from (2.19), (2.20), (2.15) that (2.30), (2.31) additionally imply

$$\begin{aligned} \widetilde{\Psi}_{C0}(\theta,\mathbf{w}) &= \widetilde{\Psi}_{C0}(\theta) + \frac{1}{2\varrho_0}\theta\mathbf{w}\cdot\mathbf{B}\mathbf{w}, \\ \widetilde{\eta}(\theta,\mathbf{w}) &= -\frac{\partial\widetilde{\Psi}(\theta,\mathbf{w})}{\partial\theta} = \widetilde{\eta}_0(\theta) - \frac{1}{2\varrho_0}\mathbf{w}\cdot\mathbf{B}\mathbf{w}, \\ \widetilde{\eta}_0(\theta) &= -\widetilde{\Psi}'_{C0}(\theta), \qquad \widetilde{\varepsilon}(\theta,\mathbf{w}) = \widetilde{\varepsilon}(\theta) = \widetilde{\Psi}_{C0}(\theta) - \theta\widetilde{\Psi}'_{C0}(\theta), \end{aligned} \tag{2.32}$$

$$\widetilde{\mathbf{q}}^{\alpha}(\theta, w^{\gamma}) = \varrho_0\theta f'(\theta)\frac{\partial\widetilde{\Psi}(\theta,\mathbf{w}_{\gamma})}{\partial w_{\alpha}} = \theta^2 f'(\theta)\mathbf{B}\mathbf{w}, \tag{2.33}$$

and the system of field equations (2.25) further simplifies

$$\begin{aligned} \varrho_0\partial_t\widetilde{\varepsilon}_0(\theta) + \operatorname{div}\varrho_0\theta^2 f'(\theta)\mathbf{B}\mathbf{w}) &= \varrho_0\widetilde{r}(\theta,\mathbf{w}), \\ \partial_t\mathbf{w} + \operatorname{div}(f(\theta)\mathbf{I}) = -\mathbf{N}(\theta)\mathbf{w} &= -\theta^2[f'(\theta)]^2\mathbf{K}^{-1}(\theta)\mathbf{B}\mathbf{w}. \end{aligned} \tag{2.34}$$

In the isotropic case (2.27), Eqs.(2.30), (2.31) yield

$$\alpha(\theta) = c_2\theta^2 f(\theta), \qquad c_2 = \text{const}, \tag{2.35}$$

$$\nu(\theta) = c_2\theta^2[f'(\theta)]^2[\kappa(\theta)]^{-1} \tag{2.36}$$

and the corrsponding field equations are

$$\begin{aligned} \varrho_0\partial_t\widetilde{\varepsilon}_0(\theta) + \operatorname{div}(c_2 f'(\theta)\mathbf{w}) &= \varrho_0\widetilde{r}(\theta,|\mathbf{w}|), \\ \partial_t\mathbf{w} + \operatorname{div}(f(\theta)\mathbf{I}) = -\nu(\theta)\mathbf{w} &= -c_2\frac{\theta^2[f'(\theta)]^2}{\kappa(\theta)}\mathbf{w}. \end{aligned} \tag{2.37}$$

In [4], the systems (2.34) and (2.37) were transformed, with the aid of (2.18), (2.24), (2.36), to the corresponding equivalent systems with respect to $\theta, \mathbf{q}$. Then in [5, 6], the transformed system (2.37) was alternatively derived as a special (linear in $\mathbf{q}$) version of the extended thermodynamics of a rigid conductor of heat, and successively applied for phenomenological modelling of some observed features of the second sound propagation in dielectric crystals.

2.2.5. Comparison with the Maxwell–Cattoneo–Vernotte equation. Assumption that $\widetilde{q}(\theta,\mathbf{w})$ is linear in $\mathbf{w}$ together with the assumption that the source term $\mathbf{b}(\theta,\mathbf{w})$ is also linear in $\mathbf{w}$, make it possible to express the evolution equation for $\mathbf{w}$ equivalently as the evolution equation for the heat flux vector $\mathbf{q}$, and to replace the corresponding constitutive functions (2.19), (2.20) and by $\widehat{\varepsilon}(\theta,\mathbf{q}) = \widetilde{\varepsilon}(\theta,\mathbf{S}^{-1}(\theta)\mathbf{q})$,

$\widehat{\eta}(\theta, \mathbf{q}) = \widetilde{\eta}(\theta, \mathbf{S}^{-1}(\theta)\mathbf{q})$, and $\widehat{\psi}_C(\theta, \mathbf{q}) = \widetilde{\Psi}_C(\theta, \mathbf{S}^{-1}(\theta)\mathbf{q})$. Substitution of (2.18) into (2.13), (2.14) yields the following evolution equation for the heat flux vector

$$(\mathbf{S}^{-1}(\theta)\mathbf{q})^{\cdot} + \operatorname{div}(f(\theta)\mathbf{I}) = -\mathbf{N}(\theta)\mathbf{S}^{-1}(\theta)\mathbf{q}. \tag{2.38}$$

This fact leads to the question, under which further additional constitutive assumptions the evolution equation for the heat flux (2.38), coincides with (or can be reduced to) the Maxwell - Cattoneo - Vernotte equation

$$\tau\dot{\mathbf{q}} + \mathbf{q} = -\kappa \operatorname{grad}\theta, \tag{2.39}$$

where $\tau > 0$ is constant thermal relaxation time and $\kappa > 0$ is constant heat conductivity, or with its generalization proposed by PAO and BANERJEE [23]

$$\mathbf{T}(\theta)\dot{\mathbf{q}} + \mathbf{q} = -\mathbf{K}(\theta) \operatorname{grad}\theta, \tag{2.40}$$

where $\mathbf{T}(\theta)$ is a temperature-dependent thermal relaxation tensor and $\mathbf{K}(\theta)$ is a temperature-dependent heat conductivity tensor. For arbitrary positive definite $\mathbf{S}(\theta)$ and for arbitrary $f(\theta)$, the evolution equation (2.38), when rearranged to the respective form similar to (2.40)

$$\begin{aligned}\mathbf{S}(\theta)\mathbf{N}^{-1}(\theta)\mathbf{S}^{-1}(\theta)\dot{\mathbf{q}} + \left\{\mathbf{I} - \mathbf{S}(\theta)\mathbf{N}^{-1}(\theta)\mathbf{S}^{-1}(\theta)\left[\frac{d}{d\theta}\mathbf{S}(\theta)\right]\mathbf{S}^{-1}(\theta)\dot{\theta}\right\}\mathbf{q} \\ = -\mathbf{S}(\theta)\mathbf{N}^{-1}f'(\theta) \operatorname{grad}\theta,\end{aligned} \tag{2.41}$$

contains a term proportional to $\dot{\theta}$. It takes the form (2.40) if

$$\mathbf{S}(\theta) = \mathbf{S}_0\,, \qquad \alpha(\theta) = \alpha_0\,, \tag{2.42}$$

and the following identification holds

$$\mathbf{T}(\theta) = \mathbf{S}_0\mathbf{N}^{-1}(\theta)\mathbf{S}_0^{-1}\,, \qquad \mathbf{K}(\theta) = f'(\theta)\mathbf{S}_0\mathbf{N}^{-1}(\theta). \tag{2.43}$$

In this case, the evolution equation (2.41) for the heat flux vector takes the form

$$\frac{1}{f'(\theta)}\mathbf{K}(\theta)\mathbf{S}_0^{-1}\dot{\mathbf{q}} + \mathbf{q} = -\mathbf{K}(\theta) \operatorname{grad}\theta, \tag{2.44}$$

while the corresponding evolution equations for the internal state variable is

$$\dot{\mathbf{w}} + \operatorname{div}(f(\theta)\mathbf{I}) = -f'(\theta)\mathbf{K}^{-1}(\theta)\mathbf{S}_0\mathbf{w} = -\mathbf{N}(\theta)\mathbf{w}. \tag{2.45}$$

The model is completely determined by prescribing $f(\theta), \widetilde{\Psi}_{C0}(\theta)$, positive definite constant tensor $\mathbf{S}_0$, the source factor $\mathbf{N}(\theta)$ related to $\mathbf{S}_0$, $\mathbf{K}(\theta)$ and $f'(\theta)$ and the source term $\widetilde{r}(\theta, \mathbf{w}, t, X^\alpha)$, and, according to (2.42), (2.43), (2.18), (2.19),

$$
(2.46)\qquad
\begin{aligned}
\widetilde{\Psi}_{C0}(\theta, \mathbf{w}) &= \widetilde{\Psi}_{C0}(\theta) + \frac{1}{2\varrho_0 \theta f'(\theta)} \mathbf{w} \cdot \mathbf{S}_0 \mathbf{w}, \\
\widetilde{\eta}(\theta, \mathbf{w}) &= \widetilde{\eta}_0(\theta) - \frac{1}{2\varrho_0} \frac{d}{d\theta} \left(\frac{1}{\theta f'(\theta)} \right) \mathbf{w} \cdot \mathbf{S}_0 \mathbf{w}, \\
\widetilde{\varepsilon}(\theta, \mathbf{w}) &= \widetilde{\varepsilon}_0(\theta) + \frac{1}{2\varrho_0} \left[\frac{1}{\theta f'(\theta)} - \theta \frac{d}{d\theta} \left(\frac{1}{\theta f'(\theta)} \right) \right] \mathbf{w} \cdot \mathbf{S}_0 \mathbf{w}, \\
\widetilde{q}^\alpha(\theta, w_\gamma) &= \varrho_0 \theta f'(\theta) \frac{\partial \widetilde{\Psi}_C(\theta, w_\gamma)}{\partial w_\alpha} = S_0^{\alpha\beta} w_\beta .
\end{aligned}
$$

It follows from (2.44) that the model of a rigid conductor of heat with vector internal state variable subject to additional constitutive assumptions (2.14), (2.18), (2.42) is not able to reproduce the generalization of the Maxwell–Cattaneo–Vernotte equation (2.40) since the relations (2.43) imply $\mathbf{T}(\theta) = \dfrac{1}{f'(\theta)} \mathbf{K}(\theta) \mathbf{S}_0^{-1}$, thus admitting only such temperature-dependent thermal relaxation tensors $\mathbf{T}(\theta)$ which are related to $\mathbf{K}(\theta)$ by scalar function of θ and positive definite constant tensor. The thermodynamic restrictions imposed on $\mathbf{T}(\theta)$ and $\mathbf{K}(\theta)$ by Clausius–Duhem entropy inequality were investigated by COLEMAN, FABRIZIO and OWEN [16, 17] and it was shown that $\mathbf{K}(\theta)$ should be positive definite and $\mathbf{K}^{-1}(\theta)\mathbf{T}(\theta)$ should be symmetric. The model (2.45), (2.46), (2.43) satisfies those conditions provided that $\mathbf{K}(\theta) = f'(\theta) \mathbf{S}_0 \mathbf{N}^{-1}(\theta)$ is positive definite (see also (2.22)) while $\mathbf{K}^{-1}(\theta)\mathbf{T}(\theta) = \dfrac{1}{f'(\theta)} \mathbf{S}_0^{-1}$ is symmetric since $\mathbf{S}_0$ is positive definite.

The Maxwell–Cattaneo–Vernotte equation (2.39) is recovered if we take the isotropic case (2.27) and put

$$
(2.47)\qquad f(\theta) = \theta, \quad \nu(\theta) = \frac{1}{\tau} = \text{const}, \quad \alpha(\theta) = \alpha_0, \quad \kappa(\theta) = \alpha_0 \tau = \kappa = \text{const}.
$$

Substitution of (2.47) into (2.19), (2.20) yields

$$
(2.48)\qquad
\begin{aligned}
\widetilde{\Psi}_C(\theta, |\mathbf{w}|) &= \widetilde{\Psi}_{C0}(\theta) + \frac{\kappa}{2\varrho_0 \theta \tau} |\mathbf{w}|^2, \\
\widetilde{\eta}(\theta, |\mathbf{w}|) &= \widetilde{\eta}_0(\theta) - \frac{\kappa}{2\varrho_0 \theta^2 \tau} |\mathbf{w}|^2, \\
\widetilde{\varepsilon}(\theta, |\mathbf{w}|) &= \widetilde{\varepsilon}_0(\theta) + \frac{\kappa}{\varrho_0 \theta \tau} |\mathbf{w}|^2, \\
\widetilde{\eta}_0(\theta) &= -\widetilde{\Psi}'_{C0}(\theta), \qquad \widetilde{\varepsilon}_0(\theta) = \widetilde{\Psi}_{C0}(\theta) - \theta \widetilde{\Psi}'_{C0}(\theta), \\
\widetilde{q}^\alpha(\theta, \mathbf{w}) &= \varrho_0 \theta \frac{\partial \widetilde{\Psi}_C(\theta, |\mathbf{w}|)}{\partial w_\alpha} = \frac{\kappa}{\tau} w^\alpha,
\end{aligned}
$$

and the system of field equations with respect to θ, $\mathbf{w}$ for a rigid conductor of heat described by the Maxwell - Cattaneo - Vernotte equation (2.39) can be obtained by substituting (2.47), $(2.48)_{3,5}$ into (2.25), (2.24)

$$\text{(2.49)}\qquad \begin{aligned} \varrho_0\partial_t\left[\varepsilon_0(\theta)+\frac{\kappa}{\varrho_0\theta\tau}|\mathbf{w}|^2\right]+\operatorname{div}\left(\frac{\kappa}{\tau}\mathbf{w}\right)&=\varrho_0\widetilde{r}(\theta,\mathbf{w},t,X^\alpha),\\ \partial_t\mathbf{w}+\operatorname{div}(\theta\mathbf{I})&=-\frac{1}{\tau}\mathbf{w}. \end{aligned}$$

This description of a rigid conductor of heat governed by the Maxwell - Cattaneo - Vernotte equation in terms of internal state variable was obtained by KOSIŃSKI [22] as a special case of the general model of a deformable (inelastic) conductor of heat with internal state variables. Substituting $(2.48)_5$ into $(2.48)_{2,3}$ we may express internal energy $\widetilde{\varepsilon}(\theta,|\mathbf{w}|)$ and entropy $\widetilde{\eta}(\theta,|\mathbf{w}|)$ as $\widehat{\varepsilon}(\theta,|\mathbf{q}|)$ and $\widehat{\eta}(\theta,|\mathbf{q}|)$, respectively, and then calculate

$$\text{(2.50)}\qquad d\widehat{\eta}=\frac{1}{\theta}\frac{\partial\widehat{\eta}}{\partial\theta}d\theta+\frac{\tau}{2\varrho_0\kappa\theta^2}q_\beta dq^\beta=\frac{1}{\theta}d\widehat{\varepsilon}-\frac{\tau}{2\varrho_0\kappa\theta^2}q_\beta dq^\beta,$$

what agrees with the result of the analysis of the Maxwell - Cattaneo - Vernotte equation performed in the framework of EIT by JOU and CASSAS–VAZQUEZ [36]. It follows from $(2.48)_3$, $(2.46)_3$ that the model of a rigid conductor of heat with vector internal state variables satisfying additional assumptions (2.14), (2.18) and subject to the requirement $\widetilde{\varepsilon}(\theta,\mathbf{w})=\widetilde{\varepsilon}_0(\theta)$ (discussed in Sec. 2.2.4.B) cannot reproduce the Maxwell - Cattaneo - Vernotte equation (2.39), (2.48), (2.49) but it can reproduce a special form of the generalized Maxwell - Cattaneo - Vernotte equation (2.40), (2.43) for $f(\theta)=-c/\theta$ (c – const) and, consequently, in this case the relation $\mathbf{T}(\theta)=\dfrac{\theta^2}{c}\mathbf{K}(\theta)\mathbf{S}_0^{-1}$ must be satisfied. This special form of the generalized Maxwell - Cattaneo - Vernotte equation was discussed in [16, 17, 23]. Hence, the model of a rigid conductor with vector internal state variable (2.32) - (2.34) based on all three assumptions (2.13), (2.14), (2.18), (2.28), developed in [4, 5, 6], cannot be considered as a "generalization" of the Maxwell - Cattaneo - Vernotte equation but should be understood as the "alternative" to the generalized Maxwell - Cattaneo - Vernotte equation (2.40).

2.2.6. Generalization of the the Maxwell–Cattoneo–Vernotte equation and other specific models through the model with internal state variables. Further analysis will be focused on the model described by the equation $(2.6)_1$ corresponding to the balance of energy, the evolution equation for the internal state variable $\mathbf{w}$ in the form (2.13) and the inequality (2.7). In order to emphasize the relation to the model of a rigid conductor of heat with "semi-empirical temperature" discussed in the next section and the resemblance to the Maxwell - Cattaneo - Vernotte equation, we denote

$$\text{(2.51)}\qquad f_1(\theta):=-\tau f(\theta),\qquad \mathbf{c}(\theta,\mathbf{w})=\tau\mathbf{b}(\theta,\mathbf{w}),\qquad \tau=\text{const}$$

and, substituting (2.51) into (2.13), $(2.6)_1$, (2.7), we rewrite the system of corresponding field equations together with Clausius–Duhem entropy inequality in the following form:

$$
\begin{aligned}
\varrho_0 \partial_t \widetilde{\varepsilon}(\theta, w_\beta) + \partial_\alpha \widetilde{q}^\alpha(\theta, w_\beta) &= \varrho_0 \widetilde{r}(\theta, w_\beta), \\
\tau \partial_t w_\gamma - \partial_\alpha [f_1(\theta) \delta^\alpha{}_\gamma] &= c_\gamma(\theta, w_\beta),
\end{aligned} \tag{2.52}
$$

$$
\varrho_0 \sigma := \varrho_0 \partial_t \widetilde{\eta} + \partial_\alpha \left[\frac{1}{\theta} \widetilde{q}^\alpha(\theta, w_\beta) \right] \geq \varrho_0 \frac{1}{\theta} \widetilde{r}(\theta, \mathbf{w}, t, X^\alpha). \tag{2.53}
$$

In (2.51), (2.52) constant $\tau > 0$ can be interpreted as a thermal relaxation time. According to the analysis performed in Secs. 2.2.3, 2.2.4, 2.2.5, the model corresponding to the field equations (2.52) and consistent with the inequality (2.53) is completely determined by prescribing $f_1(\theta)$, constant τ, free energy function $\widetilde{\Psi}_C(\theta, \mathbf{w})$ which, without loss of generality, may be assumed as

$$
\widetilde{\Psi}_C(\theta, \mathbf{w}) = \widetilde{\Psi}_{C0}(\theta) + \widetilde{\Psi}_1(\theta, \mathbf{w}), \tag{2.54}
$$

and the source terms $\widetilde{r}(\theta, w_\beta, t, X^\alpha)$, $c_\gamma(\theta, w_\beta)$. This model comprises all particular models discussed in Secs. 2.2.4, 2.2.5 as special cases corresponding to specific forms of constitutive quantities $f_1(\theta)$, $\widetilde{\Psi}_1(\theta, \mathbf{w})$ and $\mathbf{c}(\theta, \mathbf{w})$. For comparison, those specific forms are presented in the Table 1.

The system of field equations (2.52) and the entropy inequality (2.53) can be considered as a representation of the class of phenomenological models of heat transport at finite speeds containing posible generalizations of both the mentioned models of practical applicability. Constitutive functions $f_1(\theta)$, $\widetilde{\Psi}_1(\theta, \mathbf{w})$, $\mathbf{c}(\theta, \mathbf{w})$ can be arranged to the form representing explicitly the corrections to, or derivations from those models. For example, taking $f_1(\theta) = -\tau\theta + m(\theta)$, $\widetilde{\Psi}_1(\theta, \mathbf{w}) = \dfrac{\kappa}{2\varrho\theta\tau}|\mathbf{w}|^2 + \xi(\theta, \mathbf{w})$ and $\mathbf{c}(\theta, \mathbf{w}) = -\mathbf{w} + \mathbf{d}(\theta, \mathbf{w})$, we may introduce corrections to the Maxwell–Cattoneo–Vernotte equation by means of functions $m(\theta)$, $\xi(\theta, \mathbf{w})$ and $\mathbf{d}(\theta, \mathbf{w})$.

In the remaining part of this paper, the source term in $(2.52)_2$ will be assumed as

$$
\mathbf{c}(\theta, \mathbf{w}) = \mathbf{c}_1 \mathbf{w}, \qquad c_1 = \text{const}. \tag{2.55}
$$

This simplifying assumption does not restrict the generality of further analysis concerning the application of the MDR for derivation of the thermodynamic restrictions, and of the procedures of symmetrization (together with conditions of symmetric hyperbolicity) applied in Part II. To obtain the results valid for arbitrary $\mathbf{c}(\theta, \mathbf{w})$ it suffices simply to replace $c_1 \mathbf{w}$ by $\mathbf{c}(\theta, \mathbf{w})$. Hence, the symmetric systems obtained in Part II for particular source term (2.55) also apply to general $\mathbf{c}(\theta, \mathbf{w})$ if this replacement is done.

Table 1.

Specific model (additional constitutive assumption)	$f_1(\theta)$	$\widetilde{\Psi}_1(\theta,\mathbf{w})$	$\mathbf{c}(\theta,\mathbf{w})$
(2.18), (2.24) $\widetilde{q}(\theta,\mathbf{w})=\mathbf{S}(\theta)\mathbf{w}$ [4]	no specific form	$-\dfrac{\tau}{2\varrho_0\theta f_1'(\theta)}\mathbf{w}\cdot\mathbf{S}(\theta)\mathbf{w}$ (isotropic case) $-\dfrac{\tau\alpha(\theta)}{2\varrho_0\theta f_1'(\theta)}\lvert\mathbf{w}\rvert^2$	$f_1'(\theta)\mathbf{K}^{-1}(\theta)\mathbf{S}(\theta)\mathbf{w}$ (isotropic case) $\dfrac{f_1'(\theta)\alpha(\theta)}{\kappa(\theta)}\mathbf{w}$
(2.18), (2.24), (2.28) $\widetilde{q}(\theta,\mathbf{w})=\mathbf{S}(\theta)\mathbf{w}$ and $\widetilde{\varepsilon}(\theta,\mathbf{w})=\widetilde{\varepsilon}_0(\theta)$ [4, 5, 6],	no specific form	$\dfrac{1}{2\varrho_0}\theta\mathbf{w}\cdot\mathbf{B}\mathbf{w}$ (isotropic case) $\dfrac{1}{2\varrho_0}c_2\theta\lvert\mathbf{w}\rvert^2$	$-\dfrac{\theta^2[f_1'(\theta)]^2}{\tau}\mathbf{K}^{-1}(\theta)\mathbf{B}\mathbf{w}$ (isotropic case) $-\dfrac{c_2\theta^2[f_1'(\theta)]^2}{\tau\kappa(\theta)}\mathbf{w}$
(2.42), (2.43) Generalized Maxwell – Cattoneo – Vernotte equation [16, 17, 23]	no specific form	$-\dfrac{\tau}{2\varrho_0\theta f_1'(\theta)}\mathbf{w}\cdot\mathbf{S}_0\mathbf{w}$ (isotropic case) $-\dfrac{\tau\alpha_0}{2\varrho_0\theta f_1'(\theta)}\lvert\mathbf{w}\rvert^2$	$f_1'(\theta)\mathbf{K}^{-1}(\theta)\mathbf{S}_0\mathbf{w}$ (isotropic case) $\dfrac{\alpha_0 f_1'(\theta)}{\kappa(\theta)}\mathbf{w}$
(2.42), (2.43), (2.28) Generalized Maxwell – Cattoneo – Vernotte equation with $\widetilde{\varepsilon}(\theta,\mathbf{w})=\widetilde{\varepsilon}_0(\theta)$ [16, 17]	$\dfrac{c_2\tau}{\theta}$	$\dfrac{c_2}{2\varrho_0}\theta\mathbf{w}\cdot\mathbf{S}_0\mathbf{w}$ (isotropic case) $\dfrac{{c_2}^2\alpha_0\theta}{2\varrho_0}\lvert\mathbf{w}\rvert^2$	$-\dfrac{{c_2}^2}{\theta^2}\mathbf{K}^{-1}(\theta)\mathbf{S}_0$ (isotropic case) $-\dfrac{{c_2}^2\alpha_0}{\theta^2\kappa(\theta)}$
(2.48), (2.49) Maxwell – Cattoneo – Vernotte equation [22]	$-\tau\theta$	$\dfrac{\kappa}{2\varrho_0\theta\tau}\lvert\mathbf{w}\rvert^2$	$-\mathbf{w}$

2.3. Relation to the model of a rigid heat conductor with "semi-empirical temperature"

Recently in a series of papers: KOSIŃSKI [24], CIMMELLI and KOSIŃSKI [25, 26], KOSIŃSKI and SAXTON [28], CIMMELLI, KOSIŃSKI and SAXTON [27, 29], a model of heat conduction with finite wave speed based on the concept of "semi-empirical temperature" has been developed. In this approach, a scalar internal state variable β called "semi-empirical temperature" is introduced and the equation relating the evolution of β to the temperature is postulated

$$\tau\dot{\beta}=f(\theta,\beta), \tag{2.56}$$

where the presence of the constant dimension parameter τ (thermal relaxation time) is motivated by dimensional analysis [27]. According to [25, 27], β is uniquely defined by (2.56) if a suitable initial condition

$$\beta(t_0,X^\alpha)=\beta_0(X^\alpha) \tag{2.57}$$

is given, and $f(\cdot,\cdot)$ is Lipschitz continuous. It is also assumed [25, 27] that

$$\frac{\partial f}{\partial \theta} > 0, \qquad \frac{\partial f}{\partial \beta} \le 0 \tag{2.58}$$

since $(2.58)_1$ ensures the stability of the solutions of (2.56), (2.57). For $\tau = 0$, the inequality $(2.58)_2$ makes β an increasing function of θ and ensures the order relation between different temperatures.

In [25, 27], a rigid conductor of heat with "semi-empirical temperature" was considered with the following constitutive assumptions:

$$\begin{aligned} \varepsilon &= \widehat{\varepsilon}(\theta, \operatorname{grad}\beta),\\ \mathbf{q} &= \widehat{\mathbf{q}}(\theta, \operatorname{grad}\beta),\\ r &= \widehat{r}(\theta, \operatorname{grad}\beta, t, X^{\alpha}),\\ \eta &= \widehat{\eta}(\theta, \operatorname{grad}\beta), \end{aligned} \tag{2.59}$$

and the analysis performed in [27] showed that, in this case, $f(\theta,\beta)$ must be of the form

$$f(\theta,\beta) = f_1(\theta) + f_2(\beta). \tag{2.60}$$

Substitution of constitutive functions (2.59) into (2.1) and substitution of (2.60) into (2.56) yield the following system of two field equations for θ, β:

$$\begin{aligned} \varrho_0 \dot{\widehat{\varepsilon}}(\theta, \operatorname{grad}\beta) + \operatorname{div}\widehat{\mathbf{q}}(\theta, \operatorname{grad}\beta) &= \varrho_0 \widehat{r}(\theta, \operatorname{grad}\beta, t, X^{\alpha}),\\ \tau\dot{\beta} &= f_1(\theta) + f_2(\beta), \end{aligned} \tag{2.61}$$

subject to differential unilateral constraints resulting from substitution of (2.59) into the Clausius–Duhem entropy inequality (2.2)

$$\varrho_0\sigma := \varrho_0\dot{\widehat{\eta}}(\theta, \operatorname{grad}\beta) + \operatorname{div}\left[\frac{1}{\theta}\widehat{\mathbf{q}}(\theta, \operatorname{grad}\beta)\right] \ge \varrho_0\frac{1}{\theta}\widehat{r}(\theta, \operatorname{grad}\beta, t, X^{\alpha}). \tag{2.62}$$

Assumption $(2.58)_1$ together with (2.60) enables one to invert the function $f_1(\cdot)$ ($f_1^{-1}(\cdot)$ denotes the inverse) and express θ in terms of β, $\dot{\beta}$, namely $\theta = f^{-1}(\tau\dot{\beta} - f_2(\beta))$, and, consequently, substitute $\theta = \widehat{\theta}(\beta,\dot{\beta})$ into (2.61), (2.62) thus obtaining second-order nonlinear partial differential equation for β subject to the second-order unilateral differential constraints. The alternative approach is to express the system (2.61) as an equivalent first-order quasi-linear system of partial differential equations. In general case, the obtained first-order system will be subject, besides the differential inequality resulting from (2.62), also to both evolutive (involving time derivative) and involutive (involving only spatial derivatives) constraints. In [28], the corresponding first-order system was derived for particular form of $f(\cdot,\cdot)$ in (2.56) while in [27], such system was discussed in the context of comparison of the model of a rigid conductor of heat with "semi-empirical temperature" and the model proposed by MORRO and RUGGIERI [4, 5, 6] expressed in terms of θ, $\mathbf{q}$ (see Sec. 2.2.4). For this comparison, the additional constitutive assumptions $\widehat{\mathbf{q}}(\theta, \operatorname{grad}\beta) = \alpha(\theta)\operatorname{grad}\beta$ and $\widehat{\varepsilon}(\theta, \operatorname{grad}\beta) = \widetilde{\varepsilon}(\theta)$ (corresponding to (2.18) and (2.28), respectively) were introduced. In [27], the new variable

$$\mathbf{w} := \operatorname{grad}\beta, \tag{2.63}$$

which is suggested in view of constitutive assumptions (2.59), together with the "prolonged" evolution equation obtained by spatial differentiation of $(2.61)_2$

$$\partial_\alpha[\tau\partial_t\beta] = f_1'(\theta)\partial_\alpha\theta + f_2'(\beta)\partial_\alpha\beta = \tau\partial_t[\partial_\alpha\beta], \quad \alpha = 1, 2, 3 \tag{2.64}$$

were employed in derivation of the equivalent first-order system. Substituting (2.63) into (2.64) and into $(2.61)_1$ we obtain the system of 4 field equations with the left-hand side exactly the same as the left-hand side of the system (2.52) and with the right-hand side of the evolution equation for $\mathbf{w}$ (resulting from (2.63), (2.64)) of the form $f_2'(\beta)\mathbf{w}$. Hence, the condition $f_2(\beta) = c_1\beta$, c_1 – constant, ((2.58) implies $c_1 \leq 0$) enables the transformation of the system (2.61) and the inequality (2.62) into the equivalent quasi-linear first-order conservative system for $\theta, \mathbf{w}$

$$\begin{aligned} \varrho_0\dot{\widehat{\varepsilon}}(\theta, \mathbf{w}) + \operatorname{div}\widehat{\mathbf{q}}(\theta, \mathbf{w}) &= \varrho_0\widehat{r}(\theta, \mathbf{w}, t, X^\alpha), \\ \tau\dot{w} - \operatorname{div}(f_1(\theta)\mathbf{I}) &= c_1\mathbf{w}, \end{aligned} \tag{2.65}$$

subject to the inequality

$$\varrho_0\sigma := \dot{\widehat{\eta}}(\theta, \mathbf{w}) + \operatorname{div}\left[\frac{1}{\theta}\widehat{\mathbf{q}}(\theta, \mathbf{w})\right] \geq \varrho_0\frac{1}{\theta}\widetilde{r}(\theta, \mathbf{w}, t, X^\alpha) \tag{2.66}$$

and involutive constraints implied by (2.63)

$$\partial_\alpha w_\beta - \partial_\beta w_\alpha = 0, \qquad \alpha, \beta = 1, 2, 3, \quad \alpha \neq \beta. \tag{2.67}$$

The constraints (2.67) enable integration of $(2.65)_2$ to the following evolution equation for a scalar β related to $\mathbf{w}$ by (2.63)

$$\tau\partial_t\beta = f_1(\theta) + c_1\beta. \tag{2.68}$$

Since the systems (2.65), (2.66) and (2.52), (2.53) for $c_1 = \text{const}$ have exactly the same form, the system (2.52), (2.53) with $c_1 = \text{const}$, when subject to involutive constraints (2.67), can be regarded as corresponding to the model of a rigid conductor of heat with "semi-empirical temperature" such that the evolution of the "semi-empirical temperature" is governed by (2.68), and this correspondence holds without any other additional constitutive assumptions (like $\mathbf{q} = \alpha(\theta)$, $\mathbf{w} = \alpha(\theta)\operatorname{grad}\beta$, $\varepsilon = \widehat{\varepsilon}(\theta)$ employed in [4, 5, 6, 17].

Properties of quasi-linear conservative systems with involutive constraints and entropy inequality or additional conservation equation were investigated by DAFERMOS [37], GODUNOV [38], BOILLAT [39–43] and STRUMIA [44, 45]. It is well known (for example, [38, 44]) that the involutive constraints hold for solutions at each time provided that they are satisfied by initial conditions. Hence, among the solutions $\theta(t, X^\alpha)$, $\mathbf{w}(t, X^\alpha)$ of the system (2.52), (2.53), a special class corresponding to the model with "semi-empirical temperature" can be distinguished. This class corresponds to initial value problems satisfying constraints (2.67) and, as a consequence, can be alternatively expressed in terms of the solutions $\beta(t, X^\alpha)$ of the respective initial value problems (2.57) (supplemented by required initial conditions for derivatives of β) for the third order nonlinear partial differential equation resulting from substitution of (2.68) into $(2.61)_1$. The involutive constraints do not affect the propagation speeds of weak discontinuity waves but may influence the shocks [39, 43]. Therfore, both models predict the same speeds of thermal waves of weak dicontinouity but may lead to different thermal shock behaviour. The involutive constraints (2.67) are irrelevant for the symmetrization of field equations (2.65) if the symmetric system for original field variables $\theta, \mathbf{w}$ is sought, but if the system of field equations is symmetrized by means of transformation of dependent variables then the involutive constraints (2.67) should be taken into account according to the procedures derived in [38–43].

3. The "main dependency relation" (MDR) and the restrictions on constitutive functions

3.1. The MDR

The general analysis of overdetermined systems of conservation equations has been provided by FREDRICHS [7]. In order to ensure the consistency of an overdetermined system of conservation equations ($N+1$ equations for N unknowns)

$$(3.1)\qquad \begin{gathered} \partial_t g^{0\Lambda}(u^K) + \partial_\alpha g^{\alpha\Lambda}(u^K) = b^\Lambda(u^K, t, X^\gamma), \\ \Lambda = 1, 2, \ldots, N+1, \qquad K = 1, 2, \ldots, N, \qquad \alpha = 1, 2, \ldots, m \end{gathered}$$

he has introduced the MDR which requires the existence of $N+1$ functions $y_\Lambda : \mathbb{R}^N \to \mathbb{R}$, $\Lambda = 1, 2, \ldots, N+1$, not all identically zero, such that (Property CI in [7])

$$(3.2)\qquad \begin{aligned} y_\Lambda(u^K)\frac{\partial g^{0\Lambda}(u^K)}{\partial u^M}\partial_t u^M + y_\Lambda(u^K)\frac{\partial g^{\alpha\Lambda}(u^K)}{\partial u^M}\partial_\alpha u^M &\equiv 0, \\ y_\Lambda(u^K) b^K(u^K, t, X^\alpha) &\equiv 0, \end{aligned}$$

holds for all functions $u^K(t, X^\alpha)$, $K = 1, 2, \dots, N$. Since the values of $u^K(t, X^\alpha)$, $\partial_t u^K(t, X^\alpha)$ and $\partial_\alpha u^K(t, X^\alpha)$ can be taken arbitrary at each point (t, X^α), the identity $(3.2)_1$ is equivalent to the following system of identities (Property CI' in [7])

$$y_\Lambda(u^K)\frac{\partial g^{0\Lambda}(u^K)}{\partial u^M} \equiv 0, \qquad y_\Lambda(u^K)\frac{\partial g^{\alpha\Lambda}(u^K)}{\partial u^M} \equiv 0. \tag{3.3}$$

The set of $N+1$ functions $y_\Lambda(u^K)$ can be obtained as a solution of the overdetermined system of linear homogeneous equations $(3.1)_2$, (3.2) and therefore, if it exists, it is not unique.

It would be convenient to introduce the following matrix notation:

$$\begin{gathered} \mathcal{A}^0(\mathbf{u}) = [\mathcal{A}^{0\Lambda}{}_M(u^K)] := \left[\frac{\partial g^{0\Lambda}(u^K)}{\partial u^M}\right], \\ \mathcal{A}^\alpha(\mathbf{u}) = [\mathcal{A}^{\alpha\Lambda}{}_M(u^K)] := \left[\frac{\partial g^{\alpha\Lambda}(u^K)}{\partial u^M}\right], \\ \alpha = 1, 2, \dots, m, \qquad \Lambda = 1, 2, \dots, N+1, \qquad I, K = 1, 2, \dots, N, \end{gathered} \tag{3.4}$$

and treat $N+1$ functions $y_\Lambda(u^K)$ as $(N+1)$ component row vector

$$\mathbf{y}^T(\mathbf{u}) = [y_\Lambda(u^K)]. \tag{3.5}$$

In matrix notation, $(3.2)_2$, (3.3) takes the form

$$\mathbf{y}^T(\mathbf{u})\mathcal{A}^0(\mathbf{u}) \equiv \mathbf{0}, \qquad \mathbf{y}^T(\mathbf{u})\mathcal{A}^\alpha(\mathbf{u}) \equiv \mathbf{0}, \qquad \mathbf{y}^T(\mathbf{u})\mathbf{b}(\mathbf{u}, t, X^\alpha) \equiv \mathbf{0}. \tag{3.6}$$

3.2. Equivalence of the entropy inequality and the MDR

The system of field equations (2.52) is a particular case of the quasi-linear system of N conservation equations for N unknowns

$$\partial_t \mathbf{f}^0(\mathbf{u}) + \partial_\alpha \mathbf{f}^\alpha(\mathbf{u}) = \mathbf{d}(\mathbf{u}, t, \mathbf{X}), \tag{3.7}$$

while the Clausius–Duhem entropy inequality (2.53) is a particular case of imposed unilateral differential constraints

$$\partial_t h^0(\mathbf{u}) + \partial_\alpha h^\alpha(\mathbf{u}) \geq \mu(\mathbf{u}, t, \mathbf{X}), \tag{3.8}$$

where $\mathbf{X}$ stands for $[X^\alpha]$. The unknowns $\mathbf{u}$ take values in an open bounded neighbourhood $\mathcal{O}$ of the origin in $\mathbb{R}^N$, and $\mathbf{f}^0$, $\mathbf{f}^\alpha$, h^0, h^α are presumed to have continuous second derivatives with respect to their argument.

Performing the respective differentiation, we rewrite the system (3.7) in a matrix form

$$\begin{gathered} \mathbf{B}^0(\mathbf{u})\partial_t \mathbf{u} + \mathbf{B}^\alpha(\mathbf{u})\partial_\alpha \mathbf{u} = \mathbf{d}(\mathbf{u}, t, \mathbf{X}), \\ \mathbf{B}^0(\mathbf{u}) = \nabla_\mathbf{u} \mathbf{f}^0(\mathbf{u}), \qquad \mathbf{B}^\alpha(\mathbf{u}) = \nabla_\mathbf{u} \mathbf{f}^\alpha(\mathbf{u}), \end{gathered} \tag{3.9}$$

and the inequality (3.8) as

$$\begin{gathered}\mathbf{k}^{0T}(\mathbf{u})\partial_t\mathbf{u} + \mathbf{k}^{\alpha T}(\mathbf{u})\partial_\alpha\mathbf{u} \geq \mu(\mathbf{u},t,\mathbf{X}),\\ \mathbf{k}^{0T}(\mathbf{u}) = \nabla_{\mathbf{u}}h^0(\mathbf{u}), \qquad \mathbf{k}^{\alpha T}(\mathbf{u}) = \nabla_{\mathbf{u}}h^\alpha(\mathbf{u}),\end{gathered} \tag{3.10}$$

where $\nabla_{\mathbf{u}}$ denotes differentiation with respect to $\mathbf{u}$.

OBSERVATION. If the system (3.7) satisfies the condition

$$\det \mathbf{B}^0(\mathbf{u}) \neq 0 \qquad \text{for all } \mathbf{u} \tag{3.11}$$

then every Lipschitz continuous solution $\mathbf{u}(t,\mathbf{X})$ of (3.7) satisfies the inequality (3.8) if and only if the system of $N+1$ conservation equations composed of (3.7) and of the following conservation equation:

$$\begin{gathered}\partial_t h^0(\mathbf{u}) + \partial_\alpha h^\alpha(\mathbf{u}) = \chi(\mathbf{u},t,\mathbf{X}),\\ \chi(\mathbf{u},t,\mathbf{X}) = \nabla_{\mathbf{u}}h^0(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1}\cdot\mathbf{d}(\mathbf{u},t,\mathbf{X}),\end{gathered} \tag{3.12}$$

with

$$\chi(\mathbf{u},\cdot,\cdot) - \mu(\mathbf{u},\cdot,\cdot) \geq 0, \tag{3.13}$$

satisfies the MDR.

It should be noted that the condition (3.11) is necessary for hyperbolicity of the system (3.7).

The proof of the Observation is given in the Appendix.

3.3. Thermodynamic restrictions via the MDR

In wiew of the Observation, thermodynamic restrictions imposed by Clausius–Duhem entropy inequality can be obtained from the requirement that the system of 5 first-order partial differential equations for 4 unknown fields $\theta(t,X^\alpha)$, $w_\gamma(t,X^\alpha)$

$$\begin{aligned}\varrho_0\partial_t\widetilde{\varepsilon} + \partial_\alpha\widetilde{q}^\alpha &= \varrho_0\widetilde{r},\\ \tau\partial_t w_\gamma - \partial_\alpha[f_1(\theta)\delta^\alpha_\gamma] &= c_1 w_\gamma,\\ \varrho_0\partial_t\widetilde{\eta} + \partial_\alpha\left[\frac{1}{\theta}\widetilde{q}^\alpha\right] &= \varrho_0\widetilde{\sigma},\end{aligned} \tag{3.14}$$

where $\widetilde{\varepsilon}$, $\widetilde{q}^\alpha$, $\widetilde{r}$, and $\widetilde{\eta}$ are postulated as the constitutive functions (2.3) and $\widetilde{\sigma} = \widetilde{\sigma}(\theta, w_\gamma, t, X^\alpha)$, should satisfy the MDR, provided that constitutive function $\widetilde{\varepsilon}(\theta, w_\gamma)$ satisfies the condition

$$\det\begin{bmatrix}\varrho_0\dfrac{\partial\widetilde{\varepsilon}(\theta,w_\gamma)}{\partial\theta} & \varrho_0\dfrac{\partial\widetilde{\varepsilon}(\theta,w_\gamma)}{\partial w_\alpha}\\ 0 & \tau\delta^\alpha{}_\gamma\end{bmatrix} \neq 0 \quad \text{for all } \theta, w_\gamma, \tag{3.15}$$

which corresponds to (3.11), and which simply means that internal energy ε depends on temperature θ monotonically for each w_γ.

The system (3.14) can be written as an overdetermined system of conservation equations ($N+1$ equations for N unknowns)

$$\begin{gathered}\partial_t g^{0\Lambda}(u^K) + \partial_\alpha g^{\alpha\Lambda}(u^K) = b^\Lambda(u^K, t, X^\gamma),\\ \Lambda = 1,2,\ldots,N+1, \qquad K = 1,2,\ldots,N, \qquad \alpha = 1,2,\ldots,m\end{gathered} \tag{3.16}$$

where, in our case, $N=4$, $m=3$ and

$$\begin{aligned}[u^K] &= [\theta, w_\gamma],\\ [g^{0\Lambda}(u^K)] &= [\varrho_0\widetilde{\varepsilon}(\theta, w_\beta), \tau w_\gamma, \varrho_0\widetilde{\eta}(\theta, w_\beta)],\\ [g^{\alpha\Lambda}(u^K)] &= \left[\widetilde{q}^\alpha(\theta, w_\beta), -f_1(\theta)\delta^\alpha{}_\gamma, \frac{1}{\theta}\widetilde{q}^\alpha(\theta, w_\beta)\right],\\ [b^\Lambda(u^K, t, X^\alpha)] &= [\varrho_0\widetilde{r}(\theta, w_\beta, t, X^\alpha), c_1 w_\gamma, \varrho_0\widetilde{\sigma}(\theta, w_\beta, t, X^\alpha)],\\ \gamma,\ \beta &= 1,2,3.\end{aligned} \tag{3.17}$$

Performing the respective differentiation we rewrite the system (3.14) in a matrix form

$$\begin{gathered}\mathcal{A}^{0\Lambda}{}_M(u^K)\partial_t u^M + \mathcal{A}^{\alpha\Lambda}{}_M(u^K)\partial_\alpha u^M = b^\Lambda(u^K, t, X^\gamma),\\ \mathcal{A}^{0\Lambda}{}_M(u^K) = \frac{\partial g^{0\Lambda}(u^K)}{\partial u^M}, \qquad \mathcal{A}^{\alpha\Lambda}{}_M(u^K) = \frac{\partial g^{\alpha\Lambda}(u^K)}{\partial u^M},\end{gathered} \tag{3.18}$$

with the 5×4 matrices $[\mathcal{A}^{0\Lambda}{}_M]$, $[\mathcal{A}^{\alpha\Lambda}{}_M]$ of the form

$$\begin{gathered}\left[\mathcal{A}^{0\Lambda}{}_M\right] = \left[\frac{\partial g^{0\Lambda}}{\partial u^M}\right] = \begin{bmatrix} \varrho_0\dfrac{\partial\widetilde{\varepsilon}}{\partial\theta} & \varrho_0\dfrac{\partial\widetilde{\varepsilon}}{\partial w_1} & \varrho_0\dfrac{\partial\widetilde{\varepsilon}}{\partial w_2} & \varrho_0\dfrac{\partial\widetilde{\varepsilon}}{\partial w_3}\\ 0 & \tau & 0 & 0\\ 0 & 0 & \tau & 0\\ 0 & 0 & 0 & \tau\\ \varrho_0\dfrac{\partial\widetilde{\eta}}{\partial\theta} & \varrho_0\dfrac{\partial\widetilde{\eta}}{\partial w_1} & \varrho_0\dfrac{\partial\widetilde{\eta}}{\partial w_2} & \varrho_0\dfrac{\partial\widetilde{\eta}}{\partial w_3}\end{bmatrix},\\ \left[\mathcal{A}^{\alpha\Lambda}{}_M\right] = \left[\frac{\partial g^{\alpha\Lambda}}{\partial u^M}\right] = \begin{bmatrix} \dfrac{\partial\widetilde{q}^\alpha}{\partial\theta} & \dfrac{\partial\widetilde{q}^\alpha}{\partial w_1} & \dfrac{\partial\widetilde{q}^\alpha}{\partial w_2} & \dfrac{\partial\widetilde{q}^\alpha}{\partial w_3}\\ -\delta^\alpha{}_1 f_1'(\theta) & 0 & 0 & 0\\ -\delta^\alpha{}_2 f_1'(\theta) & 0 & 0 & 0\\ -\delta^\alpha{}_3 f_1'(\theta) & 0 & 0 & 0\\ -\dfrac{1}{\theta^2}\widetilde{q}^\alpha + \dfrac{1}{\theta}\dfrac{\partial\widetilde{q}^\alpha}{\partial\theta} & \dfrac{1}{\theta}\dfrac{\partial\widetilde{q}^\alpha}{\partial w_1} & \dfrac{1}{\theta}\dfrac{\partial\widetilde{q}^\alpha}{\partial w_2} & \dfrac{1}{\theta}\dfrac{\partial\widetilde{q}^\alpha}{\partial w_3}\end{bmatrix}.\end{gathered} \tag{3.19}$$

In order to investigate the MDR for the system (3.16), (3.17), we introduce the following notation for y_Λ, $[y_\Lambda] = [\lambda, z^\gamma, \mu]$ and require the existence of 5 functions $\lambda(\theta, w_\gamma)$, $z^\beta(\theta, w_\gamma)$ and $\mu(\theta, w_\gamma)$, not all identically zero, such that the identities

$$\begin{gathered}\lambda[\varrho_0\partial_t\widetilde{\varepsilon} + \partial_\alpha\widetilde{q}^\alpha] + z^\gamma\{\tau\partial_t w_\gamma - \partial_\alpha[f_1(\theta)\delta^\alpha_\gamma]\} + \mu\left[\varrho_0\partial_t\widetilde{\eta} + \partial_\alpha\left(\frac{1}{\theta}\widetilde{q}^\alpha\right)\right] \equiv 0,\\ \varrho_0\lambda\widetilde{r} + c_1 z^\gamma w_\gamma + \varrho_0\mu\widetilde{\sigma} \equiv 0,\end{gathered} \tag{3.20}$$

hold for all $[\theta, w_\gamma]$. Solving the MDR $(3.2)_2$, (3.3) for (3.20), we obtain the following relations

$$\begin{gathered}\lambda\frac{\partial\widetilde{\varepsilon}}{\partial\theta} + \mu\frac{\partial\widetilde{\eta}}{\partial\theta} = 0,\\ \lambda\frac{\partial\widetilde{\varepsilon}}{\partial w_\gamma} + \mu\frac{\partial\widetilde{\eta}}{\partial w_\gamma} + z^\gamma\frac{\tau}{\varrho_0} = 0,\\ \left(\lambda + \frac{\mu}{\theta}\right)\frac{\partial\widetilde{q}^\alpha}{\partial\theta} - \frac{\mu}{\theta^2}\widetilde{q}^\alpha - z^\alpha f_1'(\theta) = 0,\\ \left(\lambda + \frac{\mu}{\theta}\right)\frac{\partial\widetilde{q}^\alpha}{\partial w_\gamma} = 0.\end{gathered} \tag{3.21}$$

According to $(3.21)_{1,2}$, the differential of entropy $d\widetilde{\eta}$ can be expressed in terms of the differentials $d\widetilde{\varepsilon}$ and dw_γ

$$\begin{aligned}\mu d\widetilde{\eta} = \mu\frac{\partial\widetilde{\eta}}{\partial\theta}d\theta + \mu\frac{\partial\widetilde{\eta}}{\partial w_\gamma}dw_\gamma &= -\lambda\frac{\partial\widetilde{\varepsilon}}{\partial\theta}d\theta - \lambda\frac{\partial\widetilde{\varepsilon}}{\partial w_\gamma}dw_\gamma - z_\gamma\frac{\tau}{\varrho_0}dw_\gamma\\ &= -\lambda d\widetilde{\varepsilon} - z_\gamma\frac{\tau}{\varrho_0}dw_\gamma,\end{aligned} \tag{3.22}$$

and rearranged to the form of generalized Gibbs relation

$$-\frac{\mu}{\lambda}d\widetilde{\eta} = d\widetilde{\varepsilon} + \frac{\tau}{\varrho_0}\frac{1}{\lambda}z_\gamma dw_\gamma, \tag{3.23}$$

which shows that the factor $-\mu/\lambda$ corresponds to the temperature θ. This fact also follows from $(3.21)_4$.

3.4. Family of solutions of the MDR

In view of (3.21), we obtain

$$\begin{aligned}\theta &= -\frac{\mu}{\lambda},\\ z^\gamma &= -\lambda\frac{\varrho_0}{\tau}\left(\frac{\partial\widetilde{\varepsilon}}{\partial w_\gamma} - \theta\frac{\partial\widetilde{\eta}}{\partial w_\gamma}\right) = -\lambda\frac{\varrho_0}{\tau}\frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma} = \mu\frac{\varrho_0}{\tau}\frac{1}{\theta}\left(\frac{\partial\widetilde{\varepsilon}}{\partial w_\gamma} - \theta\frac{\partial\widetilde{\eta}}{\partial w_\gamma}\right)\\ &= \mu\frac{\varrho_0}{\tau}\frac{1}{\theta}\frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma} = \lambda\frac{1}{\theta f_1'(\theta)}\widetilde{q}^\gamma = -\mu\frac{1}{\theta^2 f_1'(\theta)}\widetilde{q}^\gamma,\end{aligned} \tag{3.24}$$

where the free energy $\widetilde{\Psi}_C$ is introduced by (2.8).

It follows from (3.24) that the family of solutions of the MDR can be written as

$$(3.25)\qquad [y_\Lambda] = -\lambda\left[-1,\ \frac{\varrho_0}{\tau}\frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma},\ \theta\right] = -\mu\left[\frac{1}{\theta},\ -\frac{1}{\theta}\frac{\varrho_0}{\tau}\frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma},\ -1\right],$$

where λ and μ are arbitrary functions of $[\theta, w_\gamma]$ and $\lambda=-1$, $[\widehat{y}_\Lambda]=\left[-1, \frac{\varrho_0}{\tau}\frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma}, \theta\right]$ corresponds to the case when the equation of balance of energy is treated as the additional conservation equation, while $\mu = -1$, $[\breve{y}_\Lambda] = \left[\frac{1}{\theta}, -\frac{1}{\theta}\frac{\varrho_0}{\tau}\frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma}, -1\right]$ corresponds to the case when the equation of balance of entropy is treated as the additional conservation equation.

3.5. Restrictions on constitutive functions

3.5.1. General restrictions. It follows from (3.21), (3.24) that constitutive functions $\widetilde{\varepsilon}$, $\widetilde{\eta}$ and $\widetilde{q}^\alpha$ must satisfy the following relations

$$(3.26)\qquad \begin{aligned} \widetilde{q}^\gamma &= -\frac{\varrho_0\theta f_1'(\theta)}{\tau}\left(\frac{\partial\widetilde{\varepsilon}}{\partial w_\gamma} - \theta\frac{\partial\widetilde{\eta}}{\partial w_\gamma}\right) = -\frac{\varrho_0\theta f_1'(\theta)}{\tau}\frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma},\\ \frac{\partial\widetilde{\eta}}{\partial\theta} &= \frac{1}{\theta}\frac{\partial\widetilde{\varepsilon}}{\partial\theta}, \end{aligned}$$

and

$$(3.27)\qquad \widetilde{\eta} = -\frac{\partial\widetilde{\Psi}_C}{\partial\theta},\qquad \widetilde{\varepsilon} = \widetilde{\Psi}_C - \theta\frac{\partial\widetilde{\Psi}_C}{\partial\theta}.$$

Identity $(3.20)_2$ of the MDR together with (3.24), (3.26) yields

$$(3.28)\qquad \varrho_0\frac{\widetilde{r}}{\theta} + c_1\frac{1}{\theta^2 f_1'(\theta)}\widetilde{q}^\gamma w_\gamma - \varrho_0\widetilde{\sigma} \equiv 0,$$

and, in view of (3.13), it gives the entropy production inequality in the following form:

$$(3.29)\qquad \frac{c_1}{\theta^2 f_1'(\theta)}\widetilde{q}^\gamma w_\gamma = -\frac{\varrho_0}{\tau}\frac{c_1}{\theta}\left(\frac{\partial\widetilde{\varepsilon}}{\partial w_\gamma} - \theta\frac{\partial\widetilde{\eta}}{\partial w_\gamma}\right)w_\gamma = -\frac{\varrho_0}{\tau}\frac{c_1}{\theta}\frac{\partial\widetilde{\Psi}_C}{\partial w_\gamma}w_\gamma \geq 0.$$

Of course, the restrictions (3.26) – (3.29) on constitutive functions $\widetilde{q}(\theta, \mathbf{w})$, $\widetilde{\eta}(\theta, \mathbf{w})$ and $\widetilde{\varepsilon}(\theta, \mathbf{w})$ derived here with the aid of the MDR coincide with the corresponding restrictions (2.9), (2.15), (2.16) obtained in [22, 4] as direct consequences of the Clausius – Duhen entropy inequality if we substitute $f_1(\theta) = -\tau f(\theta)$, $\mathbf{b}(\theta, \mathbf{w}) = \tau^{-1}\mathbf{c}(\theta, \mathbf{w}) = \tau^{-1}\mathbf{w}$.

3.5.2. Additional constitutive assumptions

A. $\tilde{\varepsilon}(\theta, w_\gamma) = \tilde{\varepsilon}_0(\theta)$

In Secs. 2.2.3, 2.2.4, 2.2.5, the the additional constitutive assumptions concerning the form of $\tilde{\mathbf{q}}(\theta, \mathbf{w})$ and $\tilde{\varepsilon}(\theta, \mathbf{w})$ were analysed in detail. In those considerations the condition (2.28) ($\tilde{\varepsilon}(\theta, \mathbf{w}) = \tilde{\varepsilon}_0(\theta)$) was taken as supplementary to the condition (2.18) ($\tilde{\mathbf{q}} = \mathbf{S}(\theta)\mathbf{w}$). In this section, the consequences of the assumption (2.28) are studied without prior asumption (2.18).

As a consequence of the assumption (2.28), the entropy $\tilde{\eta}(\theta, w_\gamma)$ splits into two parts

$$\tilde{\eta}(\theta, w^\gamma) = \stackrel{\approx}{\eta}_0(\theta) + \stackrel{\approx}{\eta}_1(w_\gamma), \tag{3.30}$$

since, according to (3.26), we have in this case

$$\frac{\partial \tilde{\eta}(\theta, w^\gamma)}{\partial \theta} = \frac{1}{\theta}\tilde{\varepsilon}_0'(\theta), \tag{3.31}$$

and therefore

$$\frac{\partial^2 \tilde{\eta}(\theta, w^\gamma)}{\partial \theta\, \partial w_\alpha} = 0. \tag{3.32}$$

Hence, the free energy $\tilde{\Psi}_C$ given by (2.8) also splits,

$$\begin{aligned} \tilde{\Psi}_C(\theta, w_\gamma) &= \tilde{\Psi}_{C0}(\theta) - \theta \stackrel{\approx}{\eta}_1(w_\gamma), \\ \tilde{\Psi}_{C0}(\theta) &= \stackrel{\approx}{\varepsilon}(\theta) - \theta \stackrel{\approx}{\eta}_0(\theta). \end{aligned} \tag{3.33}$$

Moreover, the assumption (2.28) together with its consequences (3.30), (3.33) yields

$$\tilde{q}^\gamma = \frac{\varrho_0 \theta^2 f_1'(\theta)}{\tau} \frac{\partial \stackrel{\approx}{\eta}_1(w_\alpha)}{\partial w_\gamma}. \tag{3.34}$$

Hence, the heat flux vector $\tilde{\mathbf{q}} = [\tilde{q}^\alpha]$ is collinear with $\nabla_{\mathbf{w}} \stackrel{\approx}{\eta}_1 = \left[\dfrac{\partial \stackrel{\approx}{\eta}_1}{\partial w_\alpha}\right]$.

B. $\tilde{\varepsilon}(\theta, w_\gamma) = \tilde{\varepsilon}_0(\theta)$ *for isotropic body*

If it is additionally assumed that the body is isotropic then

$$\stackrel{\approx}{\eta}_1(w_\alpha) = \stackrel{\approx}{\eta}_1(w^\alpha w_\alpha), \tag{3.35}$$

$$\tilde{r}(\theta, w_\gamma) = \stackrel{\approx}{r}(\theta, w_\gamma w^\gamma, t, X^\alpha), \tag{3.36}$$

where $\overset{\approx}{\eta}_1$ is a function of one variable and, as a consequence, (3.30), (3.33) and (3.34) take the form

$$
\begin{aligned}
\widetilde{\eta}(\theta, w_\alpha) &= \overset{\approx}{\eta}_0(\theta) + \overset{\approx}{\eta}_1(w^\alpha w_\alpha), \\
\widetilde{\Psi}_C(\theta, w_\alpha) &= \widetilde{\Psi}_{C0}(\theta) - \theta\, \overset{\approx}{\eta}_1(w^\alpha w_\alpha), \qquad (3.37) \\
\widetilde{q}^\gamma &= 2\frac{\varrho_0 \theta^2 f_1'(\theta)}{\tau}\, \overset{\approx}{\eta}{}_1'(w_\alpha w^\alpha) w^\gamma.
\end{aligned}
$$

Therefore, the heat flux vector is collinear with internal state variable $\mathbf{w}$ (but does not depend linearly on $\mathbf{w}$) in this case and, in view of (3.37), (3.36) and (3.29), the following entropy production inequality is obtained

$$
2\frac{\varrho_0}{\tau} c_1\, \overset{\approx}{\eta}{}_1'(w_\alpha w^\alpha) w_\gamma w^\gamma \geq 0. \qquad (3.38)
$$

The model of a rigid conductor of heat corresponding to the constitutive assumption (3.30), (3.35) can be regarded as a particular generalization of the model developed in [4] in a sense that, like in [4], internal energy depends only on the temperature and the body is isotropic, but a more general dependence $(3.37)_3$ of $\widetilde{\mathbf{q}}$ on $\mathbf{w}$ is assumed instead of (2.24).

C. $\widetilde{\varepsilon}(\theta, w_\gamma) = \widetilde{\varepsilon}_0(\theta)$, *isotropic body,* $\overset{\approx}{\eta}_1$ *quadratic in* $\mathbf{w}$

Assuming a particular form of function $\overset{\approx}{\eta}_1$ namely $\overset{\approx}{\eta}_1(\xi) = -\frac{1}{2}c_2\xi$, $c_2 > 0$, in (3.35), we obtain from (3.37), (3.38),

$$
\begin{aligned}
\widetilde{\eta}(\theta, w_\alpha) &= \overset{\approx}{\eta}_0(\theta) - \frac{1}{2}c_2 w_\gamma w^\gamma, \\
\widetilde{q}^\gamma(\theta, w_\alpha) &= -\frac{\varrho_0\theta^2 f_1'(\theta) c_2}{\tau} w^\gamma, \qquad (3.39) \\
&- \frac{c_1 c_2}{\tau} w^\alpha w_\alpha \geq 0.
\end{aligned}
$$

In this case $\widetilde{\mathbf{q}}$ is linear in $\mathbf{w}$ and it follows from (2.42) that the model developed in [4] is obtained if c_1 is replaced by $\dfrac{c_2\theta^2[f_1'(\theta)]^2}{\tau\kappa(\theta)}$.

Appendix

A.1. Results for the system (3.7) in normal form

For the system (3.7) in normal (Cauchy) form

$$
\mathbf{f}^0(\mathbf{u}) = \mathbf{u}, \qquad (A.1)
$$

KOSIŃSKI [30] has formulated the following Lemma 0 (the proof is given in [31]):

Every Lipshitz continuous solution $\mathbf{u}(t, \mathbf{X})$ of the system (3.7), (A.1) satisfies (3.8) if and only if

$$\nabla_{\mathbf{u}} h^0(\mathbf{u}) \nabla_{\mathbf{u}} \mathbf{f}^\alpha(\mathbf{u}) = \nabla_{\mathbf{u}} h^\alpha(\mathbf{u}) \tag{A.2}$$

and

$$\nabla_{\mathbf{u}} h^\alpha(\mathbf{u}) \cdot \mathbf{d}(\mathbf{u}, \cdot, \cdot) - \mu(\mathbf{u}, \cdot, \cdot) \geq 0. \tag{A.3}$$

Then, as a consequence, he proved [30, 31] (by contracting both sides of (3.7) with $\nabla_{\mathbf{u}} h^0(\mathbf{u})$) the following:

COROLLARY. Every Lipschitz continuous solution $\mathbf{u}(t, \mathbf{X})$ of the system (3.7), (A.1) consistent with (3.8) satisfies the additional conservation equation (3.11), (3.12) almost everywhere.

By "consistency" we mean that (A.2), (A.3) hold. The system (2.52) considered in this paper is not in normal (Cauchy) form and therefore Lemma 0 and Corollary do not directly apply to our case.

A.2. Generalization of Lemma 0 and Corollary to the systems (3.7) not in normal form

Assumption (3.11) implies that the mapping $\mathbf{f}^0 : \mathcal{O} \to \mathbb{R}^N$ is invertible and therefore $\mathbf{v} := \mathbf{f}^0$ can be taken as new dependent variable related to $\mathbf{u}$ by the inverse mapping $\mathbf{f}^{0^{-1}} : \mathbb{R}^N \to \mathcal{O}$, $\mathbf{u}(\mathbf{v}) = \mathbf{f}^{0^{-1}}(\mathbf{v})$. In new dependent variables, system (3.7) and the inequality (3.8) assume the following form:

$$\partial_t \mathbf{v} + \partial_\alpha \widehat{\mathbf{f}}^\alpha(\mathbf{v}) = \widehat{\mathbf{d}}(\mathbf{v}, t, \mathbf{x}), \tag{A.4}$$

$$\partial_t \widehat{h}^0(\mathbf{v}) + \partial_\alpha \widehat{h}^\alpha(\mathbf{v}) \geq \widehat{\mu}(\mathbf{v}, t, \mathbf{X}), \tag{A.5}$$

where

$$\begin{aligned}
\widehat{\mathbf{f}}^\alpha(\mathbf{v}) &= \mathbf{f}^\alpha(\mathbf{u}(\mathbf{v})) = \mathbf{f}^\alpha(\mathbf{f}^{0^{-1}}(\mathbf{v})),\\
\widehat{\mathbf{d}}(\mathbf{v}, t, \mathbf{X}) &= \mathbf{d}(\mathbf{u}(\mathbf{v}), t, \mathbf{x}) = \mathbf{d}(\mathbf{f}^{0^{-1}}(\mathbf{v}), t, \mathbf{X}),\\
\widehat{h}^0(\mathbf{v}) &= h^0(\mathbf{u}(\mathbf{v})) = h^0(\mathbf{f}^{0^{-1}}(\mathbf{v})),\\
\widehat{h}^\alpha(\mathbf{v}) &= h^\alpha(\mathbf{u}(\mathbf{v})) = h^\alpha(\mathbf{f}^{0^{-1}}(\mathbf{v})),\\
\widehat{\mu}(\mathbf{v}, t, \mathbf{X}) &= \mu(\mathbf{u}(\mathbf{v}), t, \mathbf{x}) = \mu(\mathbf{f}^{0^{-1}}(\mathbf{v}), t, \mathbf{X}).
\end{aligned} \tag{A.6}$$

Since (A.4) is in normal (Cauchy) form, Lemma 0 and Corollary do apply to (A.4), (A.5) and, as a consequence, the necessary and sufficient conditions for the Lipschitz continuous solutions $\mathbf{v}(t, \mathbf{X})$ of (A.4) to satisfy (A.5) are

$$\begin{aligned}
\nabla_{\mathbf{v}} \widehat{h}^0(\mathbf{v}) \nabla_{\mathbf{v}} \widehat{\mathbf{f}}^0(\mathbf{v}) &= \nabla_{\mathbf{v}} \widehat{h}^\alpha(\mathbf{v}),\\
\nabla_{\mathbf{v}} \widehat{h}^0(\mathbf{v}) \cdot \widehat{\mathbf{d}}(\mathbf{v}, \cdot, \cdot) - \widehat{\mu}(\mathbf{v}, \cdot, \cdot) &\geq 0,
\end{aligned} \tag{A.7}$$

and the additional conservation equation

$$\partial_t \widehat{h}^0(\mathbf{v}) + \partial_\alpha \widehat{h}^\alpha(\mathbf{v}) = \nabla_{\mathbf{v}} \widehat{\eta}(\mathbf{v}) \cdot \widehat{\mathbf{d}}(\mathbf{v}, t, \mathbf{X}), \tag{A.8}$$

where $\nabla_{\mathbf{v}}$ denotes differentiation with respect to $\mathbf{v}$, is satisfied by Lipschitz continuous solutions of (A.4). Taking into account (A.6) and employing the chain rule, we obtain from (A.7)

$$\text{(A.9)}\qquad \begin{aligned} &[\nabla_{\mathbf{u}} h^0(\mathbf{u}(\mathbf{v}))\nabla_{\mathbf{v}}\mathbf{u}(\mathbf{v})][\nabla_{\mathbf{u}}\mathbf{f}^\alpha(\mathbf{u}(\mathbf{v}))\nabla_{\mathbf{u}}\mathbf{u}(\mathbf{v})] = \nabla_{\mathbf{v}} h^\alpha(\mathbf{u}(\mathbf{v}))\nabla_{\mathbf{v}}\mathbf{u}(\mathbf{v}),\\ &\nabla_{\mathbf{u}} h^0(\mathbf{u}(\mathbf{v}))\nabla_{\mathbf{v}}\mathbf{u}(\mathbf{v})\cdot\mathbf{d}(\mathbf{u}(\mathbf{v}),\cdot,\cdot) - \mu(\mathbf{u}(\mathbf{v}),\cdot,\cdot) \geq 0. \end{aligned}$$

Returning in (A.9) to original variable $\mathbf{u}$ and taking into account that

$$\text{(A.10)}\qquad \nabla_{\mathbf{v}}\mathbf{u}(\mathbf{v}) = \nabla_{\mathbf{v}}\mathbf{f}^{0^{-1}}(\mathbf{v}) = [\nabla_{\mathbf{u}} f^0(\mathbf{u}(\mathbf{v}))]^{-1} = [B^0(\mathbf{u}(\mathbf{v}))]^{-1},$$

we obtain

$$\text{(A.11)}\qquad \begin{aligned} &\nabla_{\mathbf{u}} h^0(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1}\nabla_{\mathbf{u}}\mathbf{f}^\alpha(\mathbf{u}) = \nabla_{\mathbf{u}} h^\alpha(\mathbf{u}),\\ &\nabla_{\mathbf{u}} h^0(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1}\cdot\mathbf{d}(\mathbf{u},\cdot,\cdot) - \mu(\mathbf{u},\cdot,\cdot) \geq 0. \end{aligned}$$

Hence, the relations (A.11) are generalizations of the relations (A.2), (A.3) of Lemma 0 to the case of the system (3.7) which is not in normal (Cauchy) form and satisfies the condition (3.11) and, consequently, every Lipschitz continuous solution of the system (3.7), (3.11) satisfies (3.8) iff (A.11) holds. Similarly, taking into account (A.6), $(\text{A.11})_1$, (A.10), employing the chain rule and then returning into original dependent variable $\mathbf{u}$, we obtain from (A.8) the following conservation equation

$$\text{(A.12)}\qquad \partial_t h^0(\mathbf{u}) + \partial_\alpha h^\alpha(\mathbf{u}) = \nabla_{\mathbf{u}} h^0(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1}\cdot\mathbf{d}(\mathbf{u},t,\mathbf{X}),$$

which corresponds to contraction of both sides of (3.7) with $\nabla_{\mathbf{u}} h^0(\mathbf{u})[B^0(\mathbf{u})]^{-1}$. Hence, every Lipschitz continuous solution $\mathbf{u}(t,\mathbf{X})$ of the system (3.7) satysfying (3.11) and consitent with inequality (3.8) (that is, for which (A.11) holds) satisfies the additional conservation equation (3.12) with

$$\text{(A.13)}\qquad \chi(\mathbf{u},t,\mathbf{X}) = \nabla_{\mathbf{u}} h^0(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1}\cdot\mathbf{d}(\mathbf{u},t,\mathbf{X})$$

almost everywhere, and

$$\text{(A.14)}\qquad \chi(\mathbf{u},\cdot,\cdot) - \mu(\mathbf{u},\cdot,\cdot) \geq 0,$$

according to $(\text{A.11})_2$.

A.3. Consistency with the MDR

The system (3.7) together with the additional conservation equation (A.12), (A.13), (A.14) can be considered as the system of $N+1$ conservation equations for N unknowns (3.1) which, without loss of generality, can be written in the form

$$
\text{(A.15)}\qquad
\begin{aligned}
\mathbf{g}^0(\mathbf{u}) &= [g^{0\Lambda}(u^K)] = \begin{bmatrix}\mathbf{f}^0(\mathbf{u})\\ h^0(\mathbf{u})\end{bmatrix} = \begin{bmatrix}f^{0I}(u^K)\\ h^0(u^K)\end{bmatrix},\\
\mathbf{g}^\alpha(\mathbf{u}) &= [g^{\alpha\Lambda}(u^K)] = \begin{bmatrix}\mathbf{f}^\alpha(\mathbf{u})\\ h^\alpha(\mathbf{u})\end{bmatrix} = \begin{bmatrix}f^{\alpha I}(u^K)\\ h^\alpha(u^K)\end{bmatrix},\\
\mathbf{b}(\mathbf{u},\cdot,\cdot) &= [b^\Lambda(u^K,\cdot,\cdot)] = \begin{bmatrix}\mathbf{d}(\mathbf{u},\cdot,\cdot)\\ \chi(\mathbf{u},\cdot,\cdot)\end{bmatrix} = \begin{bmatrix}d^I(u^K,\cdot,\cdot)\\ \chi(u^K,\cdot,\cdot)\end{bmatrix},\\
& I,K = 1,2,\ldots,N, \qquad \Lambda = 1,2,\ldots,N+1.
\end{aligned}
$$

It follows from (A.11), (A.12), (A.13), (3.9), (3.10) that the system (3.1), (A.15) corresponding to (3.7), (A.12), (A.13) satisfies the conditions (3.6) of the MDR for

$$
\text{(A.16)}\qquad \mathbf{y}^T(\mathbf{u}) = [\nabla_{\mathbf{u}} h^0(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1}, -1] = [\mathbf{k}^{0^T}(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1}, -1]
$$

since, in the case of (A.15), the $(N+1)\times N$ matrices (3.1.4) $\mathcal{A}^0(\mathbf{u})$, $\mathcal{A}^\alpha(\mathbf{u})$ assume the form

$$
\text{(A.17)}\qquad
\begin{aligned}
\mathcal{A}^0(\mathbf{u}) &= \begin{bmatrix}\nabla_{\mathbf{u}}\mathbf{f}^0(\mathbf{u})\\ \nabla_{\mathbf{u}} h^0(\mathbf{u})\end{bmatrix} = \begin{bmatrix}\mathbf{B}^0(\mathbf{u})\\ \mathbf{k}^{0^T}(\mathbf{u})\end{bmatrix},\\
\mathcal{A}^\alpha(\mathbf{u}) &= \begin{bmatrix}\nabla_{\mathbf{u}}\mathbf{f}^\alpha(\mathbf{u})\\ \nabla_{\mathbf{u}}\mathbf{h}^\alpha(\mathbf{u})\end{bmatrix} = \begin{bmatrix}\mathbf{B}^\alpha(\mathbf{u})\\ \mathbf{k}^{\alpha^T}(\mathbf{u})\end{bmatrix},
\end{aligned}
$$

and (A.13) holds. Therefore, the system (3.7), (A.12), (A.13) satisfies the MDR provided that the conditions (A.11) hold. In this way we have proved that if the system (3.7) satisfies the condition (3.11) and every Lipschitz continuous solution of (3.7) satisfies the inequality (3.8), then the system (3.7), (3.12) (3.13) of $N+1$ conservation equations satisfies the MDR.

For the converse, we assume that the system (3.1), (A.15) satisfies the condition (3.11) and the MDR, and that the inequality (3.13) is satisfied. The MDR implies that there exist such $\mathbf{y}(\mathbf{u})$ with the components not all identically zero which satisfy (3.6), $(3.2)_2$ for $\mathbf{b}(\mathbf{u})$ given by $(\text{A.15})_3$ and for $\mathcal{A}^0(\mathbf{u})$, $\mathcal{A}^\alpha(\mathbf{u})$ given by (A.17). For convenience, we denote $y_{N+1}(\mathbf{u}^K) = \lambda(\mathbf{u})$ and $y_I(\mathbf{u}) = p_I(\mathbf{u})$, $I = 1,2,\ldots,N$, $\mathbf{p}^T(\mathbf{u}) = [p_I(\mathbf{u})]$. Then, it follows from (3.6), (A.15), (A.17) that

$$
\text{(A.18)}\qquad
\begin{aligned}
&\mathbf{p}^T(\mathbf{u})\mathbf{B}^0(\mathbf{u}) = -\lambda(\mathbf{u})\mathbf{k}(\mathbf{u}), \qquad \mathbf{p}^T(\mathbf{u})\mathbf{B}^\alpha(\mathbf{u}) = -\lambda(\mathbf{u})\mathbf{k}(\mathbf{u}),\\
&\mathbf{p}^T(\mathbf{u})\mathbf{d}(\mathbf{u},\cdot,\cdot) = -\chi(\mathbf{u},\cdot,\cdot).
\end{aligned}
$$

The condition (3.11) implies that $\lambda(\mathbf{u})$ is not identically zero and therefore both sides of (A.18) can be multiplied by $-[\lambda(\mathbf{u})]^{-1}$. Denoting $l_I(u^K) = -[\lambda(\mathbf{u})]^{-1}p_I(\mathbf{u})$ and taking into account (3.9), (3.10) we obtain from (A.18)

$$
\text{(A.19)}\qquad \mathbf{l}^T(\mathbf{u}) = \mathbf{k}^0(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1} = \nabla_{\mathbf{u}} h^0(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1}
$$

and the following relations corresponding to (A.11):

$$\text{(A.20)}\qquad \begin{aligned} \nabla_{\mathbf{u}} h^0(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1}\nabla_{\mathbf{u}}\mathbf{f}^\alpha(\mathbf{u}) &= \nabla_{\mathbf{u}} h^\alpha(\mathbf{u}),\\ \nabla_{\mathbf{u}} h^0(\mathbf{u})[\mathbf{B}^0(\mathbf{u})]^{-1}\mathbf{d}(\mathbf{u},\cdot,\cdot) &= \chi(\mathbf{u},\cdot,\cdot). \end{aligned}$$

It then follows from the generalization of Lemma 0 that every Lipschitz continuous solution of (3.7), (3.11) satisfies the inequality (3.8), in view of (3.13), what completes the proof.

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