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A unified theory of representations for scalar-, vector- and second order tensor-valued anisotropic functions of vectors and second order tensors

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A SUBSTANTIAL generalization of Lokhin–Sedov–Boehler–Liu's isotropic extension method for representations of anisotropic tensor functions is suggested. It is shown that every scalar-, vector- and second order tensor-valued anisotropic tensor function with vector and second order tensor variables can be extended as an isotropic tensor function merely with augmented vector and second order tensor variables through some simple polynomial vector-valued and second order tensor-valued invariant tensor functions characterizing the anisotropy group. This result circumvents the difficulty involved in the usual direct generalization of the aforementioned LSBL method due to the introduction of structural tensor variables of order higher than two, and enables us to derive complete representations for various types of anisotropic tensor functions of vectors and second order tensors directly from the well-known results for isotropic tensor functions of vectors and second order tensors. All anisotropy groups describing symmetries of solid materials, including the thirty-two crystal classes and all infinitely many noncrystal classes, are considered.

Notations

T_k – the space of kth-order tensors. In particular, $T_0 \equiv R$ (the reals), $T_1 \equiv V$,

Orth – the full orthogonal group, being a subset of T_2,

Skw, Sym – the skewsymmetric and symmetric subspaces of T_2,

$D = V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$; $\quad E = V^r \times \mathrm{Skw}^s \times \mathrm{Sym}^t$,

$\mathbf{X} = (\mathbf{v}_\alpha; \mathbf{W}_\theta; \mathbf{A}_\sigma) \equiv (\mathbf{v}_1, \ldots, \mathbf{v}_a; \mathbf{W}_1, \ldots, \mathbf{W}_b; \mathbf{A}_1, \ldots, \mathbf{A}_c) \in D$,

$(\mathbf{Q} * \mathbf{T})_{i_1 \ldots i_k} = Q_{i_1 j_1} \cdots Q_{i_k j_k} T_{j_1 \ldots j_k}$, $\quad (\mathbf{Q} \in \mathrm{Orth}, \mathbf{T} \in T_k)$, $\ \mathbf{Q} * c = c$, $\ c \in R$,

$\Gamma(\mathbf{T}) = \{\mathbf{Q} \in \mathrm{Orth} \mid \mathbf{Q} * \mathbf{T} = \mathbf{T}\}$,

$\mathbf{Q} * \mathbf{X} = (\mathbf{Q} * \mathbf{v}_\alpha; \mathbf{Q} * \mathbf{W}_\theta; \mathbf{Q} * \mathbf{A}_\sigma)$,

$G(D, M) = \{\mathbf{F} : D \to M \subset T_k \mid \mathbf{F}(\mathbf{Q} * \mathbf{X}) = \mathbf{Q} * (\mathbf{F}(\mathbf{X})), \forall \mathbf{X} \in D, \mathbf{Q} \in G\} \quad (G \subset \mathrm{Orth})$,

$(\mathbf{Q} * \mathbf{S})(\mathbf{X}) = \mathbf{Q} * (\mathbf{S}(\mathbf{Q}^{\mathrm{T}} * \mathbf{X}))$, $\forall \mathbf{X} \in D \quad (\mathbf{Q} \in \mathrm{Orth},\ \mathbf{S} : D \to E)$,

$\mathbf{S} \cap (\mathbf{Q} * \mathbf{S}) = \{\mathbf{X}_0 \in D \mid (\mathbf{Q} * \mathbf{S})(\mathbf{X}_0) = \mathbf{S}(\mathbf{X}_0)\}$,

$\overset{m}{\otimes} \mathbf{K} = \mathbf{K}_1 \otimes \mathbf{K}_2 \otimes \cdots \otimes \mathbf{K}_m$, $\ \mathbf{K}_1 = \cdots = \mathbf{K}_m = \mathbf{K} \in T_k$,

$(\mathbf{G} \odot \mathbf{Z})_{i_1 \ldots i_p} = G_{i_1 \ldots i_p j_1 \ldots j_q} Z_{j_1 \ldots j_q} \quad (\mathbf{Z} \in T_q,\ \mathbf{G} \in T_{p+q})$,

$\mathbf{G} \odot \mathbf{v} = \mathbf{G}\mathbf{v}$, $\mathbf{v} \in V$; $\mathbf{G} \odot \mathbf{B} = \mathbf{G} : \mathbf{B}$, $\mathbf{B} \in T_2$,

$D(\mathbf{u}) = \{x\mathbf{u} \mid x \in R\}^a \times \{x\mathbf{E}\mathbf{u} \mid x \in R\}^b \times \{x\mathbf{I} + y\mathbf{u} \otimes \mathbf{u} \mid x, y \in R\}^c \quad (\mathbf{0} \neq \mathbf{u} \in V)$,

$\mathbf{e}, \mathbf{n}$	two orthonormal vectors,
$\mathbf{I}, \mathbf{E}$	the second order identity tensor; the third order Eddington tensor,
$\mathbf{R}_{\mathbf{a}}^{\theta}$	the right-handed rotation through the angle θ about an axis represented by $\mathbf{0} \neq \mathbf{a} \in V$,
$S \setminus T$	the set of all elements that belong to the set S but not to the set T,
$\mathbf{u} \cdot \mathbf{v}$	the scalar product of the vectors $\mathbf{u}, \mathbf{v} \in V$,
$< \mathbf{z}, \mathbf{e} >$	the angle between the vectors $\mathbf{z}$ and $\mathbf{e}$,
$\mathbf{q}(\mathbf{A})$	a vector associated with the symmetry tensor $\mathbf{A} \in \mathrm{Sym}$, refer to (2.9),
$\mathbf{v}^{\circ}$	the perpendicular projection of the vector $\mathbf{v}$ on the $\mathbf{n}$-plane, refer to (2.21).

1. Introduction

SCALAR-, VECTOR- AND SECOND ORDER TENSOR-VALUED FUNCTIONS of vectors and second order tensors provide mathematical models for macroscopic physical behaviour of materials. The principle of material frame-indifference and material symmetry require that such tensor functions modelling material behaviours, i.e. constitutive relations of materials, possess a combined invariance under the material symmetry group. The central problem of theory of representations for tensor functions is to determine general reduced forms of tensor functions that are invariant under various given material symmetry groups and hence, it constitutes a rational basis for a consistent mathematical modelling of complex material behaviours (see RIVLIN [21], TRUESDELL and NOLL [43], MURAKAMI and SAWCZUK [18], TELEGA [42], BOEHLER [7, 8], ERINGEN and MAUGIN [10], KIRAL and ERINGEN [14], BETTEN [3], SMITH [34], and ZHENG [61], *et al.*, for some applications of tensor function representation theory in formulating constitutive equations of materials). In the past decades, representations for isotropic and anisotropic functions of vectors and second order tensors have been extensively studied and many significant results for polynomial and nonpolynomial representations have been obtained (see TRUESDELL and NOLL [43] and SPENCER [38] for the results on polynomial representations up to their respective concerned years; see BOEHLER [7, 8], KIRAL and ERINGEN [14], SMITH [34] and ZHENG [61] for the subsequent development; see RYCHLEWSKI and ZHANG [25] for a comprehensive review and comments). However, most of the established results were confined to *integrity bases* for polynomial scalar-valued functions (see PIPKIN and RIVLIN [20], ADKINS [1, 2], SPENCER and RIVLIN [39, 40, 41], SPENCER [37], SMITH and RIVLIN [36], SMITH [30, 33], SMITH and KIRAL [31], and KIRAL and SMITH [12, 13], *et al.*, for some general results of this aspect; see also SPENCER [38], KIRAL and ERINGEN [14], and SMITH [34] for details). General aspects of representation problems for most types of anisotropic functions remain open, except for isotropic, transversely isotropic and orthotropic functions and for some other particular cases, etc. (see WANG [45], SMITH [32], BOEHLER [5], PENNISI and TROVATO [19], ZHENG [59, 60], JEMIOŁO and TELEGA [11], *et al.*).

The main method in current use for deriving representations of anisotropic functions is the *Lokhin–Sedov–Boehler–Liu isotropic extension method*(1) (see LOKHIN and SEDOV [17], BOEHLER [6, 8] and LIU [16]). It was through Boehler's and Liu's works that this method became known. According to Boehler and Liu, through some vectors and second order tensors characterizing the anisotropy group, an anisotropic function can be extended as an isotropic function with augmented tensor variables and hence, the representation problem for the former can be reduced to that for the latter. For such simple anisotropy groups as transverse isotropy groups and triclinic, monoclinic and rhombic crystal classes, isotropic extension functions merely with vector and second order tensor variables can be established using the above method, as has been shown by BOEHLER [7, 8] and LIU [16]. Therefore, the well-known results for representations for isotropic functions of vectors and second order tensors (see WANG [45], SMITH [32], BOEHLER [5], and PENNISI and TROVATO [19], *et al.*) can be used to derive the desired results for representations for anisotropic functions of vectors and second order tensors relative to the foregoing anisotropy groups. However, it is known that any set of vectors and second order tensors is not enough to characterize any anisotropy group except those mentioned above, since the symmetry group of any vector or second order tensor involves only two-fold and/or ∞-fold symmetry. In view of this, a direct generalization of the aforementioned LSBL method has been suggested (see ZHANG and RYCHLEWSKI [57] and ZHENG and SPENCER [58]; see the monograph by RYCHLEWSKI [22] for a comprehensive and coherent account of this aspect), which realizes isotropic extension of anisotropic functions by means of additional tensor variables of higher order characterizing the anisotropy group. The latter were introduced earlier as *anisotropic tensors* or *structural tensors* by various authors (see SMITH and RIVLIN [35, 36], SIROTIN [28, 29], SEDOV and LOKHIN [26], *et al.*), and shown to be valid for all anisotropy groups. However, for each anisotropy group other than those mentioned before, such direct generalization of LSBL method results in isotropic extension functions whose variables include tensors of order higher than two, and representation problems for them are difficult (see the comments by ZHANG and RYCHLEWSKI [57] and RYCHLEWSKI and ZHANG [25]). In reality, even for the simplest case of this aspect, i.e. the isotropic scalar-valued function of a single fourth-order tensor, such as the elasticity tensor, a complete functional basis has not been obtained until the recent work by this author (see XIAO [52]; see also RYCHLEWSKI [23], BETTEN and HELISCH [4], and BOEHLER, KIRILLOV and ONAT [9], *et al.*, for some other results; see also the comments by RYCHLEWSKI and ZHANG [25], §5 and RYCHLEWSKI [24], §2).

Recently, this author (see XIAO and GUO [46] and XIAO [49]) has made a substantial extension of the above-mentioned LSBL method. It has been shown

(1) It seems that the expression *isotropic extension* was first introduced by RYCHLEWSKI and ZHANG in [25], which was followed in [49].

that through some vector-valued and second order tensor-valued invariant tensor functions, an anisotropic function of vector and second order tensor variables can be extended as an isotropic function whose variables consist merely of vectors and second order tensors, and hence the aforesaid difficulty involved in the aforementioned direct generalization is circumvented. In this paper, basing upon a fundamental isotropic extension theorem for anisotropic functions (see XIAO and GUO [46] and XIAO [49] and below), we shall systematically construct isotropic extension functions merely with augmented vector and second order tensor variables for scalar-, vector- and second order tensor-valued anisotropic functions of vector and second order tensor variables relative to all the thirty-two crystal classes and all noncrystal classes. Employing these results and the well-known results for representations for isotropic functions of vectors and second order tensors, one can readily derive complete or even complete irreducible representations for various types of anisotropic functions (see the recent results by this author [47–51, 53–55]).

The early forms of most of the results given in this paper were reported in a summary by this author (see [49]). In the latter, complete proofs for each presented result were sought and moreover, results for the icosahedral class I_h and the infinitely many noncrystal classes D_{2md} and S_{4m}, where $m = 2, 3, \ldots$, were left open. In this article, we present new results for subgroups of the transverse isotropy group $D_{\infty h}$, which simplify the corresponding results given in [49], and moreover, we provide results for the icosahedral class I_h and for all noncrystal classes D_{2md} and S_{4m}. Complete proofs for all these results will be given.

It should be pointed out that the commonly-considered material symmetric groups in solid mechanics are the five classes of transverse isotropy groups, the thirty-two crystal classes and the full orthogonal group etc. (see, e.g., TRUESDELL and NOLL [43] and SPENCER [38]), since for a long time it has been believed that the just-mentioned orthogonal subgroups seem to exhaust symmetries of all known solids. As a result, one may doubt the reality of any noncrystallographic point group other than those just mentioned in describing symmetry of any real solid. For this, we would call attention to the recent advances in modern crystallography, especially the discovery of *quasi-crystals* (see, e.g., VAINSHTEIN [44] and SENECHAL [27] and the references therein).

2. The fundamental isotropic extension theorem and others

Throughout this paper, vector and tensor mean a three-dimensional vector and tensor. The Schoenflies symbol will be used to denote the orthogonal subgroup classes (see SPENCER [38] and VAINSHTEIN [44] for an account of crystal classes and noncrystal classes). Moreover, M will be used to represent any of the sets R, V, Skw and Sym, unless otherwise indicated.

2.1. The fundamental isotropic extension theorem

The succeeding account will be mainly based on the following fact.

THEOREM A. (*ISOTROPIC EXTENSION THEOREM*) *Let* $G \subset \mathrm{Orth}$ *be an anisotropy group, i.e. an orthogonal subgroup other than the full and proper orthogonal groups. Let* $M \subset T_k$ *be a subspace that is invariant under the group* G. *Moreover, let*

$$\mathbf{S} : D \equiv V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c \to E \equiv V^r \times \mathrm{Skw}^s \times \mathrm{Sym}^t \tag{2.1}$$

be a set of vector-valued and second order tensor-valued functions that are invariant under the group G *and satisfy the following condition*

$$\mathbf{F}(\mathbf{Q}^{\mathrm{T}} * \mathbf{X}_0) = \mathbf{Q}^{\mathrm{T}} * (\mathbf{F}(\mathbf{X}_0)) \qquad (\forall \mathbf{F} \in G(D, M), \quad \mathbf{Q} \in \mathrm{Orth}, \quad \mathbf{X}_0 \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})) . \tag{2.2}$$

Then a tensor function $\mathbf{\Psi} : D \to M \subset T_k$ *is invariant under the group* G *iff there is an isotropic extension function* $\mathbf{\Psi}^e \in \mathrm{Orth}(D \times E, M)$ *such that* $\mathbf{\Psi}$ *is the restriction of* $\mathbf{\Psi}^e$ *on the surface or the graph* $\mathrm{Graph}(\mathbf{S}) \equiv \{(\mathbf{X}, \mathbf{S}(\mathbf{X})) \subset D \times E \mid \mathbf{X} \in D\}$, *i.e.*

$$\mathbf{\Psi}(\mathbf{X}) = \mathbf{\Psi}^e(\mathbf{X}, \mathbf{X}^e) \mid_{\mathbf{X}^e = \mathbf{S}(\mathbf{X})} = \mathbf{\Psi}^e(\mathbf{X}, \mathbf{S}(\mathbf{X})) \qquad (\forall \mathbf{X} \in D) . \tag{2.3}$$

In the above theorem, the conditions for the set **S** of invariant tensor functions are weaker than those given in Xiao and Guo [46] and Xiao [49]. In reality, the conditions for **S** in the above theorem are given by (2.2) and

$$\mathbf{Q} \in G \implies \mathbf{Q} * \mathbf{S} = \mathbf{S} , \tag{2.4}$$

while those in [46] and [49] are given by (2.2) and

$$\mathbf{Q} \in G \iff \mathbf{Q} * \mathbf{S} = \mathbf{S} . \tag{2.5}$$

The former merely requires that the set **S** of tensor functions be invariant under the group G, while the latter requires that the symmetry group of **S** be identical with the group G.

Theorem A can be proved by means of the procedure given in [49] with little change. For a set **S** of tensor functions of interest, it is easier to prove whether **S** fulfills the invariance condition (2.4) or not, whereas it is not easy to judge whether **S** obeys the stronger invariance condition (2.5) or not, since it is not easy to determine the symmetry group of the set **S** of tensor functions.

A set **S** of tensor functions from D to E (cf. (2.1)) determines a surface in a Euclidean space R^n, where $n = 3(a + b + r + s) + 6(c + t)$, refer to §3.1 in [49] for detail. This fact allows a geometrical interpretation of the above extension theorem. The latter indicates that for every anisotropic tensor function $\mathbf{\Psi}$ relative to the anisotropy group $G \subset \mathrm{Orth}$ with a set of variables pertaining to the space $D = V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$, one can find a surface $\mathbf{S}: D \to E = V^r \times \mathrm{Skw}^s \times \mathrm{Sym}^t$ in an augmented space $D \times E$ such that $\mathbf{\Psi}$ can be visualized as the restriction of an isotropic tensor function $\mathbf{\Psi}^e$ with a set of variables pertaining to the augmented space $D \times E$ on this surface, i.e. (2.3) holds. Such a surface $\mathbf{S}$ will be referred to as an *isotropic extension surface* for the anisotropic functions in $G(D, M)$ or as an *IES* for $G(D, M)$ for brevity. Necessary and sufficient that a surface $\mathbf{S}: D \to E$ is an IES for $G(D, M)$ is the condition that this surface fulfils both the invariance condition (2.4) and the consistency condition (2.2).

From the above theorem it follows that representations for the anisotropic function $\Psi \in G(D, M)$ can be obtained from those for the isotropic function $\mathbf{\Psi}^e \in \mathrm{Orth}(D \times E, M)$ merely by replacing the variables $\mathbf{X}^e \in E$ of the latter with $\mathbf{S}(\mathbf{X})$. In this sense, the above isotropic extension theorem, together with the well-known representation theorems for isotropic functions of vectors and second order tensors, constitutes a unified basis for the theory of representations for anisotropic functions of vectors and second order tensors. In the succeeding sections, for a domain $D = V^a \times \mathrm{Skw}^b \times \mathrm{Sym}^c$ for any given positive integers a, b and c, for each image set $M \in \{R, V, \mathrm{Skw}, \mathrm{Sym}\}$ and for each crystal and noncrystal class G, we shall provide a simple IES for $G(D, M)$.

2.2. A lemma

For each surface $\mathbf{S}$ that will be given, it is required to prove that the conditions (2.2) and (2.4) can be satisfied. The main difficulty arises from the consistency condition (2.2), for even for a given nontrival surface $\mathbf{S}$ it is not easy to determine the intersecting surface $\mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$, let alone the fact that we must find a suitable surface $\mathbf{S}$ such that the conditions (2.2) and (2.4) can be satisfied.

We shall attack the above problem by choosing surfaces $\mathbf{S}$ in such a manner that all nontrivial intersecting surfaces $\mathbf{S} \cap (\mathbf{Q} * \mathbf{S}) \subset D$ are exactly certain prescribed particular subsets of D, which are provided by $D(\mathbf{u})$ or union of such subsets, where $\mathbf{u}$ is a unit vector in the direction of a symmetry axis of the related anisotropy group, since the following fact holds.

LEMMA A. Let $G \in \{C_{mv}, C_{mh}, S_{2m}, D_{mh}, D_{md}\}$, where $m \geq 3$, and let the unit vector $\mathbf{n}$ be in the direction of the principal axis of the group G. Moreover, define the group $D(G)$ by

$$D(G) = \begin{cases} C_{\infty v}, & G = C_{mv}, \\ C_{\infty h}, & G = C_{mh}, S_{2m}, \\ D_{\infty h}, & G = D_{mh}, D_{md}. \end{cases} \tag{2.6}$$

Then we have

$$\mathbf{F}(\mathbf{Q}^{\mathrm{T}} * \mathbf{X}_0) = \mathbf{Q}^{\mathrm{T}} * (\mathbf{F}(\mathbf{X}_0))$$

for any $\mathbf{Q} \in D(G)$, $\mathbf{X}_0 \in D(\mathbf{n})$, $\mathbf{F} \in G(D, M)$ and for each $M \in \{R, V, \mathrm{Skw}, \mathrm{Sym}\}$.

P r o o f. For each group G in question, there is $\mathbf{R}_0 = \mathbf{R}_\mathbf{n}^{2\pi/m} \in G$. For such $\mathbf{R}_0$, we have

$$\mathbf{R}_0 * (\mathbf{F}(\mathbf{X}_0)) = \mathbf{F}(\mathbf{R}_0 * \mathbf{X}_0) = \mathbf{F}(\mathbf{X}_0)$$

for each $\mathbf{X}_0 = (a_\alpha \mathbf{n}, b_\theta \mathbf{E}\mathbf{n}, c_\sigma \mathbf{I} + d_\sigma \mathbf{n}\otimes\mathbf{n}) \in D(\mathbf{n})$ and each $\mathbf{F} \in G(D, M)$. From this we derive

$$\mathbf{F}(\mathbf{X}_0) = \begin{cases} a(\mathbf{X}_0)\mathbf{n}, & M = V\,, \\ b(\mathbf{X}_0)\mathbf{E}\mathbf{n}, & M = \mathrm{Skw}\,, \\ c(\mathbf{X}_0)\mathbf{I} + d(\mathbf{X}_0)\mathbf{n}\otimes\mathbf{n}, & M = \mathrm{Sym}\,, \end{cases} \tag{2.7}$$

where for $M = \mathrm{Sym}$ the condition $m \geq 3$ is used. From the latter and the fact that for each $\mathbf{Q} \in D(G)$ there is $\mathbf{Q}_0 \in G$ such that

$$\mathbf{Q}\mathbf{n} = \mathbf{Q}_0\mathbf{n}, \qquad \mathbf{Q} * (\mathbf{E}\mathbf{n}) = \mathbf{Q}_0 * (\mathbf{E}\mathbf{n})\,, \tag{2.8}$$

we conclude that the lemma holds. *Q.E.D.*

2.3. The vector $\mathbf{q}(\mathbf{A})$ and the angle $< \mathbf{q}(\mathbf{A}), \mathbf{e} >$

Symbols $\mathbf{n}$ and $\mathbf{e}$ are used to represent two given orthonormal vectors. For any symmetric tensor $\mathbf{A} \in \mathrm{Sym}$, we introduce the vector $\mathbf{q}(\mathbf{A})$ by

$$\mathbf{q}(\mathbf{A}) = \frac{1}{2}(\mathbf{e}\cdot\mathbf{A}\mathbf{e} - \mathbf{e}'\cdot\mathbf{A}\mathbf{e}')\mathbf{e} + (\mathbf{e}\cdot\mathbf{A}\mathbf{e}')\mathbf{e}'\,. \tag{2.9}$$

Here and hereafter

$$\mathbf{e}' = \mathbf{n}\times\mathbf{e}\,. \tag{2.10}$$

Hence, $(\mathbf{n}, \mathbf{e}, \mathbf{e}')$ constitutes an orthonormal system.

The norm of any vector $\mathbf{v}$ is denoted by $|\mathbf{v}|$. Let $\mathbf{z}$ be a vector on the $\mathbf{n}$-plane. We define the *angle* $< \mathbf{z}, \mathbf{e} >$ *formed by the two vectors* $\mathbf{z}$ *and* $\mathbf{e}$ *on the* $\mathbf{n}$*-plane* as follows:

$$\cos < \mathbf{z}, \mathbf{e} > = \mathbf{z}\cdot\mathbf{e}/|\mathbf{z}|, \qquad \sin < \mathbf{z}, \mathbf{e} > = \mathbf{z}\cdot\mathbf{e}'/|\mathbf{z}|\,, \tag{2.11}$$

for $|\mathbf{z}| \neq 0$ and $< \mathbf{z}, \mathbf{e} > = 0$ for $|\mathbf{z}| = 0$. When $|\mathbf{z}| \neq 0$, it is evident that the angle $< \mathbf{z}, \mathbf{e} >$ is determined by (2.11) within $2k\pi$.

For the vector $\mathbf{q}(\mathbf{A})$ on the $\mathbf{n}$-plane introduced before, when $|\mathbf{q}(\mathbf{A})| \neq 0$ we have

$$\begin{aligned} \cos < \mathbf{q}(\mathbf{A}), \mathbf{e} > &= \frac{1}{2}(\mathbf{e}\cdot\mathbf{A}\mathbf{e} - \mathbf{e}'\cdot\mathbf{A}\mathbf{e}')/|\mathbf{q}(\mathbf{A})|\,, \\ \sin < \mathbf{q}(\mathbf{A}), \mathbf{e} > &= (\mathbf{e}\cdot\mathbf{A}\mathbf{e}')/|\mathbf{q}(\mathbf{A})|. \end{aligned} \tag{2.12}$$

Let **a** be a unit vector on the **n**-plane. Then, applying the equalities

$$(2.13)\qquad \begin{aligned} \mathbf{R}_{\mathbf{n}}^{\theta}\mathbf{e} &= \mathbf{e}\cos\theta + \mathbf{e}'\sin\theta\,,\\ \mathbf{R}_{\mathbf{n}}^{\theta}\mathbf{e}' &= -\mathbf{e}\sin\theta + \mathbf{e}'\cos\theta\,; \end{aligned}$$

$$(2.14)\qquad \begin{aligned} \mathbf{R}_{\mathbf{a}}^{\pi}\mathbf{e} &= \mathbf{e}\cos 2 < \mathbf{a},\mathbf{e} > +\mathbf{e}'\sin 2 < \mathbf{a},\mathbf{e} >,\\ \mathbf{R}_{\mathbf{a}}^{\pi}\mathbf{e}' &= \mathbf{e}\sin 2 < \mathbf{a},\mathbf{e} > -\mathbf{e}'\cos 2 < \mathbf{a},\mathbf{e} >, \end{aligned}$$

we derive the transformation formulas (cf. XIAO [54])

$$(2.15)\qquad < \mathbf{q}(\mathbf{Q}*\mathbf{A}),\mathbf{e} > = \begin{cases} 2\theta + < \mathbf{q}(\mathbf{A}),\mathbf{e} >, & \mathbf{Q} = \delta\mathbf{R}_{\mathbf{n}}^{\theta}\,,\\ 4 < \mathbf{a},\mathbf{e} > - < \mathbf{q}(\mathbf{A}),\mathbf{e} >, & \mathbf{Q} = \delta\mathbf{R}_{\mathbf{a}}^{\pi}\,, \end{cases}$$

for $|\mathbf{q}(\mathbf{A})| \neq 0$ and

$$(2.16)\qquad |\mathbf{q}(\mathbf{Q}*\mathbf{A})| = |\mathbf{q}(\mathbf{A})|, \quad \forall\,\mathbf{Q}\in D_{\infty h}\,,$$

where $D_{\infty h}$ is the maximal transverse isotropy group with the principal axis **n** (cf. (3.1) given later). Moreover, the following formulas hold:

$$(2.17)\qquad < (\mathbf{Q}\mathbf{v})^{\circ},\mathbf{e} > = \begin{cases} (1-\delta)\pi/2 + \theta + < \mathbf{v}^{\circ},\mathbf{e} >, & \mathbf{Q} = \delta\mathbf{R}_{\mathbf{n}}^{\theta}\,,\\ (1+\delta)\pi/2 + 2 < \mathbf{a},\mathbf{e} > - < \mathbf{v}^{\circ},\mathbf{e} >, & \mathbf{Q} = \delta\mathbf{R}_{\mathbf{a}}^{\pi}\,, \end{cases}$$

$$(2.18)\qquad |(\mathbf{Q}\mathbf{v})^{\circ}| = |\mathbf{v}^{\circ}|, \quad \forall\,\mathbf{Q}\in D_{\infty h}\,,$$

for any vector $\mathbf{v}\in V$ and any $\mathbf{Q}\in D_{\infty h}$, and

$$(2.19)\qquad < ((\mathbf{Q}*\mathbf{B})\mathbf{n})^{\circ},\mathbf{e} > = \begin{cases} \theta + < (\mathbf{B}\mathbf{n})^{\circ},\mathbf{e} >, & \mathbf{Q} = \delta\mathbf{R}_{\mathbf{n}}^{\theta}\,,\\ \pi + 2 < \mathbf{a},\mathbf{e} > - < (\mathbf{B}\mathbf{n})^{\circ},\mathbf{e} >, & \mathbf{Q} = \delta\mathbf{R}_{\mathbf{a}}^{\pi}\,, \end{cases}$$

$$(2.20)\qquad |((\mathbf{Q}*\mathbf{B})\mathbf{n})^{\circ}| = |(\mathbf{B}\mathbf{n})^{\circ}|, \quad \forall\,\mathbf{Q}\in D_{\infty h}\,,$$

for any second order tensor **B** and any $\mathbf{Q}\in D_{\infty h}$. In the above, $\delta^2 = 1$. Throughout, $\mathbf{v}^{\circ}$ is used to designate the perpendicular projection of the vector **v** on the **n**-plane, i.e.

$$(2.21)\qquad \mathbf{v}^{\circ} = \mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\mathbf{n}\,.$$

For each antisymmetric tensor $\mathbf{W}\in \mathrm{Skw}$, the vector **Wn** lies on the **n**-plane, i.e.

$$(2.22)\qquad (\mathbf{W}\mathbf{n})^{\circ} = \mathbf{W}\mathbf{n}\,,$$

since the latter is normal to **n**.

Henceforth, for any two vectors $\mathbf{p}, \mathbf{q} \in V$, $\mathbf{p} \vee \mathbf{q} \in \mathrm{Sym}$ is used to signify the symmetric second order tensor defined by

$$\mathbf{p} \vee \mathbf{q} = \mathbf{p} \otimes \mathbf{q} + \mathbf{q} \otimes \mathbf{p} . \tag{2.23}$$

Moreover, we denote

$$\mathbf{D}_1 = \mathbf{e} \otimes \mathbf{e} - \mathbf{e}' \otimes \mathbf{e}', \qquad \mathbf{D}_2 = \mathbf{e} \vee \mathbf{e}' . \tag{2.24}$$

For any $\mathbf{Q} \in D_{\infty h}$, by using (2.13)–(2.14), we derive the following formulas.

$$\begin{aligned} \mathbf{R}_{\mathbf{n}}^{\theta} * \mathbf{D}_1 &= \mathbf{D}_1 \cos 2\theta + \mathbf{D}_2 \sin 2\theta , \\ \mathbf{R}_{\mathbf{n}}^{\theta} * \mathbf{D}_2 &= -\mathbf{D}_1 \sin 2\theta + \mathbf{D}_2 \cos 2\theta ; \end{aligned} \tag{2.25}$$

$$\begin{aligned} \mathbf{R}_{\mathbf{a}}^{\pi} * \mathbf{D}_1 &= \mathbf{D}_1 \cos 4 < \mathbf{a}, \mathbf{e} > + \mathbf{D}_2 \sin 4 < \mathbf{a}, \mathbf{e} > , \\ \mathbf{R}_{\mathbf{a}}^{\pi} * \mathbf{D}_2 &= \mathbf{D}_1 \sin 4 < \mathbf{a}, \mathbf{e} > - \mathbf{D}_2 \cos 4 < \mathbf{a}, \mathbf{e} > . \end{aligned} \tag{2.26}$$

3. Improper subgroups of the transverse isotropy group $D_{\infty h}$

Prior to the succeeding account, we would point out the following fact: According to Theorems 2.1 and 2.2 given in [49], representations for anisotropic functions relative to a *rotation subgroup* $G \subset \mathrm{Orth}$ can be obtained from those for anisotropic functions relative to the centrosymmetric group

$$\bar{G} = \{\pm \mathbf{Q} \mid \mathbf{Q} \in G\}.$$

As a result, henceforth we need only to take the *improper subgroups* of Orth into account.

3.1. Transverse isotropy groups

$$D_{\infty h} = \{\pm \mathbf{R}_{\mathbf{n}}^{\theta},\ \pm \mathbf{R}_{\mathbf{a}}^{\pi} \mid \mathbf{a} = \mathbf{R}_{\mathbf{n}}^{\theta} \mathbf{e},\ \theta \in R\} , \tag{3.1}$$

$$C_{\infty v} = \{\mathbf{R}_{\mathbf{n}}^{\theta}, -\mathbf{R}_{\mathbf{a}}^{\pi} \mid \mathbf{a} = \mathbf{R}_{\mathbf{n}}^{\theta} \mathbf{e},\ \theta \in R\} , \tag{3.2}$$

$$C_{\infty h} = \{\pm \mathbf{R}_{\mathbf{n}}^{\theta} \mid \theta \in R\} . \tag{3.3}$$

According to BOEHLER [6] and LIU [16], the following offer an IES for $G(D, M)$ for each $G \in \{D_{\infty h},\ C_{\infty v},\ C_{\infty h}\}$.

$$D_{\infty h} \ : \ \mathbf{S}(\mathbf{X}) = (\mathbf{n} \otimes \mathbf{n}) , \tag{3.4}$$

$$C_{\infty v} \ : \ \mathbf{S}(\mathbf{X}) = (\mathbf{n}) , \tag{3.5}$$

$$C_{\infty h} \ : \ \mathbf{S}(\mathbf{X}) = (\mathbf{E}\mathbf{n}) . \tag{3.6}$$

In reality, the following equalities hold (cf. LIU [16]).

(3.7) $$\Gamma(\mathbf{n}\otimes\mathbf{n}) = D_{\infty h}\,, \qquad \Gamma(\mathbf{n}) = C_{\infty v}\,, \qquad \Gamma(\mathbf{E}\mathbf{n}) = C_{\infty h}\,.$$

Hence, trivially, each surface $\mathbf{S}(\mathbf{X})$ given above satisfies the conditions (2.2) and (2.5).

Only for the anisotropic functions relative to such simple anisotropy groups as the transverse isotropy groups as well as triclinic, monoclinic and rhombic groups, trivial IESes such as those shown above (for the results concerning the latter groups, refer to BOEHLER [6, 8] and LIU [16]), which consist merely of some constant vectors and second order tensors, i.e. trivial vector- and second order tensor-valued tensor functions, can be found. For anisotropic functions concerning any other anisotropy group, nontrivial IESes have to be constructed, as will be done in the succeeding sections.

3.2. Classes D_{2m+1d}, C_{2m+1v} and S_{4m+2} for $m \geq 1$

(3.8) $$D_{2m+1d} = \{\pm\mathbf{R}_{\mathbf{n}}^{2k\pi/2m+1},\ \pm\mathbf{R}_{\mathbf{l}_k}^{\pi} \mid \mathbf{l}_k = \mathbf{R}_{\mathbf{n}}^{2k\pi/2m+1}\mathbf{e},\ k = 0, 1, 2, \ldots, 2m\},$$

(3.9) $$C_{2m+1v} = C_{\infty v} \cap D_{2m+1d}; \qquad S_{4m+2} = C_{\infty h} \cap D_{2m+1d}\,.$$

The above classes include the trigonal crystal classes D_{3d}, C_{3v} and S_6 as the particular case when $m = 1$.

Henceforth, we denote

(3.10) $$\mathbf{D}(G) = \begin{cases} \mathbf{n}, & G = C_{rv}\,, \\ \mathbf{E}\mathbf{n}, & G = C_{rh}, S_{2r}\,, \\ \mathbf{n}\otimes\mathbf{n}, & G = D_{rh}, D_{rd}\,, \end{cases}$$

for each $r \geq 2$.

THEOREM 1. *Let $G \in \{D_{2m+1d}, C_{2m+1v}, S_{4m+2}\}$. Then the surface*

(3.11) $$\mathbf{S}(\mathbf{X}) = (\mathbf{D}(G); \mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{v}_\alpha^{\circ}); \mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{W}_\theta\mathbf{n}); \mathbf{E}\boldsymbol{\eta}_{2m}((\mathbf{A}_\sigma\mathbf{n})^{\circ}), \mathbf{E}\boldsymbol{\eta}_{m}(\mathbf{q}(\mathbf{A}_\sigma)))$$

is an IES for $G(D, M)$, where $\mathbf{D}(G)$ is given by (3.10) *and*

(3.12) $$\boldsymbol{\eta}_r(\mathbf{z}) = |\mathbf{z}|^r(\mathbf{e}\cos r < \mathbf{z}, \mathbf{e} > -\mathbf{e}'\sin r < \mathbf{z}, \mathbf{e} >)$$

for any vector $\mathbf{z}$ *on the* $\mathbf{n}$*-plane and for each integer* $r \geq 1$.

P r o o f. First, we prove that the given surface $\mathbf{S}(\mathbf{X})$ obeys the invariance requirement (2.4). Applying the formulas (2.13) and $(2.17)_1$ and (2.18), for $\mathbf{Q} = \pm\mathbf{R}_{\mathbf{n}}^{\theta}$ we infer

$$\begin{aligned} \mathbf{Q}^{\mathrm{T}} * (\mathbf{E}\boldsymbol{\eta}_{2m}((\mathbf{Q}\mathbf{v})^{\circ})) &= |\mathbf{v}^{\circ}|^{2m}\mathbf{Q}^{\mathrm{T}} * (\mathbf{E}(\mathbf{e}\cos(2m\theta + x) - \mathbf{e}'\sin(2m\theta + x))) \\ &= |\mathbf{v}^{\circ}|^{2m}\mathbf{E}(\mathbf{e}\cos((2m+1)\theta + x) - \mathbf{e}'\sin((2m+1)\theta + x))\,, \end{aligned}$$

where $x = 2m < \mathbf{v}^{\circ}, \mathbf{e} >$. Hence, we have

$$\mathbf{Q}^{\mathrm{T}} * (\mathbf{E}\boldsymbol{\eta}_{2m}((\mathbf{Q}\mathbf{v})^{\circ})) = \mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{v}^{\circ}), \qquad \mathbf{Q} = \pm\mathbf{R}_{\mathbf{n}}^{2\pi/2m+1}.$$

Moreover, applying $(2.17)_2$, (2.18), and (2.14) for $\mathbf{a} = \mathbf{e}$, we deduce

$$\mathbf{R}_{\mathbf{e}}^{\pi} * (\mathbf{E}\boldsymbol{\eta}_{2m}((\mathbf{R}_{\mathbf{e}}^{\pi}\mathbf{v})^{\circ})) = \mathbf{R}_{\mathbf{e}}^{\pi} * (\mathbf{E}(\mathbf{e}\cos(2m\pi - x) - \mathbf{e}'\sin(2m\pi - x))) = \mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{v}^{\circ}).$$

From the above facts we derive that the tensor function $\mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{v}^{\circ})$ is invariant under the group D_{2m+1d}, since the three orthogonal tensors $\pm\mathbf{R}_{\mathbf{n}}^{2\pi/2m+1}$ and $\mathbf{R}_{\mathbf{e}}^{\pi}$ can generate the group D_{2m+1d}. Similarly, by applying the formulas (2.13)–(2.16) and (2.19)–(2.20) we can prove that each of the other tensor functions in the given surface $\mathbf{S}(\mathbf{X})$ is also invariant under the group D_{2m+1d}. Thus, the given surface $\mathbf{S}(\mathbf{X})$ obeys (2.4).

Next, we prove that the surface $\mathbf{S}(\mathbf{X})$ satisfies the condition (2.2). We have

$$\mathbf{S} \cap (\mathbf{Q} * \mathbf{S}) = \begin{cases} D, & \mathbf{Q} \in G, \\ \emptyset, & \mathbf{Q} \in \mathrm{Orth}\backslash\Gamma(\mathbf{D}(G)), \end{cases}$$

for each $\mathbf{Q} \in \mathrm{Orth}\backslash(\Gamma(\mathbf{D}(G))\backslash G)$, where the symmetry groups $\Gamma(\mathbf{D}(G))$ are given by (3.10) and (3.7), and moreover $\emptyset$ is used to denote the empty set. Trivially, the condition (2.2) is satisfied for each $\mathbf{Q} \in \mathrm{Orth}\backslash(\Gamma(\mathbf{D}(G))\backslash G)$.

Moreover, for each $\mathbf{Q} \in \Gamma(\mathbf{D}(G)) \setminus G \subset D_{\infty h} \setminus G$, the intersecting point $\mathbf{X}_0 = (\mathbf{v}_{\alpha}, \mathbf{W}_{\theta}, \mathbf{A}_{\sigma}) \in \mathbf{S}\cap(\mathbf{Q}*\mathbf{S})$ is determined by the system of tensor equations of the forms (A.1)–(A.4), where the variables are: $\mathbf{v} = \mathbf{v}_1, \ldots, \mathbf{v}_a$; $\mathbf{W} = \mathbf{W}_1, \ldots, \mathbf{W}_b$; $\mathbf{A} = \mathbf{A}_1, \ldots, \mathbf{A}_c$. By Theorem A.1 we know that $\mathbf{S} \cap (\mathbf{Q} * \mathbf{S}) = D(\mathbf{n})$ for each $\mathbf{Q} \in \Gamma(\mathbf{D}(G)) \setminus G$. Then, from this fact and Lemma A we deduce that the condition (2.2) is also satisfied for each $\mathbf{Q} \in \Gamma(\mathbf{D}(G))\backslash G$. *Q.E.D.*

3.3. Classes D_{2m+2h}, C_{2m+2v} and C_{2m+2h} for $m \geq 1$

$$\text{(3.13)} \qquad D_{2m+2h} = \{\pm\mathbf{R}_{\mathbf{n}}^{k\pi/m+1}, \ \pm\mathbf{R}_{\mathbf{l}_k}^{\pi} \mid \mathbf{l}_k = \mathbf{R}_{\mathbf{n}}^{k\pi/2m+2}\mathbf{e}, \ k = 0, 1, 2, \ldots, 2m+1\},$$

$$\text{(3.14)} \qquad C_{2m+2v} = C_{\infty v} \cap D_{2m+2h}, \qquad C_{2m+2h} = C_{\infty h} \cap D_{2m+2h}.$$

The above classes include the tetragonal and hexagonal crystal classes D_{4h}, D_{6h}, C_{4v}, C_{6v}, C_{4h} and C_{6h} as the particular cases when $m = 1, 2$.

THEOREM 2. *Let $G \in \{D_{2m+2h}, C_{2m+2v}, C_{2m+2h}\}$. Then the surface*

$$\text{(3.15)} \qquad \mathbf{S}(\mathbf{X}) = (\mathbf{D}(G); \boldsymbol{\Phi}_{2m}(\mathbf{v}_{\alpha}^{\circ}); \boldsymbol{\Phi}_{2m}(\mathbf{W}_{\theta}\mathbf{n}); \boldsymbol{\Phi}_{2m}((\mathbf{A}_{\sigma}\mathbf{n})^{\circ}), \boldsymbol{\Phi}_{m}(\mathbf{q}(\mathbf{A}_{\sigma})))$$

is an IES for $G(D, M)$, where the tensor $\mathbf{D}(G)$ is given by (3.10) *and*

$$\text{(3.16)} \qquad \boldsymbol{\Phi}_r(\mathbf{z}) = |\mathbf{z}|^r(\mathbf{D}_1 \cos r < \mathbf{z}, \mathbf{e} > - \mathbf{D}_2 \sin r < \mathbf{z}, \mathbf{e} >)$$

for any vector $\mathbf{z}$ on the $\mathbf{n}$-plane and any integer $r \geq 1$, and the tensors $\mathbf{D}_1$ and $\mathbf{D}_2$ are given by (2.24).

P r o o f. The proof concerning the condition (2.2) is the same as that of Theorem 1, except for the fact that Eqs. (A.7)–(A.10) and Theorem A.2 is used instead of Eqs. (A.1)–(A.4) and Theorem A.1. Hence, in the following we need only to prove that the given surface $\mathbf{S}(\mathbf{X})$ obeys the invariance condition (2.4). In reality, applying the formulas (2.25) and $(2.17)_1$ and (2.18), for $\mathbf{Q} = \pm\mathbf{R}_{\mathbf{n}}^{\theta}$ we deduce

$$\begin{aligned}\mathbf{Q}^{\mathrm{T}} * (\mathbf{\Phi}_{2m}((\mathbf{Q}\mathbf{v})^{\mathrm{o}})) &= |\mathbf{v}^{\mathrm{o}}|^{2m}\mathbf{Q}^{\mathrm{T}} * (\mathbf{D}_1 \cos(2m\theta + x) - \mathbf{D}_2 \sin(2m\theta + x)) \\ &= |\mathbf{v}^{\mathrm{o}}|^{2m}(\mathbf{D}_1 \cos((2m+2)\theta + x) - \mathbf{D}_2 \sin((2m+2)\theta + x)),\end{aligned}$$

where $x = 2m < \mathbf{v}^{\mathrm{o}}, \mathbf{e} >$. Hence, we have

$$\mathbf{Q}^{\mathrm{T}} * (\mathbf{\Phi}_{2m}((\mathbf{Q}\mathbf{v})^{\mathrm{o}})) = \mathbf{\Phi}_{2m}(\mathbf{v}^{\mathrm{o}}), \qquad \mathbf{Q} = \pm\mathbf{R}_{\mathbf{n}}^{\pi/m+1}.$$

Moreover, applying $(2.17)_2$, (2.18), and (2.26) for $\mathbf{a} = \mathbf{e}$, we infer

$$\mathbf{R}_{\mathbf{e}}^{\pi} * (\mathbf{\Phi}_{2m}((\mathbf{R}_{\mathbf{e}}^{\pi}\mathbf{v})^{\mathrm{o}})) = |\mathbf{v}^{\mathrm{o}}|^{2m}\mathbf{R}_{\mathbf{e}}^{\pi} * (\mathbf{D}_1 \cos(2m\pi - x) - \mathbf{D}_2 \sin(2m\pi - x)) = \mathbf{\Phi}_{2m}(\mathbf{v}^{\mathrm{o}}).$$

From the above facts we derive that the tensor function $\mathbf{\Phi}_{2m}(\mathbf{v}^{\mathrm{o}})$ is invariant under the group D_{2m+2h}, since the three orthogonal tensors $\pm\mathbf{R}_{\mathbf{n}}^{\pi/m+1}$ and $\mathbf{R}_{\mathbf{e}}^{\pi}$ can generate the group D_{2m+2h}. Similarly, by applying the formulas (2.15)–(2.16), (2.19)–(2.20) and (2.25)–(2.26) we can prove that each of the other tensor functions in the given surface $\mathbf{S}(\mathbf{X})$ is also invariant under the group D_{2m+2h}. Thus, we conclude that the given surface $\mathbf{S}(\mathbf{X})$ obeys (2.4). *Q.E.D.*

3.4. Classes D_{2m+1h} and C_{2m+1h} for $m \geq 1$

(3.17) $$D_{2m+1h} = \{(-1)^k\mathbf{R}_{\mathbf{n}}^{k\pi/2m+1},\ (-1)^k\mathbf{R}_{\mathbf{l}_k}^{\pi} \mid \mathbf{l}_k = \mathbf{R}_{\mathbf{n}}^{k\pi/4m+2}\mathbf{e},\ k = 0, 1, 2, \ldots, 4m+1\},$$

(3.18) $$C_{2m+1h} = C_{\infty h} \cap D_{2m+1h}\,.$$

The above classes include the hexagonal crystal classes D_{3h} and C_{3h} as the particular case when $m = 1$. Note that $\mathbf{e} = \mathbf{l}_0$ is a *two-fold rotation axis* of D_{2m+1h}.

THEOREM 3. *Let $G \in \{D_{2m+1h}, C_{2m+1h}\}$. Then the surface*

(3.19) $$\mathbf{S}(\mathbf{X}) = (\mathbf{D}(G); \boldsymbol{\eta}_{2m}(\mathbf{v}_{\alpha}^{\mathrm{o}}); \boldsymbol{\eta}_{2m}(\mathbf{W}_{\theta}\mathbf{n}); \boldsymbol{\eta}_{2m}((\mathbf{A}_{\sigma}\mathbf{n})^{\mathrm{o}}), \boldsymbol{\eta}_{m}(\mathbf{q}(\mathbf{A}_{\sigma})))$$

is an IES for $G(D, M)$, where $\mathbf{D}(G)$ is given by (3.10) *and the vector-valued function $\boldsymbol{\eta}_r(\mathbf{z})$ is given by* (3.12) *for any vector $\mathbf{z}$ on the $\mathbf{n}$-plane and each integer $r \geq 1$.*

P r o o f. The proof concerning the condition (2.2) is the same as that of Theorem 1, except for the fact that Eqs. (A.11)–(A.14) and Theorem A.3 are used instead of Eqs. (A.1)–(A.4) and Theorem A.1. In the following, we prove that the given surface obeys the invariance condition (2.4).

Applying the formulas (2.13) and $(2.17)_1$ and (2.18), for $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{n}}^{\theta}$, $\delta^2 = 1$, we deduce

$$\begin{aligned}\mathbf{Q}^{\mathrm{T}}(\boldsymbol{\eta}_{2m}((\mathbf{Qv})^{\circ})) &= |\mathbf{v}^{\circ}|^{2m}\mathbf{Q}^{\mathrm{T}}(\mathbf{e}\cos(2m\theta + x) - \mathbf{e}'\sin(2m\theta + x)) \\ &= |\mathbf{v}^{\circ}|^{2m}\delta(\mathbf{e}\cos((2m+1)\theta + x) - \mathbf{e}'\sin((2m+1)\theta + x)),\end{aligned}$$

where $x = 2m < \mathbf{v}^{\circ}, \mathbf{e} >$. Hence, we have

$$\mathbf{Q}^{\mathrm{T}}(\boldsymbol{\eta}_{2m}((\mathbf{Qv})^{\circ})) = \boldsymbol{\eta}_{2m}(\mathbf{v}^{\circ}), \qquad \mathbf{Q} = -\mathbf{R}_{\mathbf{n}}^{\pi/2m+1}.$$

Moreover, applying $(2.17)_2$, (2.18), and (2.14) for $\mathbf{a} = \mathbf{e}$, we infer

$$\mathbf{R}_{\mathbf{e}}^{\pi}(\boldsymbol{\eta}_{2m}((\mathbf{R}_{\mathbf{e}}^{\pi}\mathbf{v})^{\circ})) = \mathbf{R}_{\mathbf{e}}^{\pi}(\mathbf{e}\cos(2m\pi - x) - \mathbf{e}'\sin(2m\pi - x)) = \boldsymbol{\eta}_{2m}(\mathbf{v}^{\circ}).$$

From the above facts we derive that the tensor function $\boldsymbol{\eta}_{2m}(\mathbf{v}^{\circ})$ is invariant under the group D_{2m+1h}, since the two orthogonal tensors $-\mathbf{R}_{\mathbf{n}}^{\pi/2m+1}$ and $\mathbf{R}_{\mathbf{e}}^{\pi}$ can generate the group D_{2m+1h}. Similarly, by applying the formulas (2.13)–(2.16) and (2.19)–(2.20) we can prove that each of the other tensor functions in the given surface $\mathbf{S}(\mathbf{X})$ is also invariant under the group D_{2m+1h}. Thus, we conclude that the given surface $\mathbf{S}(\mathbf{X})$ obeys (2.4). *Q.E.D.*

3.5. Classes D_{2md} for $m \geq 2$

$$D_{2md} = \{(-1)^k\mathbf{R}^{k\pi/2m},\ (-1)^k\mathbf{R}_{\mathbf{l}_k}^{\pi} \mid \mathbf{l}_k = \mathbf{R}_{\mathbf{n}}^{k\pi/4m}\mathbf{e},\quad k = 0, 1, 2, \ldots, 4m-1\}. \tag{3.20}$$

Note that $\mathbf{e} = \mathbf{l}_0$ is a *two-fold rotation axis* of D_{2md}.

THEOREM 4. *The surface*

$$\begin{aligned}\mathbf{S}(\mathbf{X}) = (&\mathbf{n}\otimes\mathbf{n}; \mathbf{n}\vee\boldsymbol{\eta}_{2m-1}(\mathbf{v}_{\alpha}^{\circ}); \boldsymbol{\eta}_{2m-1}(\mathbf{W}_{\theta}\mathbf{n}); \boldsymbol{\eta}_{2m-1}((\mathbf{A}_{\sigma}\mathbf{n})^{\circ}), f_m(\mathbf{A}_{\sigma})\mathbf{n}, \\ &\boldsymbol{\Phi}_{2m-1}(\mathbf{q}(\mathbf{A}_{\sigma})); (\mathbf{n}\cdot\mathbf{v}_{\alpha})\boldsymbol{\Phi}_{m-1}(\mathbf{q}(\mathbf{A}_{\sigma})); (\mathbf{e}\cdot\mathbf{W}_{\theta}\mathbf{e}')g_m(\mathbf{A}_{\sigma})\mathbf{n})\end{aligned} \tag{3.21}$$

is an IES for $D_{2md}(D, M)$, where the tensor-valued function $\boldsymbol{\Phi}_r(\mathbf{z})$ and the vector-valued function $\boldsymbol{\eta}_r(\mathbf{z})$ are given by (3.16) and (3.12) for any vector $\mathbf{z}$ on the $\mathbf{n}$-plane and each integer $r \geq 1$, respectively, and moreover

$$\begin{aligned}f_m(\mathbf{A}_{\sigma}) &= |\mathbf{q}(\mathbf{A}_{\sigma})|^m \sin m < \mathbf{q}(\mathbf{A}_{\sigma}), \mathbf{e} >, \\ g_m(\mathbf{A}_{\sigma}) &= |\mathbf{q}(\mathbf{A}_{\sigma})|^m \cos m < \mathbf{q}(\mathbf{A}_{\sigma}), \mathbf{e} > .\end{aligned} \tag{3.22}$$

Proof. First, we prove that the given surface $\mathbf{S}(\mathbf{X})$ obeys the invariance condition (2.4). Applying the formulas (2.13) and $(2.17)_1$ and (2.18), for $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{n}}^{\theta}$, $\delta^2 = 1$, we deduce

$$\begin{aligned}\mathbf{Q}^{\mathrm{T}} * (\mathbf{n} \vee \boldsymbol{\eta}_N((\mathbf{Q}\mathbf{v})^{\circ})) &= |\mathbf{v}^{\circ}|^N \mathbf{Q}^{\mathrm{T}} * \left(\mathbf{n} \vee \left(\mathbf{e}\cos\left(\frac{1-\delta}{2}\pi + N\theta + x\right)\right.\right.\\ &\qquad\left.\left. - \mathbf{e}'\sin\left(\frac{1-\delta}{2}\pi + N\theta + x\right)\right)\right)\\ &= |\mathbf{v}^{\circ}|^N \mathbf{n} \vee \left(\mathbf{e}\cos\left(\frac{1-\delta}{2}\pi + 2m\theta + x\right)\right.\\ &\qquad\left. - \mathbf{e}'\sin\left(\frac{1-\delta}{2}\pi + 2m\theta + x\right)\right),\end{aligned}$$

where $N = 2m-1$ and $x = (2m-1) < \mathbf{v}^{\circ}, \mathbf{e} >$. Hence, we have

$$\mathbf{Q}^{\mathrm{T}} * (\mathbf{n} \vee \boldsymbol{\eta}_{2m-1}((\mathbf{Q}\mathbf{v})^{\circ})) = \mathbf{n} \vee \boldsymbol{\eta}_{2m-1}(\mathbf{v}^{\circ}), \qquad \mathbf{Q} = -\mathbf{R}_{\mathbf{n}}^{\pi/2m}.$$

Moreover, by applying $(2.17)_2$, (2.18), and (2.14) for $\mathbf{a} = \mathbf{e}$ we infer

$$\begin{aligned}&\mathbf{R}_{\mathbf{e}}^{\pi} * (\mathbf{n} \vee \boldsymbol{\eta}_N((\mathbf{R}_{\mathbf{e}}^{\pi}\mathbf{v})^{\circ}))\\ &\quad = |\mathbf{v}^{\circ}|^N \mathbf{R}_{\mathbf{e}}^{\pi} * (\mathbf{n} \vee (\mathbf{e}\cos(N\pi - x) - \mathbf{e}'\sin(N\pi - x))) = \mathbf{n} \vee \boldsymbol{\eta}_N(\mathbf{v}^{\circ}).\end{aligned}$$

From the above facts we conclude that the tensor function $\mathbf{n} \vee \boldsymbol{\eta}_{2m-1}(\mathbf{v}^{\circ})$ is invariant under the group D_{2md}, since the two orthogonal tensors $-\mathbf{R}_{\mathbf{n}}^{\pi/2m}$ and $\mathbf{R}_{\mathbf{e}}^{\pi}$ can generate the group D_{2md}. Similarly, by using the formulas (2.13)–(2.16), (2.19)–(2.20) and (2.25)–(2.26) we can prove that each of the other tensor functions in the given surface $\mathbf{S}(\mathbf{X})$ is also invariant under the group D_{2md}. Thus, we conclude that the given surface obeys (2.4).

Next, we prove that the given surface $\mathbf{S}(\mathbf{X})$ satisfies the condition (2.2). It can be readily verified that the condition (2.2) is satisfied for each $\mathbf{Q} \in \mathrm{Orth} \setminus (D_{\infty h} \setminus D_{2md})$ by using $(3.7)_1$. Thus, the rest is to prove that the condition (2.2) is satisfied for each $\mathbf{Q} \in D_{\infty h} \setminus D_{2md}$. To this end, consider two cases. First, for each $\mathbf{Q} \in D_{\infty h} \setminus D_{4mh}$, the intersection point $\mathbf{X}_0 = (\mathbf{v}_{\alpha}, \mathbf{W}_{\theta}, \mathbf{A}_{\sigma}) \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is determined by the system of tensor equations of the forms (A.15)–(A.17) and (A.10), in the latter m being replaced by $2m-1$, and moreover

$$\begin{aligned}f_m(\mathbf{Q}^{\mathrm{T}} * \mathbf{A})\mathbf{Q}\mathbf{n} &= f_m(\mathbf{A})\mathbf{n},\\ (\mathbf{n}\cdot(\mathbf{Q}^{\mathrm{T}}\mathbf{v}))\mathbf{Q} * (\boldsymbol{\Phi}_{m-1}(\mathbf{q}(\mathbf{Q}^{\mathrm{T}} * \mathbf{A}))) &= (\mathbf{n}\cdot\mathbf{v})\boldsymbol{\Phi}_{m-1}(\mathbf{q}(\mathbf{A})),\\ (\mathbf{e}\cdot(\mathbf{Q}^{\mathrm{T}} * \mathbf{W})\mathbf{e}')g_m(\mathbf{Q}^{\mathrm{T}} * \mathbf{A})\mathbf{Q}\mathbf{n} &= (\mathbf{e}\cdot\mathbf{W}\mathbf{e}')g_m(\mathbf{A})\mathbf{n},\end{aligned}$$

where the variables are: $\mathbf{v} = \mathbf{v}_1, \ldots, \mathbf{v}_a$; $\mathbf{W} = \mathbf{W}_1, \ldots, \mathbf{W}_b$; $\mathbf{A} = \mathbf{A}_1, \ldots, \mathbf{A}_c$. From Theorem A.4 and the proof of Theorem A.2 (cf. $(\mathrm{A}.6)_4$) we derive that $\mathbf{S} \cap (\mathbf{Q} * \mathbf{S}) = D(\mathbf{n})$, and furthermore the point $\mathbf{X}_0 \in D(\mathbf{n})$ satisfies the above three equations. Then, by noticing $2m \geq 4$ and applying Lemma A we conclude that the condition (2.2) is satisfied for each $\mathbf{Q}$ in question.

Moreover, for each $\mathbf{Q} \in D_{4mh} \setminus D_{2md}$, by using the fact

$$\mathbf{Q} \in D_{4mh} \setminus D_{2md} \Rightarrow -\mathbf{Q} \in D_{2md} \tag{3.23}$$

we infer that the intersection point $\mathbf{X}_0 = (\mathbf{v}_\alpha, \mathbf{W}_\theta, \mathbf{A}_\sigma) \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is determined by

$$\boldsymbol{\eta}_{2m-1}(\mathbf{z}) = \mathbf{0}, \qquad f_m(\mathbf{A}_\sigma) = 0,$$
$$(\mathbf{v}_\alpha \cdot \mathbf{n})\boldsymbol{\Phi}_{m-1}(\mathbf{A}_\sigma) = \mathbf{O}, \qquad (\mathbf{e} \cdot \mathbf{W}_\theta \mathbf{e}')g_m(\mathbf{A}_\sigma) = 0,$$

where $\mathbf{z} = \mathbf{v}_\alpha^o, \mathbf{W}_\theta \mathbf{n}, (\mathbf{A}_\sigma \mathbf{n})^o$. The first two equations yield

$$\begin{aligned} \mathbf{v}_\alpha &= a_\alpha \mathbf{n}, \qquad \mathbf{W}_\theta = b_\theta \mathbf{E}\mathbf{n}, \\ \mathbf{A}_\sigma &= c_\sigma \mathbf{I} + d_\sigma \mathbf{n} \otimes \mathbf{n} + h_\sigma(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2), \end{aligned} \tag{3.24}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are two orthonormal vectors in the $\mathbf{n}$-plane and will be given later. Substituting the above results into the last two equations given before, we derive

$$a_\alpha h_\sigma = 0, \qquad b_\theta h_\sigma = 0.$$

Thus, for each $\mathbf{Q} \in D_{4mh} \setminus D_{2md}$ the point $\mathbf{X}_0 \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is given by $\mathbf{X}_0 \in D(\mathbf{n})$ or

$$\begin{aligned} &\mathbf{v}_\alpha = \mathbf{0}, \quad \mathbf{W}_\theta = \mathbf{O}, \quad \mathbf{A}_\sigma = c_\sigma \mathbf{I} + d_\sigma \mathbf{n} \otimes \mathbf{n} + h_\sigma(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2), \\ &\mathbf{e}_1 = \mathbf{l}_{2k}, \quad \mathbf{e}_2 = \mathbf{n} \times \mathbf{e}_1 = \mathbf{l}_{2k+2m}, \quad k = 0, 1, 2, \ldots, m-1. \end{aligned} \tag{3.25}$$

For the point $\mathbf{X}_0 \in D(\mathbf{n})$, by noting $2m \geq 4$ and using Lemma A we deduce that the condition (2.2) can be satisfied. For the point $\mathbf{X}_0$ given by (3.25), from the facts

$$\begin{aligned} &\mathbf{R}_0 * \mathbf{X}_0 = \mathbf{X}_0, \quad \mathbf{R}_0 \in \{\mathbf{R}_{\mathbf{e}_1}^\pi, \mathbf{R}_{\mathbf{e}_2}^\pi, \mathbf{R}_{\mathbf{n}}^\pi\} \subset D_{2md} \\ &\qquad \Rightarrow \forall \mathbf{F} \in D_{2md}(D, M): \ \mathbf{F}(\mathbf{X}_0) = \mathbf{F}(\mathbf{R}_0 * \mathbf{X}_0) = \mathbf{R}_0 * (\mathbf{F}(\mathbf{X}_0)), \end{aligned}$$

we derive

$$\mathbf{F}(\mathbf{X}_0) = \begin{cases} \mathbf{0}, & M = V, \\ \mathbf{O}, & M = \mathrm{Skw}, \\ c_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + c_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + c_3 \mathbf{n} \otimes \mathbf{n}, & M = \mathrm{Sym}, \end{cases}$$

where $c_i = c_i(\mathbf{X}_0)$. Hence, by means of the above fact and (3.23) we infer

$$\begin{aligned} \mathbf{F}(\mathbf{Q}^{\mathrm{T}} * \mathbf{X}_0) &= \mathbf{F}(\mathbf{Q}_0{}^{\mathrm{T}} * \mathbf{X}_0) \qquad (\mathbf{Q}_0 = -\mathbf{Q}) \\ &= \mathbf{Q}_0{}^{\mathrm{T}} * (\mathbf{F}(\mathbf{X}_0)) = \mathbf{Q}^{\mathrm{T}} * (\mathbf{F}(\mathbf{X}_0)) \end{aligned}$$

for any $\mathbf{Q} \in D_{4mh} \setminus D_{2md}$ and any $\mathbf{F} \in D_{2md}(D, M)$, i.e. the condition (2.2) is fulfilled for each $\mathbf{Q} \in D_{4mh} \setminus D_{2md}$. *Q.E.D.*

If the tensor functions of the form $(\mathbf{e}\cdot\mathbf{W}\mathbf{e}')g_m(\mathbf{A})\mathbf{n}$ are removed from the surface $\mathbf{S}(\mathbf{X})$ given by (3.21), then from the above proof we know that the condition (2.2) is still satisfied for each $\mathbf{Q} \in D_{\infty h} \setminus D_{4mh}$. On the other hand, for each $\mathbf{Q} \in D_{4mh} \setminus D_{2md}$, the intersection point $\mathbf{X}_0 \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is given by $\mathbf{X}_0 \in D(\mathbf{n})$ or by $(3.25)_2$ and (3.24) with each $a_\alpha = 0$. For the latter, with the aid of (3.23), one may readily verify that the condition (2.2) is fulfilled for scalar-valued and second order tensor-valued functions, i.e. for each image set $M \in \{R, \mathrm{Skw}, \mathrm{Sym}\}$. This shows that the tensor functions mentioned before is needed only for vector-valued functions, i.e. only for the image set $M = V$. Thus, we arrive at the following simplified result.

COROLLARY. The surface

$$\mathbf{S}(\mathbf{X}) = (\mathbf{n}\otimes\mathbf{n}; \mathbf{n} \vee \boldsymbol{\eta}_{2m-1}(\mathbf{v}_\alpha^{\circ}); \boldsymbol{\eta}_{2m-1}(\mathbf{W}_\theta\mathbf{n}); \boldsymbol{\eta}_{2m-1}((\mathbf{A}_\sigma\mathbf{n})^{\circ}), f_m(\mathbf{A}_\sigma)\mathbf{n}, \boldsymbol{\Phi}_{2m-1}(\mathbf{q}(\mathbf{A}_\sigma)); (\mathbf{n}\cdot\mathbf{v}_\alpha)\boldsymbol{\Phi}_{m-1}(\mathbf{q}(\mathbf{A}_\sigma))) \tag{3.26}$$

is an IES for $D_{2md}(D, M)$ for each $M \in \{R, \mathrm{Skw}, \mathrm{Sym}\}$.

3.6. Classes S_{4m} for $m \geq 2$

$$S_{4m} = C_{\infty h} \cap D_{2md} = \{(-1)^k \mathbf{R}_{\mathbf{n}}^{k\pi/2m} \mid k = 0, 1, 2, \ldots, 4m-1\}. \tag{3.27}$$

THEOREM 5. *Let* $\mathbf{e}$ *be any given unit vector on the* $\mathbf{n}$*-plane. Then the surface*

$$\mathbf{S}(\mathbf{X}) = (\mathbf{E}\mathbf{n}; \mathbf{n} \vee \boldsymbol{\eta}_{2m-1}(\mathbf{v}_\alpha^{\circ}); \boldsymbol{\eta}_{2m-1}(\mathbf{W}_\theta\mathbf{n}); \boldsymbol{\eta}_{2m-1}((\mathbf{A}_\sigma\mathbf{n})^{\circ}), f_m(\mathbf{A}_\sigma)\mathbf{n}, g_m(\mathbf{A}_\sigma)\mathbf{n}), \tag{3.28}$$

is an IES for $S_{4m}(D, M)$, *where* $\boldsymbol{\eta}_r(\mathbf{z})$ *is given by* (3.12) *for any vector* $\mathbf{z}$ *on the* $\mathbf{n}$*-plane and each integer* $r \geq 1$, *and moreover* $f_m(\mathbf{A}_\sigma)$ *and* $g_m(\mathbf{A}_\sigma)$ *are given by* (3.22).

Proof. It can easily be verified that the tensor functions $f_m(\mathbf{A})\mathbf{n}$ and $g_m(\mathbf{A})\mathbf{n}$ are invariant under the group S_{4m} by using the formula $(2.15)_1$. Moreover, it is known that each of the tensor functions $\boldsymbol{\eta}_{2m-1}(\mathbf{z})$, $\mathbf{z} = \mathbf{v}^{\circ}$, $\mathbf{W}\mathbf{n}$, $(\mathbf{A}\mathbf{n})^{\circ}$ is invariant under the group $D_{2md}(\supset S_{4m})$ (cf. the former part of the proof for Theorem 4). Thus, we conclude that the given surface $\mathbf{S}(\mathbf{X})$ obeys the invariance condition (2.4).

Now consider the condition (2.2). It is readily verified that the latter can be satisfied for each $\mathbf{Q} \in \mathrm{Orth} \setminus (C_{\infty h} \setminus S_{4m})$ by using $(3.7)_3$. Moreover, for each $\mathbf{Q} \in C_{\infty h} \setminus S_{4m}$, the intersection point $\mathbf{X}_0 = (\mathbf{v}_\alpha, \mathbf{W}_\theta, \mathbf{A}_\sigma) \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is determined by the system of tensor equations of the forms (A.15)–(A.17) and

$$f_m(\mathbf{Q}^{\mathrm{T}} * \mathbf{A})\mathbf{Q}\mathbf{n} = f_m(\mathbf{A}), \qquad g_m(\mathbf{Q}^{\mathrm{T}} * \mathbf{A})\mathbf{Q}\mathbf{n} = g_m(\mathbf{A})\mathbf{n},$$

where the variables are: $\mathbf{v} = \mathbf{v}_1, \ldots, \mathbf{v}_a$; $\mathbf{W} = \mathbf{W}_1, \ldots, \mathbf{W}_b$; $\mathbf{A} = \mathbf{A}_1, \ldots, \mathbf{A}_c$. From the latter and Theorems A.4 we infer that $\mathbf{S} \cap (\mathbf{Q} * \mathbf{S}) = D(\mathbf{n})$ for each $\mathbf{Q} \in C_{\infty h} \setminus S_{4m}$. Then, by this fact and Lemma A we infer that the condition (2.2) is satisfied for each $\mathbf{Q} \in C_{\infty h} \setminus S_{4m}$ and each $m \geq 2$. *Q.E.D.*

3.7. The tetragonal crystal class D_{2d}

$$D_{2d} = \{(-1)^k \mathbf{R}_{\mathbf{n}}^{k\pi/2},\ (-1)^k \mathbf{R}_{\mathbf{l}_k}^{\pi} \mid \mathbf{l}_k = \mathbf{R}_{\mathbf{n}}^{k\pi/4}\mathbf{e},\ k = 0,1,2,3\}. \tag{3.29}$$

Note that the orthonormal vectors $\mathbf{l}_0 = \mathbf{e}$ and $\mathbf{l}_2 = \mathbf{e}'$ represent the two two-fold rotation axes of D_{2d}.

THEOREM 6. *The surface*

$$\begin{aligned}\mathbf{S}(\mathbf{X}) = (\mathbf{n}\otimes\mathbf{n};\ \mathbf{n}\vee\boldsymbol{\eta}_1(\mathbf{v}^{\circ}) + (\mathbf{v}\cdot\mathbf{n})\mathbf{D}_1;\ \boldsymbol{\eta}_1(\mathbf{W}_\theta\mathbf{n});\ \boldsymbol{\eta}_1((\mathbf{A}\mathbf{n})^{\circ}) + f_1(\mathbf{A})\mathbf{n},\\ \boldsymbol{\Phi}_1(\mathbf{q}(\mathbf{A}_\sigma));\ (\mathbf{e}\cdot\mathbf{W}_\theta\mathbf{e}')g_1(\mathbf{A}_\sigma)\mathbf{n})\end{aligned} \tag{3.30}$$

is an IES for $D_{2d}(D, M)$, where the tensor functions $\boldsymbol{\eta}_1(\mathbf{z})$, $\boldsymbol{\Phi}_1(\mathbf{z})$, $f_1(\mathbf{A})$ and $g_1(\mathbf{A})$ are obtained by taking $m = 1$ in (3.12), (3.16) *and* (3.22), *respectively.*

P r o o f. First, we prove that the given surface $\mathbf{S}(\mathbf{X})$ meets the invariance condition (2.4). To this end, it suffices to prove that the tensor function $(\mathbf{v}\cdot\mathbf{n})\mathbf{D}_1$ is invariant under the group D_{2d}, since each other tensor function in the surface $\mathbf{S}(\mathbf{X})$ given here is included in the IES given by Theorem 4, where $m = 1$, and is invariant under the group D_{2d}. By using (2.25) we infer $((\mathbf{Q}^{\mathrm{T}}\mathbf{v})\cdot\mathbf{n})\mathbf{Q} * \mathbf{D}_1 = \delta(\mathbf{v}\cdot\mathbf{n})(\mathbf{D}_1\cos 2\theta + \mathbf{D}_2 \sin 2\theta)$ for $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{n}}^{\theta}$, $\delta^2 = 1$. Hence, we have

$$((\mathbf{Q}^{\mathrm{T}}\mathbf{v})\cdot\mathbf{n})\mathbf{Q} * \mathbf{D}_1 = (\mathbf{v}\cdot\mathbf{n})\mathbf{D}_1, \quad \forall\, \mathbf{Q} \in S_4.$$

Moreover, we have

$$((\mathbf{R}_{\mathbf{e}}^{\pi}\mathbf{v})\cdot\mathbf{n})\mathbf{R}_{\mathbf{e}}^{\pi} * \mathbf{D}_1 = (\mathbf{v}\cdot\mathbf{n})\mathbf{D}_1 .$$

Thus, $(\mathbf{v}\cdot\mathbf{n})\mathbf{D}_1$ is invariant under the group D_{2d}, since S_4 and $\mathbf{R}_{\mathbf{e}}^{\pi}$ generate the latter.

Next, we prove that the given surface obeys the condition (2.2). It can be readily verified that for each $\mathbf{Q} \in \mathrm{Orth} \setminus (D_{\infty h} \setminus D_{2d})$ the condition (2.2) can be satisfied by using $(3.7)_1$.

Moreover, for each $\mathbf{Q} \in D_{\infty h} \setminus D_{4h}$, the intersection point $\mathbf{X}_0 = (\mathbf{v}_\alpha, \mathbf{W}_\theta, \mathbf{A}_\sigma) \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is determined by (A.10) (for $m = 1$), (A.15)–(A.17) (for $m = 1$) and

$$\begin{aligned}((\mathbf{Q}^{\mathrm{T}}\mathbf{v})\cdot\mathbf{n})\mathbf{Q} * \mathbf{D}_1 &= (\mathbf{v}\cdot\mathbf{n})\mathbf{D}_1 ,\\ f_1(\mathbf{Q}^{\mathrm{T}} * \mathbf{A})\mathbf{Q}\mathbf{n} &= f_1(\mathbf{A})\mathbf{n},\\ (\mathbf{e}\cdot(\mathbf{Q}^{\mathrm{T}}\mathbf{W})\mathbf{e}')g_1(\mathbf{Q}^{\mathrm{T}} * \mathbf{A})\mathbf{Q}\mathbf{n} &= (\mathbf{e}\cdot\mathbf{W}\mathbf{e}')g_1(\mathbf{A})\mathbf{n},\end{aligned}$$

where the variables are: $\mathbf{v} = \mathbf{v}_1, \ldots, \mathbf{v}_a$; $\mathbf{W} = \mathbf{W}_1, \ldots, \mathbf{W}_b$; $\mathbf{A} = \mathbf{A}_1, \ldots, \mathbf{A}_c$. From the first equation above and Theorems A.4 and the proof of Theorem A.2 (cf. $(\mathrm{A}.6)_4$) we derive

$$\mathbf{v}_\alpha = \mathbf{0}, \qquad \mathbf{W}_\theta = b_\theta\mathbf{E}\mathbf{n}, \qquad \mathbf{A}_\sigma = c_\sigma\mathbf{I} + d_\sigma\mathbf{n}\otimes\mathbf{n} \tag{3.31}$$

for each $\mathbf{Q} \in D_{\infty h} \setminus D_{4h}$, and moreover the point $\mathbf{X}_0$ given above satisfies the last two equations given before. Evidently,

$$\mathbf{Q}_0 * \mathbf{X}_0 = \mathbf{X}_0, \qquad \mathbf{Q}_0 = -\mathbf{R}_{\mathbf{n}}^{\pi/2} \in D_{2d}$$

for any point $\mathbf{X}_0 \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$. Then we have

$$\mathbf{Q}_0 * (\mathbf{F}(\mathbf{X}_0)) = \mathbf{F}(\mathbf{Q}_0 * \mathbf{X}_0) = \mathbf{F}(\mathbf{X}_0)$$

for any $\mathbf{F} \in D_{2d}(D, M)$, $\mathbf{X}_0 \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ and $\mathbf{Q} \in D_{\infty h} \setminus D_{4h}$. Thus, we deduce (2.7) with $a(\mathbf{X}_0) = 0$. Applying the latter fact and the fact that

$$\forall\, \mathbf{Q} \in D_{\infty h}, \exists \mathbf{Q}' \in D_{2d}: \ \mathbf{Q} * (\mathbf{En}) = \mathbf{Q}' * (\mathbf{En}), \ \mathbf{Q} * (\mathbf{n} \otimes \mathbf{n}) = \mathbf{Q}' * (\mathbf{n} \otimes \mathbf{n}),$$

we deduce that the condition (2.2) is satisfied for each $\mathbf{Q} \in D_{\infty h} \setminus D_{4h}$.

Finally, for each $\mathbf{Q} \in D_{4h} \setminus D_{2d}$, by means of (3.23), where $m = 1$, we infer that the intersection point $\mathbf{X}_0 = (\mathbf{v}_\alpha, \mathbf{W}_\theta, \mathbf{A}_\sigma) \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is of the form

$$\mathbf{v}_\alpha = \mathbf{0}, \qquad \mathbf{W}_\theta = b_\theta \mathbf{En}, \qquad \mathbf{A}_\sigma = c_\sigma \mathbf{I} + d_\sigma \mathbf{n} \otimes \mathbf{n} + h_\sigma \mathbf{D}_1$$

with $b_\theta h_\sigma = 0$ for $\theta = 1, \ldots, b$ and $\sigma = 1, \ldots, c$. Hence, for each $\mathbf{Q} \in D_{4h} \setminus D_{2d}$, the point $\mathbf{X}_0$ is given by (3.31) or by (3.25) with $\mathbf{e}_1 = \mathbf{e}$ and $\mathbf{e}_2 = \mathbf{e}'$. From the argument given above for the corresponding case, we know that the condition (2.2) is satisfied for the point $\mathbf{X}_0$ given by (3.31). On the other hand, from the latter part of the proof for Theorem 4, we know that the condition (2.2) is also satisfied for the point $\mathbf{X}_0$ given by (3.25). Thus, we conclude that the given surface $\mathbf{S}(\mathbf{X})$ also fulfils the condition (2.2) for each $\mathbf{Q} \in D_{4h} \setminus D_{2d}$. *Q.E.D.*

By virtue of the same argument as that used to derive the corollary of Theorem 4, we arrive at the following simplified result.

COROLLARY. The surface

$$\text{(3.32)} \qquad \mathbf{S}(\mathbf{X}) = (\mathbf{n} \otimes \mathbf{n};\ \mathbf{n} \vee \boldsymbol{\eta}_1(\mathbf{v}^o) + (\mathbf{v} \cdot \mathbf{n})\mathbf{D}_1;\ \boldsymbol{\eta}_1(\mathbf{W}_\theta \mathbf{n});$$
$$\boldsymbol{\eta}_1((\mathbf{An})^o) + f_1(\mathbf{A})\mathbf{n}, \boldsymbol{\Phi}_1(\mathbf{q}(\mathbf{A}_\sigma)))$$

is an IES for $D_{2d}(D, M)$ for each $M \in \{R, \mathrm{Skw}, \mathrm{Sym}\}$.

3.8. The tetragonal crystal class S_4

$$\text{(3.33)} \qquad S_4 = D_{2d} \cap C_{\infty h} = \{(-1)^k \mathbf{R}_{\mathbf{n}}^{k\pi/2} \mid k = 0, 1, 2, 3\}.$$

THEOREM 7. *Let* $\mathbf{e}$ *and* $\mathbf{e}'$ *be any two orthonormal vectors on the* $\mathbf{n}$*-plane. Then the surface*

$$\text{(3.34)} \qquad \mathbf{S}(\mathbf{X}) = (\mathbf{En};\ \mathbf{n} \vee \boldsymbol{\eta}_1(\mathbf{v}^o) + (\mathbf{v} \cdot \mathbf{n})\mathbf{D}_1;\ \boldsymbol{\eta}_1(\mathbf{W}_\theta \mathbf{n});$$
$$\boldsymbol{\eta}_1((\mathbf{An})^o) + f_1(\mathbf{A})\mathbf{n},\ g_1(\mathbf{A}_\sigma)\mathbf{n})$$

is an IES for $S_4(D, M)$*, where each tensor function is given in Theorem 6.*

P r o o f. From the former part of the proof for Theorem 5 we know that each tensor function except $(\mathbf{v}\cdot\mathbf{n})\mathbf{D}_1$ is invariant under the group S_4. Moreover, it is known that the tensor function just indicated is invariant under the group $D_{2d}(\supset S_4)$ (cf. the former part of the proof for Theorem 6). Thus, we conclude that the given surface $\mathbf{S}(\mathbf{X})$ meets the invariance condition (2.4).

In what follows we prove that the given surface $\mathbf{S}(\mathbf{X})$ obeys the condition (2.2). It can be easily verified by using $(3.7)_3$ that for each $\mathbf{Q} \in \mathrm{Orth} \setminus (C_{\infty h} \setminus S_4)$ the condition (2.2) can be satisfied. Moreover, for each $\mathbf{Q} \in C_{\infty h}\setminus S_4$, the intersection point $\mathbf{X}_0 = (\mathbf{v}_\alpha, \mathbf{W}_\theta, \mathbf{A}_\sigma) \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is determined by (A.15)–(A.17) (for $m = 1$) and

$$((\mathbf{Q}^{\mathrm{T}}\mathbf{v})\cdot\mathbf{n})\mathbf{Q} * \mathbf{D}_1 = (\mathbf{v}\cdot\mathbf{n})\mathbf{D}_1\,,$$
$$f_1(\mathbf{Q}^{\mathrm{T}} * \mathbf{A})\mathbf{Q}\mathbf{n} = f_1(\mathbf{A})\mathbf{n}, \qquad g_1(\mathbf{Q}^{\mathrm{T}} * \mathbf{A})\mathbf{Q}\mathbf{n} = g_1(\mathbf{A})\mathbf{n},$$

where the variables are: $\mathbf{v} = \mathbf{v}_1, \ldots, \mathbf{v}_a$; $\mathbf{W} = \mathbf{W}_1, \ldots, \mathbf{W}_b$; $\mathbf{A} = \mathbf{A}_1, \ldots, \mathbf{A}_c$. From Theorem A.4 we derive

$$\mathbf{v}_\alpha = a_\alpha \mathbf{n}, \qquad \mathbf{W}_\sigma = b_\sigma \mathbf{E}\mathbf{n}, \qquad \mathbf{A}_\sigma = c_\sigma \mathbf{I} + d_\sigma \mathbf{n}\otimes\mathbf{n} + p_\sigma \mathbf{D}_1 + q_\sigma \mathbf{D}_2\,.$$

From the last three equations given before we further derive $a_\alpha = p_\sigma = q_\sigma = 0$ and hence the intersection point $\mathbf{X}_0$ is given by (3.31). Thus, from the corresponding case in the proof of Theorem 6 we conclude that the condition (2.2) can be satisfied for each $\mathbf{Q} \in C_{\infty h}\setminus S_4$. *Q.E.D.*

3.9. Remark

In each IES given in this section, each vector and each second order tensor, except the constant tensor $\mathbf{D}(G)$, is a *homogeneous polynomial function* of some components of the vector variable and/or the second order tensor variable concerned. In reality, the trigonometric functions $\cos r\theta$ and $\sin r\theta$ for each integer $r \geq 1$ are associated with the following two kinds of Tschebysheff polynomials.

$$H_r(\cos\theta) = \cos r\theta, \qquad T_r(\sin\theta) = \frac{\sin(r+1)\theta}{\cos\theta}\,. \tag{3.35}$$

Let $C_r(x) \in \{H_r(x), T_r(x)\}$. Then we have

$$C_r(x) = \begin{cases} \displaystyle\sum_{k=0}^{n} c_{2k}x^{2k} & \text{if} \quad r = 2n\,, \\ \displaystyle\sum_{k=1}^{n} c_{2k-1}x^{2k-1} & \text{if} \quad r = 2n-1\,, \end{cases} \tag{3.36}$$

where each c_k is a constant. Hence, with the aid of the above formulas and (2.11), we infer that for any vector $\mathbf{z}$ on the $\mathbf{n}$-plane, the functions $|\mathbf{z}|^r \cos r < \mathbf{z}, \mathbf{e} >$

and $|\mathbf{z}|^r \sin r < \mathbf{z}, \mathbf{e} >$ for each $r \geq 1$, which are used to construct each presented IES, are homogeneous polynomials of degree r in the components $\mathbf{z}\cdot\mathbf{e}$ and $\mathbf{z}\cdot\mathbf{e}'$, where $\mathbf{z} = \mathbf{v}^o$, $\mathbf{Wn}$, $(\mathbf{An})^o$, $\mathbf{q}(\mathbf{A})$.

The results given in this section simplify the corresponding ones given in [49]. In reality, in each IES given here, each tensor function is presented in concise and clear forms, while in each IES given in [49], each tensor function is given in a somewhat implicit and complicated summation form.

Other remarks will be given in Sec. 6.

4. Cubic crystal classes O_h, T_d and T_h

$$O_h = \left(\bigcup_{k=1}^{3}(C_{4h}(\mathbf{n}_k) \cup C_{2h}(\mathbf{p}_k) \cup C_{2h}(\mathbf{q}_k))\right) \cup \left(\bigcup_{t=1}^{4} S_6(\mathbf{r}_t)\right), \tag{4.1}$$

$$T_d = \left(\bigcup_{k=1}^{3}(S_4(\mathbf{n}_k) \cup C_{1h}(\mathbf{p}_k) \cup C_{1h}(\mathbf{q}_k))\right) \cup \left(\bigcup_{t=1}^{4} C_3(\mathbf{r}_t)\right), \tag{4.2}$$

$$T_h = \left(\bigcup_{k=1}^{3} C_{2h}(\mathbf{n}_k)\right) \cup \left(\bigcup_{t=1}^{4} S_6(\mathbf{r}_t)\right), \tag{4.3}$$

where $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ are three orthonormal vectors and

$$\begin{aligned} \sqrt{2}\mathbf{p}_1 &= \mathbf{n}_3 + \mathbf{n}_2, & \sqrt{2}\mathbf{p}_2 &= \mathbf{n}_1 + \mathbf{n}_3, & \sqrt{2}\mathbf{p}_3 &= \mathbf{n}_2 + \mathbf{n}_1, \\ \sqrt{2}\mathbf{q}_1 &= \mathbf{n}_3 - \mathbf{n}_2, & \sqrt{2}\mathbf{q}_2 &= \mathbf{n}_1 - \mathbf{n}_3, & \sqrt{2}\mathbf{q}_3 &= \mathbf{n}_2 - \mathbf{n}_1; \end{aligned} \tag{4.4}$$

$$\begin{aligned} \sqrt{3}\mathbf{r}_1 &= \mathbf{n}_1 - \mathbf{n}_2 - \mathbf{n}_3, & \sqrt{3}\mathbf{r}_2 &= \mathbf{n}_2 - \mathbf{n}_3 - \mathbf{n}_1, \\ \sqrt{3}\mathbf{r}_3 &= \mathbf{n}_3 - \mathbf{n}_1 - \mathbf{n}_2, & \sqrt{3}\mathbf{r}_4 &= \mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3. \end{aligned} \tag{4.5}$$

Each $\mathbf{n}_k$ is called a four-fold axis of either of the groups O_h and T_d or a two-fold axis of the group T_h, and each $\mathbf{r}_t$ is called a three-fold axis of each of the groups O_h and T_d and T_h.

Here and hereafter, for any unit vector $\mathbf{u}$ and each integer $m \geq 2$, $S_{2m}(\mathbf{u})$ and $C_{2mh}(\mathbf{u})$ are used to denote the groups obtained by the replacement of $\mathbf{n}$ with $\mathbf{u}$ in (3.8) and $(3.9)_2$, (3.13) and $(3.14)_2$, and (3.27), respectively. Moreover, $C_3(\mathbf{u})$ is used to denote the rotation subgroup of $S_6(\mathbf{u})$. Finally,

$$C_{1h}(\mathbf{u}) = \{\mathbf{I}, -\mathbf{R}_{\mathbf{u}}^{\pi}\}, \qquad C_{2h}(\mathbf{u}) = \{\pm\mathbf{I}, \pm\mathbf{R}_{\mathbf{u}}^{\pi}\}.$$

4.1. The class O_h

The following fourth-order tensor is invariant under the group O_h:

$$\mathbf{O}_h = \sum_{k=1}^{3}(\overset{4}{\otimes}\, \mathbf{n}_k). \tag{4.6}$$

In this section and the next section, each presented surface $\mathbf{S}(\mathbf{X})$ is formed by tensor functions of the form

$$\mathbf{G} \odot (\overset{s}{\otimes} \mathbf{Z}),$$

where the tensor $\mathbf{G}$ is invariant under the anisotropy group G concerned, and $\mathbf{Z}$ is one of the vector variables and the second order tensor variables. Evidently, each such tensor function is invariant under the group G concerned and therefore the given surface $\mathbf{S}(\mathbf{X})$ meets the invariance condition (2.4). As a result, henceforth only the invariance of the tensor $\mathbf{G}$ is indicated and the invariance condition (2.4) is no longer mentioned.

THEOREM 8. *The surface*

$$\mathbf{S}(\mathbf{X}) = (\mathbf{O}_h : (\overset{2}{\otimes} \mathbf{v}_\alpha);\ \mathbf{O}_h : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W}_\theta));\ \mathbf{O}_h : \mathbf{A}_\sigma,\ \mathbf{O}_h : \mathbf{A}_\sigma^2) \tag{4.7}$$

is an IES for $O_h(D, M)$.

P r o o f. It is evident that the condition (2.2) can be satisfied for each $\mathbf{Q} \in O_h$. On the other hand, for each $\mathbf{Q} \in \mathrm{Orth}\backslash O_h$, by Theorem A.5 we know that the point $\mathbf{X}_0 = (\mathbf{v}_\alpha, \mathbf{W}_\theta, \mathbf{A}_\sigma) \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is given by the following cases.

CASE 1. If there exist $\mathbf{u}$, $\mathbf{v} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ or $\mathbf{u}$, $\mathbf{v} \in \{\mathbf{r}_1, \ldots, \mathbf{r}_4\}$ such that $\overset{2}{\otimes} (\mathbf{Q}^{\mathrm{T}}\mathbf{u}) = \overset{2}{\otimes} \mathbf{v}$, then $\mathbf{X}_0 \in D(\mathbf{u})$;

CASE 2. If $\overset{2}{\otimes} (\mathbf{Q}^{\mathrm{T}}\mathbf{u}) \neq \overset{2}{\otimes} \mathbf{v}$ for any $\mathbf{u}, \mathbf{v} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ and any $\mathbf{u}, \mathbf{v} \in \{\mathbf{r}_1, \ldots, \mathbf{r}_4\}$, then $\mathbf{v}_\alpha = \mathbf{0}$, $\mathbf{W}_\theta = \mathbf{O}$, $\mathbf{A}_\sigma = c_\sigma \mathbf{I}$.

For Case 1, for each $\mathbf{F} \in O_h(D, M)$ we have

$$\mathbf{R}_0 * (\mathbf{F}(\mathbf{X}_0)) = \mathbf{F}(\mathbf{R}_0 * \mathbf{X}_0) = \mathbf{F}(\mathbf{X}_0),$$

where

$$\mathbf{R}_0 = \mathbf{R}_{\mathbf{u}}^\theta \in O_h, \qquad \theta = \begin{cases} \pi/2, & \mathbf{u} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}, \\ 2\pi/3, & \mathbf{u} \in \{\mathbf{r}_1, \ldots, \mathbf{r}_4\}. \end{cases}$$

From these we deduce

$$\mathbf{F}(\mathbf{X}_0) = \begin{cases} a(\mathbf{X}_0)\mathbf{u}, & M = V, \\ b(\mathbf{X}_0)\mathbf{E}\mathbf{u}, & M = \mathrm{Skw}, \\ c(\mathbf{X}_0)\mathbf{I} + d(\mathbf{X}_0)\mathbf{u} \otimes \mathbf{u}, & M = \mathrm{Sym}. \end{cases} \tag{4.8}$$

Then, by using the latter and the fact that for each $\mathbf{Q}$ in question, there exists $\mathbf{Q}_0 \in O_h$ such that

$$\mathbf{Q}^{\mathrm{T}}\mathbf{u} = \mathbf{Q}_0{}^{\mathrm{T}}\mathbf{u} \qquad \text{and} \qquad \mathbf{Q}^{\mathrm{T}} * (\mathbf{E}\mathbf{u}) = \mathbf{Q}_0{}^{\mathrm{T}} * (\mathbf{E}\mathbf{u}),$$

we infer

$$\mathbf{F}(\mathbf{Q}^{\mathrm{T}} * \mathbf{X}_0) = \mathbf{F}({\mathbf{Q}_0}^{\mathrm{T}} * \mathbf{X}_0) = {\mathbf{Q}_0}^{\mathrm{T}} * (\mathbf{F}(\mathbf{X}_0)) = \mathbf{Q}^{\mathrm{T}} * (\mathbf{F}(\mathbf{X}_0))$$

for any $\mathbf{F} \in O_h(D, M)$ and therefore the condition (2.2) is satisfied.

Moreover, for Case 2, we have

$$\mathbf{R}_0 * (\mathbf{F}(\mathbf{X}_0)) = \mathbf{F}(\mathbf{R}_0 * \mathbf{X}_0) = \mathbf{F}(\mathbf{X}_0)$$

for any $\mathbf{F} \in O_h(D, M)$ and any $\mathbf{R}_0 \in O_h$. From this we derive

$$\mathbf{F}(\mathbf{X}_0) = \begin{cases} \mathbf{0}, & M = V\,, \\ \mathbf{O}, & M = \mathrm{Skw}\,, \\ c(\mathbf{X}_0)\mathbf{I}, & M = \mathrm{Sym}\,, \end{cases}$$

and hence for any $\mathbf{Q} \in \mathrm{Orth}$,

$$\mathbf{F}(\mathbf{Q}^{\mathrm{T}} * \mathbf{X}_0) = \mathbf{F}(\mathbf{X}_0) = \mathbf{Q}^{\mathrm{T}} * (\mathbf{F}(\mathbf{X}_0)),$$

i.e. the condition (2.2) is fulfilled. *Q.E.D.*

4.2. The class T_d

The following third-order tensor is invariant under the group T_d:

$$\mathbf{T}_d = \sum_{k=1}^{3} \boldsymbol{\omega}_k \otimes \mathbf{n}_k = \sum_{k=1}^{3} \mathbf{n}_k \otimes \boldsymbol{\omega}_k\,, \tag{4.9}$$

where

$$\boldsymbol{\omega}_1 = \mathbf{n}_2 \vee \mathbf{n}_3\,, \qquad \boldsymbol{\omega}_2 = \mathbf{n}_3 \vee \mathbf{n}_1\,, \qquad \boldsymbol{\omega}_3 = \mathbf{n}_1 \vee \mathbf{n}_2\,. \tag{4.10}$$

THEOREM 9. *The surface*

$$\begin{aligned} \mathbf{S}(\mathbf{X}) = (&\mathbf{T}_d\mathbf{v}_\alpha,\ \mathbf{O}_h : (\overset{2}{\otimes}\, \mathbf{v}_\alpha); \mathbf{T}_d : (\overset{2}{\otimes}\, (\mathbf{E} : \mathbf{W}_\theta)), \\ &\mathbf{O}_h : (\overset{2}{\otimes}\, (\mathbf{E} : \mathbf{W}_\theta));\ \mathbf{T}_d : \mathbf{A}_\sigma,\ \mathbf{O}_h : \mathbf{A}_\sigma;\ \mathbf{T}_d : (\mathbf{W}_\theta\mathbf{A}_\sigma)) \end{aligned} \tag{4.11}$$

is an IES for $T_d(D, M)$.

P r o o f. It is evident that for each $\mathbf{Q} \in T_d$ the condition (2.2) can be satisfied. In what follows we prove that the condition (2.2) can also be satisfied for each $\mathbf{Q} \in \mathrm{Orth} \setminus T_d$. First, for each $\mathbf{Q} \in \mathrm{Orth} \setminus O_h$, by using Theorem A.6 we know that the point $\mathbf{X}_0 = (\mathbf{v}_\alpha,\ \mathbf{W}_\theta,\ \mathbf{A}_\sigma) \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is given by the following cases.

CASE 1. $\mathbf{v}_\alpha = 0$, $\mathbf{W}_\sigma = b_\theta \mathbf{E}\mathbf{u}$, $\mathbf{A}_\sigma = c_\sigma \mathbf{I} + d_\sigma \mathbf{u} \otimes \mathbf{u}$ if

$$\exists\, \mathbf{u}, \mathbf{v} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\} : \overset{2}{\otimes} (\mathbf{Q}^{\mathrm{T}}\mathbf{u}) = \overset{2}{\otimes} \mathbf{v};$$

CASE 2. $\mathbf{X}_0 \in D(\mathbf{u})$ if

$$\exists\, \mathbf{u},\ \mathbf{v} \in \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\} :\ \mathbf{Q}^{\mathrm{T}}\mathbf{u} = \mathbf{v}\,; \tag{4.12}$$

CASE 3. $\mathbf{v}_\alpha = \mathbf{0}$, $\mathbf{W}_\theta = \mathbf{O}$, $\mathbf{A}_\sigma = c_\sigma \mathbf{I}$, if $\mathbf{Q}$ obeys

$$\forall\, \mathbf{u},\ \mathbf{v} \in \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\} :\ \mathbf{Q}^{\mathrm{T}}\mathbf{u} \neq \mathbf{v} \quad \text{and} \quad \forall\, \mathbf{u}, \mathbf{v} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\} : \overset{2}{\otimes} (\mathbf{Q}^{\mathrm{T}}\mathbf{u}) \neq \overset{2}{\otimes} \mathbf{v}\,. \tag{4.13}$$

It is readily verfied that the condition (2.2) can be satisfied for Case 3 and Case 1. For Case 2, we have

$$\mathbf{R}_0 * (F(\mathbf{X}_0)) = F(\mathbf{R}_0 * \mathbf{X}_0) = F(\mathbf{X}_0), \qquad \mathbf{R}_0 = \mathbf{R}_{\mathbf{u}}^{2\pi/3} \in T_d$$

for each $\mathbf{F} \in T_d(D, M)$. From this we derive (4.8). Then by using the fact that for each $\mathbf{Q}$ satisfying (4.12) there is $\mathbf{Q}_0 \in T_d$ such that

$$\mathbf{Q}^{\mathrm{T}}\mathbf{u} = \mathbf{Q}_0{}^{\mathrm{T}}\mathbf{u}, \qquad \mathbf{Q}^{\mathrm{T}} * (\mathbf{E}\mathbf{u}) = \mathbf{Q}_0^{\mathrm{T}} * (\mathbf{E}\mathbf{u})\,,$$

we infer

$$\mathbf{F}(\mathbf{Q}^{\mathrm{T}} * \mathbf{X}_0)) = \mathbf{F}(\mathbf{Q}_0^{\mathrm{T}} * \mathbf{X}_0) = \mathbf{Q}_0^{\mathrm{T}} * (\mathbf{F}(\mathbf{X}_0)) = \mathbf{Q}^{\mathrm{T}} * (\mathbf{F}(\mathbf{X}_0)).$$

Thus the condition (2.2) is satisfied for Case 2.

Next, for each $\mathbf{Q} \in O_h \setminus T_d$, the point $\mathbf{X}_0 = (\mathbf{v}_\alpha,\ \mathbf{W}_\theta,\ \mathbf{A}_\sigma) \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is determined by

$$\mathbf{T}_d \mathbf{v}_\alpha = \mathbf{O}; \qquad \mathbf{T}_d : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W}_\theta)) = \mathbf{0}; \qquad \mathbf{T}_d : \mathbf{A}_\sigma = \mathbf{0}, \qquad \mathbf{T}_d : (\mathbf{W}_\theta \mathbf{A}_\sigma)) = 0,$$

where $\alpha = 1, 2, \ldots, a$, $\theta = 1, 2, \ldots, b$, $\sigma = 1, 2, \ldots, c$. The first three equations yield

$$\mathbf{v}_\alpha = \mathbf{0}, \qquad \mathbf{W}_\theta = b_\theta \mathbf{E}\mathbf{u}, \qquad \mathbf{u} \in \{\mathbf{n}_1,\ \mathbf{n}_2,\ \mathbf{n}_3\},$$
$$\mathbf{A}_\sigma = a_\sigma \mathbf{n}_1 \otimes \mathbf{n}_1 + b_\sigma \mathbf{n}_2 \otimes \mathbf{n}_2 + c_\sigma \mathbf{n}_3 \otimes \mathbf{n}_3\,;$$

and the last equation further produces

$$\mathbf{v}_\alpha = \mathbf{0}, \qquad \mathbf{W}_\theta = b_\theta \mathbf{E}\mathbf{u}, \qquad \mathbf{A}_\sigma = c_\sigma \mathbf{I} + d_\sigma \mathbf{u} \otimes \mathbf{u}, \qquad \mathbf{u} \in \{\mathbf{n}_1,\ \mathbf{n}_2,\ \mathbf{n}_3\}\,;$$

or

$$\mathbf{v}_\alpha = \mathbf{0}, \qquad \mathbf{W}_\theta = \mathbf{O}, \qquad \mathbf{A}_\sigma = a_\sigma \mathbf{n}_1 \otimes \mathbf{n}_1 + b_\sigma \mathbf{n}_2 \otimes \mathbf{n}_2 + c_\sigma \mathbf{n}_3 \otimes \mathbf{n}_3\,.$$

For the former, we have

$$\mathbf{R}_0 * (F(\mathbf{X}_0)) = F(\mathbf{R}_0 * \mathbf{X}_0) = F(\mathbf{X}_0), \qquad \mathbf{R}_0 = -\mathbf{R}_{\mathbf{u}}^{\pi/2} \subset T_d\,,$$

for each $F \in T_d(D, M)$. From the above we derive (4.8) with $a(\mathbf{X}_0) = 0$. Then by using the latter and the fact that $\mathbf{Q}_0 = -\mathbf{Q} \in T_d$ for each $\mathbf{Q} \in O_h \setminus T_d$ we infer that the condition (2.2) is satisfied for the case at issue. For the latter case for $\mathbf{X}_0$ we have

$$\mathbf{R}_0 * (\mathbf{F}(\mathbf{X}_0)) = \mathbf{F}(\mathbf{R}_0 * \mathbf{X}_0) = \mathbf{F}(\mathbf{X}_0), \qquad \mathbf{R}_0 \in \{\mathbf{R}_{\mathbf{n}_1}^{\pi}, \mathbf{R}_{\mathbf{n}_2}^{\pi}, \mathbf{R}_{\mathbf{n}_3}^{\pi}\} \subset T_d$$

for each $\mathbf{F} \in T_d(D, M)$. From this we derive

$$F(X_0) = \begin{cases} \mathbf{0}, & M = V\,, \\ \mathbf{O}, & M = \mathrm{Skw}\,, \\ a(\mathbf{X}_0)\mathbf{n}_1 \otimes \mathbf{n}_1 + b(\mathbf{X}_0)\mathbf{n}_2 \otimes \mathbf{n}_2 + c(\mathbf{X}_0)\mathbf{n}_3 \otimes \mathbf{n}_3\,, & M = \mathrm{Sym}\,. \end{cases}$$

Then, using the latter and the fact that $-\mathbf{Q} \in T_d$ for each $\mathbf{Q} \in O_h \setminus T_d$, one may easily deduce that the condition (2.2) is also satisfied for the case in question. *Q.E.D.*

4.3. The class T_h

The following two fourth-order tensors are invariant under the group T_h:

$$\mathbf{T}_h^a = \sum_{k=1}^{3} \mathbf{E}\mathbf{n}_k \otimes \boldsymbol{\omega}_k\,, \tag{4.14}$$

$$\begin{aligned} \mathbf{T}_h^s &= (\mathbf{N}_2 - \mathbf{N}_3) \otimes \mathbf{N}_1 + (\mathbf{N}_3 - \mathbf{N}_1) \otimes \mathbf{N}_2 + (\mathbf{N}_1 - \mathbf{N}_2) \otimes \mathbf{N}_3\,, \\ \mathbf{N}_k &= \mathbf{n}_k \otimes \mathbf{n}_k\,, \qquad k = 1, 2, 3\,, \end{aligned} \tag{4.15}$$

where $\boldsymbol{\omega}_k$, $k = 1, 2, 3$, are given by (4.10).

THEOREM 10. *The surface*

$$\mathbf{S}(\mathbf{X}) = \Big(\mathbf{T}_h^a : (\overset{2}{\otimes}\, \mathbf{v}_\alpha), \mathbf{T}_h^s : (\overset{2}{\otimes}\, \mathbf{v}_\alpha);\ \mathbf{T}_h^a : (\overset{2}{\otimes}\, (\mathbf{E} : \mathbf{W}_\theta)), \\ \mathbf{T}_h^s : (\overset{2}{\otimes}\, (\mathbf{E} : \mathbf{W}_\theta)); \mathbf{T}_h^a : \mathbf{A}_\sigma, \mathbf{T}_h^s : \mathbf{A}_\sigma\Big) \tag{4.16}$$

is an IES for $T_h(D, M)$.

P r o o f. It is evident that the condition (2.2) can be satisfied for each $\mathbf{Q} \in T_h$. Moreover, for each $\mathbf{Q} \in \mathrm{Orth} \setminus T_h$, by Theorem A.7 we infer that the point $\mathbf{X}_0 \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$ is given by the two cases

CASE 1. $\mathbf{X}_0 \in D(\mathbf{u})$ if $\exists \mathbf{u}, \mathbf{v} \in \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$: $\mathbf{Q}^T\mathbf{u} = (\det \mathbf{Q})\mathbf{v}$;

CASE 2. $\mathbf{v}_\alpha = \mathbf{0}, \mathbf{W}_\theta = \mathbf{O}, \mathbf{A}_\sigma = c_\sigma \mathbf{I}$ if $\forall \mathbf{u}, \mathbf{v} \in \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$: $\mathbf{Q}^T\mathbf{u} \neq (\det \mathbf{Q})\mathbf{v}$.

It can be easily proved that the condition (2.2) can be satisfied for Case 2. Moreover, by means of the similar procedure used in the proof for Case 2 of Theorem 9, it can be verified that the condition (2.2) can also be satisfied for Case 1 shown above. *Q.E.D.*

5. The icosahedral group I_h

The icosahedral group I_h is the most complicated yet intriguing one in all subgroups of Orth, which characterizes the symmetry of the icosahedron. In a famous lecture delivered in 1884, F. KLEIN [15] presented a comprehensive account of the icosahedron and the icosahedral group. According to classical crystallography, there exists no solid whose symmetry is described by the icosahedral group or any other non-crystallographic point group except the transverse isotropy groups and the full and proper orthogonal groups. However, such a traditional viewpoint has been proved to be too narrow by the recent discovery of quasicrystals (cf. Vainshtein [44] and Senechal [27] and the related literature therein). The latter possess symmetries forbidden by the classical crystallography rule, such as five-, eight-, and ten-fold symmetries etc. Of them, the icosahedral quasicrystal is the one which has received much attention.

The icosahedral group I_h is of the form

$$I_h = \left(\bigcup_{s=1}^{6} S_{10}(\mathbf{n}_s)\right) \cup \left(\bigcup_{t=1}^{10} S_6(\mathbf{r}_t)\right) \cup \left(\bigcup_{c=1}^{15} C_{2h}(\mathbf{a}_c)\right), \tag{5.1}$$

where the groups $S_{10}(\mathbf{u})$, $S_6(\mathbf{u})$ and $C_{2h}(\mathbf{u})$ for any unit vector $\mathbf{u}$ are indicated at the start of §4.

The unit vectors $\mathbf{n}_\alpha$, $\mathbf{r}_\sigma$, $\mathbf{a}_\tau$, $\alpha = 1, \ldots, 6$; $\sigma = 1, \ldots, 10$; $\tau = 1, \ldots, 15$ are used to represent the six five-fold axes, the ten three-fold axes and the fifteen two-fold axes, respectively. Let $\mathbf{n}$ and $\mathbf{e}$ be two orthonormal vectors. Then the six five-fold axes of I_h are expressible in the form (cf. XIAO [53])

$$\begin{aligned} \mathbf{n}_6 &= \mathbf{n}, \\ \mathbf{n}_k &= (\mathbf{n} + 2\mathbf{l}_k)/\sqrt{5} = \mathbf{R}_\mathbf{n}^{2k\pi/5}\mathbf{n}_5, \\ \mathbf{l}_k &= \mathbf{R}_\mathbf{n}^{2k\pi/5}\mathbf{e}, \qquad k = 1, \ldots, 5, \end{aligned} \tag{5.2}$$

with the property

$$(\mathbf{n}_i \cdot \mathbf{n}_j)^2 = \frac{1}{5} + \frac{4}{5}\delta_{ij}, \qquad i, j = 1, \ldots, 6. \tag{5.3}$$

Moreover, each three-fold axis $\mathbf{r}_t$ and each two-fold axis $\mathbf{l}_c$ can be determined by the five-fold axes $\mathbf{n}_s$, refer to XIAO [52] for detail.

The follwing three tensors are invariant under the group I_h:

$$\mathbf{I}_h^r = \sum_{k=1}^{6} \left(\overset{2r+4}{\otimes} \mathbf{n}_k \right), \qquad r = 1, 2, 3. \tag{5.4}$$

THEOREM 11. *The surface*

$$\mathbf{S}(\mathbf{X}) = \Big(\mathbf{I}_h^1 \odot (\overset{4}{\otimes} \mathbf{v}_\alpha), \mathbf{I}_h^2 \odot (\overset{6}{\otimes} \mathbf{v}_\alpha), \mathbf{I}_h^3 \odot (\overset{8}{\otimes} \mathbf{v}_\alpha); \mathbf{I}_h^1 \odot (\overset{4}{\otimes} (\mathbf{E} : \mathbf{W}_\theta)), \tag{5.5}$$

$$\mathbf{I}_h^2 \odot (\overset{6}{\otimes} (\mathbf{E} : \mathbf{W}_\theta)), \mathbf{I}_h^3 \odot (\overset{8}{\otimes} (\mathbf{E} : \mathbf{W}_\theta)); \mathbf{I}_h^1 \odot (\overset{2}{\otimes} \mathbf{A}_\sigma), \mathbf{I}_h^2 \odot (\overset{3}{\otimes} \mathbf{A}_\sigma), \mathbf{I}_h^3 \odot (\overset{4}{\otimes} \mathbf{A}_\sigma) \Big)$$

is an IES for $I_h(D, M)$.

P r o o f. It suffices to prove that (2.2) holds for each $\mathbf{Q} \in \mathrm{Orth} \setminus I_h$. For $\mathbf{X}_0 = (\mathbf{v}_\alpha, \mathbf{W}_\theta, \mathbf{A}_\sigma) \in \mathbf{S} \cap (\mathbf{Q} * \mathbf{S})$, $\mathbf{Q} \in \mathrm{Orth} \setminus I_h$, by applying Theorem A.8 we infer that $\mathbf{X}_0 \in D(\mathbf{u})$ if $\mathbf{Q}$ satisfies (A.62) and that $\mathbf{v}_\alpha = 0$, $\mathbf{W}_\theta = \mathbf{O}$, $\mathbf{A}_\sigma = c_\sigma \mathbf{I}$ if $\mathbf{Q}$ satisfies (A.63). By means of these facts and the procedure used in the proof of Theorem 8, it can be proved that the condition (2.2) is satisfied. *Q.E.D.*

6. Examples and concluding remarks

Employing the results presented in the previous sections as well as the well-known representation theorems for isotropic functions of vectors and second order tensors, one can derive complete representations for any type of scalar-, vector- and second order tensor-valued anisotropic functions of vectors and second order tensors merely replacing some variables of the former with $\mathbf{S}(\mathbf{X})$ (cf. (2.3)). It should be noted, however, that representations obtained in this manner are generally not irreducible. To obtain complete irreducible representations, further effort should be made. Recently, the general results given here have been used to investigate various kinds of anisotropic functions of vectors and second order tensors. Simple irreducible functional bases and generating sets for scalar-valued and symmetric second order tensor-valued anisotropic functions of a single symmetric second order tensor have been obtained for all thirty-two crystal classes (cf. XIAO [47, 50, 51] and all noncrystal classes (cf. XIAO [53, 54]). Moreover, irreducible representations for scalar-, vector- and second order tensor-valued anisotropic functions of any finite number of vectors and second order tensors have been derived for all kinds of subgroups of the transverse isotropy group $C_{\infty h}$ (cf. XIAO [55]).

The extension theorems presented in the previous sections are concerned with anisotropic functions with an arbitrary number of vector and second order tensor variables. Recently, this author (see XIAO [48]) has proved that representation problems for rth-order tensor-valued isotropic or anisotropic tensor functions with an arbitrary number of vector and second order tensor variables can be reduced to those for certain rth-order tensor-valued isotropic or anisotropic tensor functions merely with not more than three (for $r \geq 1$) or four (for $r = 0$) vector and/or second order tensor variables (see [56] for further results). According to this fact, to derive a complete representation for any given type of anisotropic functions of vectors and second order tensors, it suffices to apply the corresponding extension theorem given here to treat the related anisotropic functions of not more than three or four vectors and/or second order tensors.

As an example, we apply Theorem 3 to derive irreducible nonpolynomial representations for scalar-valued and vector-valued anisotropic functions of any finite number of vectors relative to the group D_{2m+1h} for each integer $m \geq 1$.

According to Theorem A and Theorem 3, anisotropic functions of the a vector variables $\mathbf{v}_1, \ldots, \mathbf{v}_a$ relative to the group D_{2m+1h} can be extended as isotropic functions of the extended variables $(\mathbf{v}_\alpha, \boldsymbol{\eta}_{2m}(\mathbf{v}^\circ_\alpha), \mathbf{N})$, where $\mathbf{N} = \mathbf{n} \otimes \mathbf{n}$. Thus, applying the well-known results for representations of isotropic functions (cf. Wang [45] and Smith [32], *et al.*), we obtain a functional basis and a generating set for scalar-valued and vector-valued anisotropic functions of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_a$ relative to the group D_{2m+1h} as follows.

Functional basis:

$$|\mathbf{v}|^2,\ \mathbf{u}\cdot\mathbf{v},\ \mathbf{v}\cdot\mathbf{N}\mathbf{v},\ \mathbf{v}\cdot\mathbf{N}^2\mathbf{v}, \mathbf{u}\cdot\mathbf{N}\mathbf{v},\ \mathbf{u}\cdot\mathbf{N}^2\mathbf{v};$$

Generating set:

$$\mathbf{v}, \mathbf{N}\mathbf{v}, \mathbf{N}^2\mathbf{v},$$

where $\mathbf{u}, \mathbf{v} = \mathbf{v}_1, \ldots, \mathbf{v}_a, \boldsymbol{\eta}_{2m}(\mathbf{v}^\circ_1), \ldots, \boldsymbol{\eta}_{2m}(\mathbf{v}^\circ_a)$, $\mathbf{u} \neq \mathbf{v}$ and $\boldsymbol{\eta}_{2m}(\mathbf{v}^\circ)$ is given by (3.12).

Since

$$\mathbf{N}\boldsymbol{\eta}_{2m}(\mathbf{v}^\circ) = \mathbf{0}, \qquad \mathbf{N}^2 = \mathbf{N},$$

either of the above two sets includes a large number of obviously redundant elements. Removing the latter and noticing the identity

$$|\mathbf{v}|^2 = |\mathbf{v}^\circ|^2 + (\mathbf{v}\cdot\mathbf{n})^2,$$

we arrive at the following simplified results.

Functional basis:

$$(\mathbf{u}\cdot\mathbf{n})(\mathbf{v}\cdot\mathbf{n}), \mathbf{u}^\circ\cdot\mathbf{v}^\circ, |\mathbf{u}^\circ|\cdot|\mathbf{v}^\circ|^{2m}\cos(<\mathbf{u}^\circ,\mathbf{e}> +2m <\mathbf{v}^\circ,\mathbf{e}>). \tag{6.1}$$

Generating set:

$$(\mathbf{v}\cdot\mathbf{n})\mathbf{n}, \mathbf{v}^\circ, |\mathbf{v}^\circ|^{2m}(\mathbf{e}\cos 2m <\mathbf{v}^\circ,\mathbf{e}> -\mathbf{e}'\sin 2m <\mathbf{v}^\circ,\mathbf{e}>). \tag{6.2}$$

In the above, $\mathbf{u}, \mathbf{v} = \mathbf{v}_1, \ldots, \mathbf{v}_a$, the unit vector $\mathbf{e}$ may be any two-fold rotation axis of D_{2m+1h} and $\mathbf{e}'$ is given by (2.10). In deriving the former, the invariants of the form

$$\boldsymbol{\eta}_{2m}(\mathbf{u}^o)\cdot\boldsymbol{\eta}_{2m}(\mathbf{v}^o) = (|\mathbf{u}^o|\cdot|\mathbf{v}^o|)^{2m} \cos 2m < \mathbf{u}^o, \mathbf{v}^o >,$$

which seems not obviously to be redundant, are also removed. In reality, by virtue of $(3.36)_1$ we know that the above invariant is expressible as a polynomial of degree $2m$ in $\mathbf{u}^o\cdot\mathbf{v}^o$ and $|\mathbf{u}|^o\cdot|\mathbf{v}^o|$ with constant coefficients. It can easily be proved that the functional basis given is irreducible, and moreover, that the generating set given is minimal.

It is worthwhile to point out the fact that the results derived above are valid for all *infinitely many* classes D_{2m+1h}. They provide all the desired representations *in a unified form*, while usually each anisotropy group has to be dealt with separately. The fact just indicated is also true for other kinds of subgroups of $D_{\infty h}$. Thus, as far as infinitely many classes of subgroups of $D_{\infty h}$ are concerned, *universal representations* may be derived by applying the extension theorems given in §3, as is done in the above and in [53 – 55].

Appendix A. General solutions to some related systems of polynomial tensor equations

In this appendix, we offer general solutions to some systems of polynomial tensor equations associated with the isotropic extension surfaces given in the previous sections. These results are used to determine the intersecting surface $\mathbf{S} \cap (\mathbf{Q} * \mathbf{S}) \subset D$ for each presented IES $\mathbf{S}$.

Henceforth, δ is used to represent $+1$ or -1, i.e. $\delta^2 = 1$; m is used to signify any given positive integer; and $\mathbf{v}, \mathbf{x} \in V$, $\mathbf{W} \in$ Skw and $\mathbf{A} \in$ Sym are used to designate vector variable, skewsymmetric tensor variable and symmetric tensor variable, respectively.

A.1. Polynomial tensor equations: subgroups of $D_{\infty h}$

THEOREM A.1. *Let* $\boldsymbol{\eta}_r(\mathbf{z})$ *be the vector-valued function given by* (3.12) *for any vector* $\mathbf{z}$ *on the* $\mathbf{n}$*-plane and each integer* $r \geq 1$. *Then, for each* $\mathbf{Q} \in D_{\infty h} \setminus D_{2m+1d}$, *the general solution to the system of tensor equations*

(A.1) $$\mathbf{Q} * (\mathbf{E}\boldsymbol{\eta}_{2m}((\mathbf{Q}^T\mathbf{v})^o)) = \mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{v}^o),$$

(A.2) $$\mathbf{Q} * (\mathbf{E}\boldsymbol{\eta}_{2m}((\mathbf{Q}^T * \mathbf{W})\mathbf{n})) = \mathbf{E}\boldsymbol{\eta}_{2m}(\mathbf{W}\mathbf{n}),$$

(A.3) $$\mathbf{Q} * (\mathbf{E}\boldsymbol{\eta}_{2m}(((\mathbf{Q}^T * \mathbf{A})\mathbf{n})^o)) = \mathbf{E}\boldsymbol{\eta}_{2m}((\mathbf{A}\mathbf{n})^o),$$

(A.4) $$\mathbf{Q} * (\mathbf{E}\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{Q}^T * \mathbf{A}))) = \mathbf{E}\boldsymbol{\eta}_m(\mathbf{q}(\mathbf{A})),$$

is given by

(A.5) $$\mathbf{v} = x\mathbf{n}, \qquad \mathbf{W} = y\mathbf{E}\mathbf{n}, \qquad \mathbf{A} = z\mathbf{I} + w\mathbf{n}\otimes\mathbf{n} \quad (\forall x, y, z, w \in R).$$

P r o o f. Let $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{n}}^{\theta}$. Then, by applying the formulas (2.13), $(2.15)_1$, $(2.17)_1$ and $(2.18)_1$ we convert Eqs. (A.1)–(A.4) to

$$|\mathbf{z}|^{2m}(\mathbf{e}\cos\Theta - \mathbf{e}'\sin\Theta) = |\mathbf{z}|^{2m}(\mathbf{e}\cos 2m < \mathbf{z},\mathbf{e} > - \mathbf{e}'\sin 2m < \mathbf{z},\mathbf{e} >),$$
$$|\mathbf{q}(\mathbf{A})|^{m}(\mathbf{e}\cos\Theta' - \mathbf{e}'\sin\Theta') = |\mathbf{q}(\mathbf{A})|^{m}(\mathbf{e}\cos m < \mathbf{q}(\mathbf{A}),\mathbf{e} > -\mathbf{e}'\sin m < \mathbf{q}(\mathbf{A}),\mathbf{e} >),$$

where

$$\Theta = -(2m+1)\theta + 2m < \mathbf{z},\mathbf{e} >, \qquad \mathbf{z} = \mathbf{v}^{\circ}, \mathbf{Wn}, (\mathbf{An})^{\circ},$$
$$\Theta' = -(2m+1)\theta + m < \mathbf{q}(\mathbf{A}),\mathbf{e} > .$$

Since $\mathbf{Q} \notin D_{2m+1d}$, we have $(2m+1)\theta \neq 2k\pi$. Then we derive

$$|\mathbf{v}^{\circ}| = |\mathbf{Wn}| = |(\mathbf{An})^{\circ}| = |\mathbf{q}(\mathbf{A})| = 0. \tag{A.6}$$

Hence (A.5) holds for each $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{n}}^{\theta} \in D_{\infty h} \setminus D_{2m+1d}$.

Next, let $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{a}}^{\pi}$. Then, by applying the formulas (2.14), $(2.15)_2$, $(2.17)_2$ and $(2.18)_2$ we recast Eqs. (A.1)–(A.4) in the form

$$|\mathbf{z}|^{2m}(\mathbf{e}\cos\Theta + \mathbf{e}'\sin\Theta) = |\mathbf{z}|^{2m}(\mathbf{e}\cos 2m < \mathbf{z},\mathbf{e} > - \mathbf{e}'\sin 2m < \mathbf{z},\mathbf{e} >),$$
$$|\mathbf{q}(\mathbf{A})|^{m}(\mathbf{e}\cos\Theta' + \mathbf{e}'\sin\Theta') = |\mathbf{q}(\mathbf{A})|^{m}(\mathbf{e}\cos m < \mathbf{q}(\mathbf{A}),\mathbf{e} > -\mathbf{e}'\sin m < \mathbf{q}(\mathbf{A}),\mathbf{e} >),$$

where

$$\Theta = (4m+2) < \mathbf{a},\mathbf{e} > -2m < \mathbf{z},\mathbf{e} >, \qquad \mathbf{z} = \mathbf{v}^{\circ}, \mathbf{Wn}, (\mathbf{An})^{\circ},$$
$$\Theta' = (4m+2) < \mathbf{a},\mathbf{e} > -m < \mathbf{q}(\mathbf{A}),\mathbf{e} > .$$

Since $\mathbf{Q} \notin D_{2m+1d}$, we have $(4m+2) < \mathbf{a},\mathbf{e} > \neq 2k\pi$. Then we derive (A.6). Hence (A.5) also holds for each $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{a}}^{\pi} \in D_{\infty h} \setminus D_{2m+1d}$. *Q.E.D.*

THEOREM A.2. *Let $\mathbf{\Phi}_r(\mathbf{z})$ be the symmetric second order tensor-valued function given by* (3.16) *for any vector $\mathbf{z}$ on the $\mathbf{n}$-plane and each integer $r \geq 1$. Then, for each $\mathbf{Q} \in D_{\infty h} \setminus D_{2m+2h}$, the general solution to the system of tensor equations*

$$\mathbf{Q} * (\mathbf{\Phi}_{2m}((\mathbf{Q}^{\mathrm{T}}\mathbf{v})^{\circ})) = \mathbf{\Phi}_{2m}(\mathbf{v}^{\circ}), \tag{A.7}$$
$$\mathbf{Q} * (\mathbf{\Phi}_{2m}((\mathbf{Q}^{\mathrm{T}} * \mathbf{W})\mathbf{n})) = \mathbf{\Phi}_{2m}(\mathbf{Wn}), \tag{A.8}$$
$$\mathbf{Q} * (\mathbf{\Phi}_{2m}(((\mathbf{Q}^{\mathrm{T}} * \mathbf{A})\mathbf{n})^{\circ})) = \mathbf{\Phi}_{2m}((\mathbf{An})^{\circ}), \tag{A.9}$$
$$\mathbf{Q} * (\mathbf{\Phi}_{m}(\mathbf{q}(\mathbf{Q}^{\mathrm{T}} * \mathbf{A}))) = \mathbf{\Phi}_{m}(\mathbf{q}(\mathbf{A})), \tag{A.10}$$

is given by (A.5).

P r o o f. Let $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{n}}^{\theta}$. By using the formulas (2.13), $(2.15)_1$, $(2.17)_1$ and $(2.18)_1$, we convert Eqs. (A.7)–(A.10) to the form

$$|\mathbf{z}|^{2m}(\mathbf{D}_1\cos\Theta - \mathbf{D}_2\sin\Theta) = |\mathbf{z}|^{2m}(\mathbf{D}_1\cos 2m < \mathbf{z},\mathbf{e} > -\mathbf{D}_2\sin 2m < \mathbf{z},\mathbf{e} >),$$
$$|\mathbf{q}(\mathbf{A})|^{m}(\mathbf{D}_1\cos\Theta' - \mathbf{D}_2\sin\Theta') = |\mathbf{q}(\mathbf{A})|^{m}(\mathbf{D}_1\cos m < \mathbf{q}(\mathbf{A}),\mathbf{e} > - \mathbf{D}_2\sin m < \mathbf{q}(\mathbf{A}),\mathbf{e} >),$$

where

$$\Theta = -(2m+2)\theta + 2m < \mathbf{z}, \mathbf{e} >, \qquad \mathbf{z} = \mathbf{v}^{\circ}, \mathbf{W}\mathbf{n}, (\mathbf{A}\mathbf{n})^{\circ},$$
$$\Theta' = -(2m+2)\theta + m < \mathbf{q}(\mathbf{A}), \mathbf{e} > .$$

Since $\mathbf{Q} \in D_{2m+2h}$, we have $(2m+2)\theta \neq 2k\pi$. Then we derive (A.6) and therefore (A.5) holds for each $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{n}}^{\theta} \in D_{\infty h} \setminus D_{2m+2h}$.

Next, let $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{a}}^{\pi}$. Then, by applying the formulas (2.14), $(2.15)_2$, $(2.17)_2$ and $(2.18)_2$ we recast Eqs. (A.7)–(A.10) in the form

$$|\mathbf{z}|^{2m}(\mathbf{D}_1 \cos\Theta + \mathbf{D}_2 \sin\Theta) = |\mathbf{z}|^{2m}(\mathbf{D}_1 \sin 2m < \mathbf{z}, \mathbf{e} > - \mathbf{D}_2 \sin 2m < \mathbf{z}, \mathbf{e} >),$$
$$|\mathbf{q}(\mathbf{A})|^{m}(\mathbf{D}_1 \cos\Theta' + \mathbf{D}_2 \sin\Theta') = |\mathbf{q}(\mathbf{A})|^{m}(\mathbf{D}_1 \sin m < \mathbf{q}(\mathbf{A}), \mathbf{e} > - \mathbf{D}_2 \sin m < \mathbf{q}(\mathbf{A}), \mathbf{e} >),$$

where

$$\Theta = (4m+4) < \mathbf{a}, \mathbf{e} > -2m < \mathbf{z}, \mathbf{e} >, \qquad \mathbf{z} = \mathbf{v}^{\circ}, \mathbf{W}\mathbf{n}, (\mathbf{A}\mathbf{n})^{\circ},$$
$$\Theta' = (4m+4) < \mathbf{a}, \mathbf{e} > -m < \mathbf{q}(\mathbf{A}), \mathbf{e} > .$$

Since $\mathbf{Q} \notin D_{2m+2h}$, we have $(4m+4) < \mathbf{a}, \mathbf{e} > \neq 2k\pi$. Then we derive (A.6). Hence (A.5) also holds for each $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{a}}^{\pi} \in D_{\infty h} \setminus D_{2m+2h}$. *Q.E.D.*

THEOREM A.3. *Let* $\boldsymbol{\eta}_r(\mathbf{z})$ *be the vector-valued function given by* (3.12). *Then, for each* $\mathbf{Q} \in D_{\infty h} \setminus D_{2m+1h}$, *the general solution to the system of tensor equations*

$$\text{(A.11)} \qquad \mathbf{Q}(\boldsymbol{\eta}_{2m}((\mathbf{Q}^{\mathrm{T}}\mathbf{v})^{\circ})) = \boldsymbol{\eta}_{2m}(\mathbf{v}^{\circ}),$$
$$\text{(A.12)} \qquad \mathbf{Q}(\boldsymbol{\eta}_{2m}((\mathbf{Q}^{\mathrm{T}} * \mathbf{W})\mathbf{n})) = \boldsymbol{\eta}_{2m}(\mathbf{W}\mathbf{n}),$$
$$\text{(A.13)} \qquad \mathbf{Q}(\boldsymbol{\eta}_{2m}(((\mathbf{Q}^{\mathrm{T}} * \mathbf{A})\mathbf{n})^{\circ})) = \boldsymbol{\eta}_{2m}((\mathbf{A}\mathbf{n})^{\circ}),$$
$$\text{(A.14)} \qquad \mathbf{Q}(\boldsymbol{\eta}_{m}(\mathbf{q}(\mathbf{Q}^{\mathrm{T}} * \mathbf{A}))) = \boldsymbol{\eta}_{m}(\mathbf{q}(\mathbf{A})),$$

is given by (A.5).

The proof of this theorem is similar to that of Theorem A.1, except for the fact that the factor δ plays no role in the latter, while it comes into play in the former (cf. the proof for the next theorem).

THEOREM A.4. *Let* $\boldsymbol{\eta}_r(\mathbf{z})$ *be the vector-valued function given by* (3.12) *for any vector* $\mathbf{z}$ *on the* $\mathbf{n}$*-plane and each integer* $r \geq 1$. *Then, for each* $\mathbf{Q} \in D_{\infty h} \setminus D_{2md}$, *the general solution to the system of tensor equations*

$$\text{(A.15)} \qquad \mathbf{Q} * (\mathbf{n} \vee \boldsymbol{\eta}_{2m-1}((\mathbf{Q}^{\mathrm{T}}\mathbf{v})^{\circ})) = \mathbf{n} \vee \boldsymbol{\eta}_{2m-1}(\mathbf{v}^{\circ}),$$
$$\text{(A.16)} \qquad \mathbf{Q}^{\mathrm{T}}(\boldsymbol{\eta}_{2m-1}((\mathbf{Q} * \mathbf{W})\mathbf{n})) = \boldsymbol{\eta}_{2m-1}(\mathbf{W}\mathbf{n}),$$
$$\text{(A.17)} \qquad \mathbf{Q}^{\mathrm{T}}(\boldsymbol{\eta}_{2m-1}(((\mathbf{Q} * \mathbf{A})\mathbf{n})^{\circ})) = \boldsymbol{\eta}_{2m-1}((\mathbf{A}\mathbf{n})^{\circ}),$$

is given by

(A.18) $$\mathbf{v} = x\mathbf{n}, \qquad \mathbf{W} = y\mathbf{E}\mathbf{n}, \qquad (\mathbf{A}\mathbf{n})^{\circ} = \mathbf{0}.$$

P r o o f. Let $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{n}}^{\theta}$. Then, by using the formulas (2.13), $(2.17)_1$, $(2.19)_1$, (2.18) and (2.20) we infer

$$|\mathbf{z}|^{2m-1}(\mathbf{e}\cos\Theta - \mathbf{e}'\sin\Theta) = |\mathbf{z}|^{2m-1}(\mathbf{e}\cos\Theta_0 - \mathbf{e}'\sin\Theta_0),$$

where

$$\Theta_0 = (2m-1) < \mathbf{z}, \mathbf{e} >, \qquad \Theta = \Theta_0 - 2m\theta - \frac{1}{2}(1-\delta)\pi,$$
$$\mathbf{z} = \mathbf{v}^{\circ}, \ \mathbf{W}\mathbf{n}, \ (\mathbf{A}\mathbf{n})^{\circ}.$$

Since $\mathbf{Q} \notin D_{2md}$, we have

$$2m\theta + \frac{1}{2}(1-\delta)\pi \neq 2k\pi.$$

Hence, we deduce $\mathbf{z} = \mathbf{0}$, $\mathbf{z} = \mathbf{v}^{\circ}$, $\mathbf{W}\mathbf{n}$, $(\mathbf{A}\mathbf{n})^{\circ}$, i.e. (A.18) holds for each $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{n}}^{\theta} \in D_{\infty h} \setminus D_{2md}$.

Let $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{a}}^{\pi}$. Then, by using the formulas (2.14), $(2.17)_2$, $(2.19)_2$, (2.18) and (2.20) we infer

$$|\mathbf{z}|^{2m-1}(\mathbf{e}\cos\Theta + \mathbf{e}'\sin\Theta) = |\mathbf{z}|^{2m-1}(\mathbf{e}\cos\Theta_0 - \mathbf{e}'\sin\Theta_0),$$

where

$$\Theta_0 = (2m-1) < \mathbf{z}, \mathbf{e} >, \qquad \Theta = 4m < \mathbf{a}, \mathbf{e} > + \frac{1}{2}(1-\delta)\pi - \Theta_0\,,$$
$$\mathbf{z} = \mathbf{v}^{\circ}, \ \mathbf{W}\mathbf{n}, \ (\mathbf{A}\mathbf{n})^{\circ}.$$

Since $\mathbf{Q} \notin D_{2md}$, we have

$$4m < \mathbf{a}, \mathbf{e} > +(1-\delta)\pi \neq 2k\pi.$$

Hence, we deduce $\mathbf{z} = \mathbf{0}$, $\mathbf{z} = \mathbf{v}^{\circ}$, $\mathbf{W}\mathbf{n}$, $(\mathbf{A}\mathbf{n})^{\circ}$, i.e. (A.18) also holds for each $\mathbf{Q} = \delta\mathbf{R}_{\mathbf{a}}^{\pi} \in D_{\infty h} \setminus D_{2md}$. *Q.E.D.*

A.2. Polynomial tensor equations: cubic crystal classes

THEOREM A.5. *Let* $\mathbf{O}_h$ *be the tensor given by* (4.6), *which is invariant under the group* O_h. *Then for each* $\mathbf{Q} \in \mathrm{Orth} \setminus O_h$, *the solution to the system of polynomial tensor equations*

(A.19) $$(\mathbf{Q} * \mathbf{O}_h) : (\mathbf{x} \otimes \mathbf{x}) = \mathbf{O}_h : (\mathbf{x} \otimes \mathbf{x}),$$

(A.20) $$(\mathbf{Q} * \mathbf{O}_h) : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W})) = \mathbf{O}_h : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W})),$$

(A.21) $$(\mathbf{Q} * \mathbf{O}_h) : \mathbf{A} = \mathbf{O}_h : \mathbf{A},$$

(A.22) $$(\mathbf{Q} * \mathbf{O}_h) : \mathbf{A}^2 = \mathbf{O}_h : \mathbf{A}^2,$$

are as follows:

Case 1. *If there are* $\mathbf{u}, \mathbf{v} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ *or* $\mathbf{u}, \mathbf{v} \in \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$ *such that*

$$\text{(A.23)} \qquad \overset{2}{\otimes} (\mathbf{Q}^{\mathrm{T}}\mathbf{u}) = \overset{2}{\otimes} \mathbf{v}\,,$$

then

$$\text{(A.24)} \qquad \mathbf{x} = a\mathbf{u}, \qquad \mathbf{W} = b\mathbf{E}\mathbf{u}, \qquad \mathbf{A} = c\mathbf{I} + d\mathbf{u}\otimes\mathbf{u} \quad (\forall\, a,\, b,\, c,\, d \in R)\,.$$

Case 2. *If for any* $\mathbf{u},\ \mathbf{v} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ *and* $\mathbf{u},\ \mathbf{v} \in \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$,

$$\text{(A.25)} \qquad \overset{2}{\otimes} (\mathbf{Q}^{\mathrm{T}}\mathbf{u}) \neq \overset{2}{\otimes} \mathbf{v},$$

then

$$\text{(A.26)} \qquad \mathbf{x} = \mathbf{0}, \qquad \mathbf{W} = \mathbf{O}, \qquad \mathbf{A} = c\mathbf{I}.$$

In the above, each $\mathbf{n}_k$ *and each* $\mathbf{r}_i$ *are a four-fold axis and a three-fold axis of* O_h, *respectively (cf.* (4.1) *and* (4.5)).

Proof. First, suppose that there be $\mathbf{u}, \mathbf{v} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ such that (A.23) holds. Then there are permutations $\sigma, \tau \in P_3$, where P_3 is the symmetric group on three letters, such that

$$\text{(A.27)} \qquad \begin{aligned} \mathbf{Q}\mathbf{n}_{\sigma(1)} &= \mathbf{n}_{\tau(1)} \cos\theta + \mathbf{n}_{\tau(2)} \sin\theta\,, \\ \mathbf{Q}\mathbf{n}_{\sigma(2)} &= -\mathbf{n}_{\tau(1)} \sin\theta + \mathbf{n}_{\tau(2)} \cos\theta\,, \\ \mathbf{Q}\mathbf{n}_{\sigma(3)} &= r\mathbf{n}_{\tau(3)}, \qquad r^2 = 1. \end{aligned}$$

Substituting the above into the equivalent form of (A.21):

$$\text{(A.28)} \qquad \sum_{k=1}^{3} \left(\hat{\mathbf{n}}_{\sigma(k)} \cdot \mathbf{A}\hat{\mathbf{n}}_{\sigma(k)}\right) \hat{\mathbf{n}}_{\sigma(k)} \otimes \hat{\mathbf{n}}_{\sigma(k)} = \sum_{k=1}^{3} \left(\mathbf{n}_{\tau(k)} \cdot \mathbf{A}\mathbf{n}_{\tau(k)}\right) \mathbf{n}_{\tau(k)} \otimes \mathbf{n}_{\tau(k)} (\equiv \mathbf{C}),$$

where $\hat{\mathbf{n}}_{\sigma(k)} = \mathbf{Q}\mathbf{n}_{\sigma(k)}$, we derive

$$\begin{aligned} (A_{\tau(2)\tau(2)} - A_{\tau(1)\tau(1)}) \sin^2 2\theta + A_{\tau(1)\tau(2)} \sin 4\theta &= 0, \\ (A_{\tau(2)\tau(2)} - A_{\tau(1)\tau(1)}) \sin 4\theta - 4A_{\tau(1)\tau(2)} \sin^2 2\theta &= 0. \end{aligned}$$

Since $\mathbf{Q} \notin O_h$, i.e. $\theta \neq k\pi/2$, the above system of homogeneous equations has merely a trivial solution, i.e.

$$\text{(A.29)} \qquad A_{\tau(1)\tau(2)} = 0, \qquad A_{\tau(1)\tau(1)} = A_{\tau(2)\tau(2)}\,,$$

where $A_{ij} = \mathbf{n}_i \cdot \mathbf{A}\mathbf{n}_j = A_{ji}$. Consequently, the equations (A.19) and (A.20) yield

$$(\mathbf{x}\cdot\mathbf{n}_{\tau(1)})(\mathbf{x}\cdot\mathbf{n}_{\tau(2)}) = 0, \qquad (\mathbf{x}\cdot\mathbf{n}_{\tau(1)})^2 = (\mathbf{x}\cdot\mathbf{n}_{\tau(2)})^2\,,$$

$$(\mathbf{y}\cdot\mathbf{n}_{\tau(1)})(\mathbf{y}\cdot\mathbf{n}_{\tau(2)}) = 0, \qquad (\mathbf{y}\cdot\mathbf{n}_{\tau(1)})^2 = (\mathbf{y}\cdot\mathbf{n}_{\tau(2)})^2, \qquad \mathbf{y} = \mathbf{E} : \mathbf{W},$$

and the equations (A.21) and (A.22) produce (A.29) and

$$B_{\tau(1)\tau(2)} = 0, \qquad B_{\tau(1)\tau(1)} = B_{\tau(2)\tau(2)}, \qquad \mathbf{B} = \mathbf{A}^2.$$

From these and the fact stated at the end of this proof we infer that the solution of Eqs. (A.19)–(A.22) is provided by (A.24) for each $\mathbf{Q} \in \mathrm{Orth} \setminus O_h$ satisfying (A.23) for $\mathbf{u}, \mathbf{v} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$.

Next, suppose that for any $\mathbf{u}$, $\mathbf{v} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$, (A.25) holds. Since (A.21), i.e. (A.28) offers two spectral representations of the same symmetric tensor $\mathbf{C} \in \mathrm{Sym}$; we infer that the two sets of eigenvalues, $\{\hat{\mathbf{n}}_k\cdot\mathbf{A}\hat{\mathbf{n}}_k\}$ and $\{\mathbf{n}_k\cdot\mathbf{A}\mathbf{n}_k\}$, coincide and their subordinate eigenprojections coincide. Taking this fact and the condition

(A.30) $$\overset{2}{\otimes}\, \mathbf{Q}^{\mathrm{T}}\mathbf{u} \neq \overset{2}{\otimes}\, \mathbf{v} \qquad (\forall \mathbf{u}, \ \mathbf{v} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\})$$

into account, we infer that $\mathbf{C} = \bar{c}\mathbf{I}$ and hence that

(A.31) $$(\mathbf{Q}\mathbf{n}_k)\cdot\mathbf{A}(\mathbf{Q}\mathbf{n}_k) = \mathbf{n}_k\cdot\mathbf{A}\mathbf{n}_k = \bar{c}, \qquad k = 1, 2, 3.$$

Moreover, letting the symmetric tensor $\mathbf{A} \in \mathrm{Sym}$ take the particular forms $\mathbf{x}\otimes\mathbf{x}$ and $\mathbf{y}\otimes\mathbf{y}$, $\mathbf{y} = \mathbf{E} : \mathbf{W}$, respectively, we infer that Eqs. (A.19), (A.20) and (A.22) yield

(A.32) $$(\mathbf{x}\cdot(\mathbf{Q}\mathbf{n}_k))^2 = (\mathbf{x}\cdot\mathbf{n}_k)^2 = \bar{a}^2, \qquad k = 1, 2, 3,$$

(A.33) $$(\mathbf{y}\cdot(\mathbf{Q}\mathbf{n}_k))^2 = (\mathbf{y}\cdot\mathbf{n}_k)^2 = \bar{b}^2, \qquad k = 1, 2, 3,$$

(A.34) $$(\mathbf{Q}\mathbf{n}_k)\cdot\mathbf{A}^2(\mathbf{Q}\mathbf{n}_k) = \mathbf{n}_k\cdot\mathbf{A}^2\mathbf{n}_k = \bar{d}^2, \qquad k = 1, 2, 3.$$

From (4.5), (A.31)–(A.34) and $\mathbf{Q} \notin O_h$ and the facts

$$\mathbf{p}\cdot(\mathbf{Q}\mathbf{q}) = (\mathbf{Q}^{\mathrm{T}}\mathbf{p})\cdot\mathbf{q}; \qquad (\mathbf{Q}\mathbf{p})\cdot\mathbf{B}(\mathbf{Q}\mathbf{q}) = \mathbf{p}\cdot(\mathbf{Q}^{\mathrm{T}} * \mathbf{B})\mathbf{q},$$

$$\mathbf{n}_k\cdot\mathbf{B}\mathbf{n}_k = c \ \& \ \mathbf{n}_k\cdot\mathbf{B}^2\mathbf{n}_k = d^2 \neq 0, \quad k = 1, 2, 3$$
$$\Longrightarrow \quad \exists \mathbf{u} \in \{\mathbf{r}_1, \dots, \mathbf{r}_4\} : \ \mathbf{B} = x\mathbf{I} + y\mathbf{u}\otimes\mathbf{u}, \quad y \neq 0,$$

for any $\mathbf{p}, \mathbf{q} \in V$ and $\mathbf{B} \in \mathrm{Sym}$, we derive (A.24) if $\bar{a}^2 + \bar{b}^2 + \bar{d}^2 \neq 0$ holds, i.e. there are $\mathbf{u}$, $\mathbf{v} \in \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$ such that (A.23) holds. Moreover, we derive (A.26) if (A.25) holds for any $\mathbf{u}$, $\mathbf{v} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ and any $\mathbf{u}$, $\mathbf{v} \in \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$, i.e. $\bar{a} = \bar{b} = \bar{d} = 0$ holds. In deriving the former, the following fact is used: if an orthogonal tensor $\mathbf{Q}$ transforms any two given three-fold axes of O_h into three-fold axes of O_h, then $\mathbf{Q} \in O_h$. *Q.E.D.*

THEOREM A.6. *Let* $\mathbf{T}_d$ *and* $\mathbf{O}_h$ *be the tensors given by* (4.9)–(4.10) *and* (4.6), *which are invariant under the groups* $T_d \subset O_h$ *and* O_h, *respectively. Then for each* $\mathbf{Q} \in \mathrm{Orth} \setminus O_h$, *the solution to the system of polynomial tensor equations*

$$(\mathbf{Q} * \mathbf{T}_d)\mathbf{x} = \mathbf{T}_d\mathbf{x}; \tag{A.35}$$

$$(\mathbf{Q} * \mathbf{O}_h) : (\mathbf{x} \otimes \mathbf{x}) = \mathbf{O}_h : (\mathbf{x} \otimes \mathbf{x}); \tag{A.36}$$

$$(\mathbf{Q} * \mathbf{T}_d) : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W})) = \mathbf{T}_d : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W})); \tag{A.37}$$

$$(\mathbf{Q} * \mathbf{O}_h) : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W})) = \mathbf{O}_h : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W})); \tag{A.38}$$

$$(\mathbf{Q} * \mathbf{T}_d) : \mathbf{A} = \mathbf{T}_d : \mathbf{A}; \tag{A.39}$$

$$(\mathbf{Q} * \mathbf{O}_h) : \mathbf{A} = \mathbf{O}_h : \mathbf{A}; \tag{A.40}$$

are as follows:

CASE 1. $\mathbf{x} = \mathbf{0}$, $\mathbf{W} = b\mathbf{E}\mathbf{n}$, $\mathbf{A} = c\mathbf{I} + d\mathbf{n} \otimes \mathbf{n}$ *if* $\exists\ \mathbf{u},\ \mathbf{v} \in \{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$: $\overset{2}{\otimes} (\mathbf{Q}^T\mathbf{u}) = \overset{2}{\otimes} \mathbf{v}$.

CASE 2. *If* $\exists \mathbf{u},\ \mathbf{v} \in \{\mathbf{r}_1, \ldots, \mathbf{r}_4\}$: $\mathbf{Q}^T\mathbf{u} = \mathbf{v}$, *then the solutions are given by* (A.24).

CASE 3. *If*

$$\forall\, \mathbf{u},\ \mathbf{v} \in \{\mathbf{r}_1, \ldots, \mathbf{r}_4\} \ : \ \mathbf{Q}^T\mathbf{u} \neq \mathbf{v} \quad \& \quad \forall\, \mathbf{u},\ \mathbf{v} \in \{\mathbf{n}_1,\ \mathbf{n}_2,\ \mathbf{n}_3\} : \ \overset{2}{\otimes} (\mathbf{Q}^T\mathbf{u}) \neq \overset{2}{\otimes} \mathbf{v}, \tag{A.41}$$

then the solution is given by (A.26).

P r o o f. Consider two cases. First, for each $\mathbf{Q}$ given by (A.27), from the proof of Theorem A.5 we know that Eqs. (A.40), (A.38) and (A.36) yield (A.29) and

$$\mathbf{x} = a\mathbf{n}_{\tau(3)}, \qquad \mathbf{W} = b\mathbf{E}\mathbf{n}_{\tau(3)}. \tag{A.42}$$

Moreover, for each $\mathbf{Q}$ given by (A.27), Eqs. (A.35) and (A.39) further yield (Eq. (A.37) provides no further restriction for $\mathbf{W}$)

$$a \sin 2\theta = 0, \qquad a(1 - r\cos 2\theta) = 0,$$
$$A_{\tau(2)\tau(3)} \sin 2\theta + A_{\tau(1)\tau(3)}(\cos 2\theta - r) = 0,$$
$$A_{\tau(2)\tau(3)}(\cos 2\theta - r) - A_{\tau(1)\tau(3)} \sin 2\theta = 0.$$

Thus, by using $\mathbf{Q} \notin O_h$, i.e. $\theta \neq k\pi/2$ we infer

$$a = 0, \qquad A_{\tau(1)\tau(1)} - A_{\tau(2)\tau(2)} = A_{\tau(i)(\tau(j)} = 0, \qquad i,\ j = 1, 2, 3, \quad i \neq j,$$

where (A.29) is incorporated. Hence Case 1 holds.

Next, for each $\mathbf{Q}$ satisfying (A.30), from the proof of Theorem A.5 we know that Eqs. (A.36), (A.38) and (A.40) yield (A.31)–(A.33). Substituting (A.32) and (A.33) into (A.35) and (A.37) respectively, we obtain

$$\bar{a}(\hat{r}_1\hat{\boldsymbol{\omega}}_1 + \hat{r}_2\hat{\boldsymbol{\omega}}_2 + \hat{r}_3\hat{\boldsymbol{\omega}}_3) = \bar{a}(r_1\boldsymbol{\omega}_1 + r_2\boldsymbol{\omega}_2 + r_3\boldsymbol{\omega}_3), \qquad \hat{\boldsymbol{\omega}}_k = \mathbf{Q} * \boldsymbol{\omega}_k,$$

i.e.

$$\text{(A.43)} \qquad \begin{gathered} \bar{a}\hat{f}(\overset{2}{\otimes}(\hat{r}_1\hat{\mathbf{n}}_1 + \hat{r}_2\hat{\mathbf{n}}_2 + \hat{r}_3\hat{\mathbf{n}}_3)) = \bar{a}f(\overset{2}{\otimes}(r_1\mathbf{n}_1 + r_2\mathbf{n}_2 + r_3\mathbf{n}_3)), \\ \hat{f} = \hat{r}_1\hat{r}_2\hat{r}_3\,, \qquad f = r_1 r_2 r_3\,, \end{gathered}$$

and

$$\text{(A.44)} \qquad \begin{gathered} \bar{b}^2\hat{g}(\hat{s}_1\hat{\mathbf{n}}_1 + \hat{s}_2\hat{\mathbf{n}}_2 + \hat{s}_3\hat{\mathbf{n}}_3) = \bar{b}^2 g(s_1\mathbf{n}_1 + s_2\mathbf{n}_2 + s_3\mathbf{n}_3), \qquad \hat{\mathbf{n}}_k = \mathbf{Q}\mathbf{n}_k\,, \\ \hat{g} = \hat{s}_1\hat{s}_2\hat{s}_3, \qquad g = s_1 s_2 s_3\,, \end{gathered}$$

where $\boldsymbol{\omega}_k$ are given by (4.10) and moreover

$$\begin{aligned} \mathbf{x}\cdot\mathbf{n}_k &= \bar{a}r_k\,, & \mathbf{x}\cdot(\mathbf{Q}\mathbf{n}_k) &= \bar{a}\hat{r}_k\,, & r_k^2 = \hat{r}_k^2 &= 1, & k &= 1,2,3, \\ (\mathbf{E}:\mathbf{W})\cdot\mathbf{n}_k &= \bar{b}s_k\,, & (\mathbf{E}:\mathbf{W})\cdot(\mathbf{Q}\mathbf{n}_k) &= \bar{b}\hat{s}_k\,, & s_k^2 = \hat{s}_k^2 &= 1, & k &= 1,2,3. \end{aligned}$$

By using (A.43)–(A.44), (4.5) and the fact that

$$\mathbf{Q}\mathbf{u},\ \mathbf{Q}\mathbf{v} \in \{\mathbf{r}_1, \ldots, \mathbf{r}_4\} \iff \mathbf{Q} \in T_d$$

for any given $\mathbf{u}, \mathbf{v} \in \{\mathbf{r}_1, \ldots, \mathbf{r}_4\}$ and $\mathbf{u} \neq \mathbf{v}$, we infer that $\mathbf{x} = a\mathbf{u}$, $\mathbf{W} = b\mathbf{E}\mathbf{u}$, if there exist $\mathbf{u}$, $\mathbf{v} \in \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$ such that $\mathbf{Q}^{\mathrm{T}}\mathbf{u} = \mathbf{v}$; and that $\mathbf{x} = 0$, $\mathbf{W} = 0$, if (A.41) holds.

On the other hand, let $w_k = \mathbf{A} : \boldsymbol{\omega}_k$ and $\hat{w}_k = \mathbf{A} : \hat{\boldsymbol{\omega}}_k$. Then Eqs. (A.39) and (A.40) may be rewritten in the forms

$$\text{(A.45)} \qquad \sum_{k=1}^{3} \hat{w}_k\hat{\mathbf{n}}_k = \sum_{k=1}^{3} w_k\mathbf{n}_k \quad \text{i.e.} \quad \hat{\mathbf{q}} = \mathbf{q}\,,$$

$$\text{(A.46)} \qquad \sum_{k=1}^{3} \hat{w}_k\hat{\boldsymbol{\omega}}_k = \sum_{k=1}^{3} w_k\boldsymbol{\omega}_k \quad \text{i.e.} \quad \hat{\mathbf{B}} = \mathbf{B}\,.$$

For the latter, the identity

$$\text{(A.47)} \qquad \mathbf{Q} * \left(\mathbf{O}_h + \frac{1}{2}\sum_{k=1}^{3} \boldsymbol{\omega}_k \otimes \boldsymbol{\omega}_k\right) = \mathbf{O}_h + \frac{1}{2}\sum_{k=1}^{3} \boldsymbol{\omega}_k \otimes \boldsymbol{\omega}_k \quad (\forall\, \mathbf{Q} \in \mathrm{Orth})$$

is used. From $\hat{\mathbf{q}}\cdot\hat{\mathbf{B}}\hat{\mathbf{q}} = \mathbf{q}\cdot\mathbf{B}\mathbf{q}$ and $\overset{2}{\otimes}(\hat{\mathbf{B}}\hat{\mathbf{q}}) = \overset{2}{\otimes}(\mathbf{B}\mathbf{q})$ we derive

$$\text{(A.48)} \qquad \sum_{k=1}^{3} (\hat{C}_k)^2\hat{\mathbf{n}}_k \otimes \hat{\mathbf{n}}_k = \sum_{k=1}^{3} (C_k)^2\mathbf{n}_k \otimes \mathbf{n}_k\,,$$

where

$$C_1 = w_2 w_3\,, \qquad C_2 = w_3 w_1\,, \qquad C_3 = w_1 w_2\,;$$
$$\hat{C}_1 = \hat{w}_2 \hat{w}_3\,, \qquad \hat{C}_2 = \hat{w}_3 \hat{w}_1\,, \qquad \hat{C}_3 = \hat{w}_1 \hat{w}_2\,.$$

By (A.30) and (A.48) we infer

$$(\hat{C}_k)^2 = (C_k)^2 = c, \qquad k = 1, 2, 3\,, \tag{A.49}$$

and then by the latter and (A.45) we infer that $\mathbf{A} = c\mathbf{I} + d\mathbf{u} \otimes \mathbf{u}$ if there exist $\mathbf{u},\ \mathbf{v} \in \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}$ such that $\mathbf{Q}^{\mathrm{T}}\mathbf{u} = \mathbf{v}$, and that $\mathbf{A} = c\mathbf{I}$ if (A.41) holds.

Finally, combining the facts derived above and the property of the group T_d stated before, we conclude that Theorem A.6 holds. *Q.E.D.*

THEOREM A.7. *Let* $\mathbf{T}_h^a$ *and* $\mathbf{T}_h^s$ *be the two tensors given by* (4.14) *and* (4.15), *which are invariant under the group* T_h. *Then for each* $\mathbf{Q} \in \mathrm{Orth}\backslash T_h$, *the solution to the system of polynomial tensor* equations

$$(\mathbf{Q} * \mathbf{T}_h^a) : (\mathbf{x} \otimes \mathbf{x}) = \mathbf{T}_h^a : (\mathbf{x} \otimes \mathbf{x}), \tag{A.50}$$
$$(\mathbf{Q} * \mathbf{T}_h^s) : (\mathbf{x} \otimes \mathbf{x}) = \mathbf{T}_h^s : (\mathbf{x} \otimes \mathbf{x}); \tag{A.51}$$
$$(\mathbf{Q} * \mathbf{T}_h^a) : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W})) = \mathbf{T}_h^a : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W})), \tag{A.52}$$
$$(\mathbf{Q} * \mathbf{T}_h^s) : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W})) = \mathbf{T}_h^s : (\overset{2}{\otimes} (\mathbf{E} : \mathbf{W})); \tag{A.53}$$
$$(\mathbf{Q} * \mathbf{T}_h^a) : \mathbf{A} = \mathbf{T}_h^a : \mathbf{A}, \tag{A.54}$$
$$(\mathbf{Q} * \mathbf{T}_h^s) : \mathbf{A} = \mathbf{T}_h^s : \mathbf{A}; \tag{A.55}$$

are as follows:

CASE 1. *If*

$$\exists\, \mathbf{u},\ \mathbf{v} \in \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\} \ :\ \mathbf{Q}^{\mathrm{T}}\mathbf{u} = (\det \mathbf{Q})\mathbf{v}\,, \tag{A.56}$$

then the solutions are given by (A.24).

CASE 2. *If*

$$\forall\, \mathbf{u},\ \mathbf{v} \in \{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\} \ :\ \mathbf{Q}^{\mathrm{T}}\mathbf{u} \neq (\det \mathbf{Q})\mathbf{v}\,, \tag{A.57}$$

then the solutions are given by (A.26).

P r o o f. First, for each $\mathbf{Q}$ given by (A.27), Eqs. (A.54)–(A.55) yield

$$\begin{aligned}
(A_{\tau(2)\tau(2)} - A_{\tau(1)\tau(1)}) \sin 2\theta + 2A_{\tau(1)\tau(2)}(\cos 2\theta - 1) &= 0,\\
(A_{\tau(2)\tau(2)} - A_{\tau(1)\tau(1)})(1 - \cos 2\theta) + 2A_{\tau(1)\tau(2)} \sin 2\theta &= 0,\\
A_{\tau(2)\tau(3)}(\cos 2\theta - 1) - A_{\tau(1)\tau(3)} \sin 2\theta &= 0,\\
A_{\tau(2)\tau(3)} \sin 2\theta + A_{\tau(1)\tau(3)}(\cos 2\theta - 1) &= 0,\\
(A_{\tau(2)\tau(2)} - A_{\tau(3)\tau(3)})(\cos 2\theta - 1) - 2A_{\tau(1)\tau(2)} \sin 2\theta &= 0,\\
(A_{\tau(3)\tau(3)} - A_{\tau(1)\tau(1)})(\cos 2\theta - 1) - 2A_{\tau(1)\tau(2)} \sin 2\theta &= 0.
\end{aligned}$$

By using $\mathbf{Q} \notin T_h$, i.e. $\theta \neq k\pi$, from the above we derive

$$A_{11} = A_{22} = A_{33}\,, \qquad A_{12} = A_{23} = A_{31} = 0.$$

Moreover, letting the tensor $\mathbf{A} \in \mathrm{Sym}$ take the particular forms $\mathbf{x} \otimes \mathbf{x}$ and $\overset{2}{\otimes}(\mathbf{E} : \mathbf{W})$, respectively, from Eqs. (A.50)–(A.53) we derive

$$x_1^2 = x_2^2 = x_3^2\,, \qquad x_1x_2 = x_2x_3 = x_3x_1 = 0,$$

$$y_1^2 = y_2^2 = y_3^2\,, \qquad y_1y_2 = y_2y_3 = y_3y_1 = 0, \qquad \mathbf{y} = \mathbf{E} : \mathbf{W}.$$

Thus, we conclude that the Case 2 holds for each $\mathbf{Q} \in \mathrm{Orth}\backslash T_h$ satisfying (A.27).

Next, for each $\mathbf{Q}$ satisfying (A.30), since the two sides of Eq. (A.55) provide two spectral representations of the same symmetric second order tensor, we deduce that either of the two involved sets of eigenvalues must be triply coalescent, or else (A.30) will be violated. Hence, we have

$$A_{11} = A_{22} = A_{33} = \bar{A}_{11} = \bar{A}_{22} = \bar{A}_{33} = c.$$

From these and the identity (A.47) we infer that (A.46) holds. Moreover, (A.54) can be recast in the form

$$\sum_{k=1}^{3} w_k \mathbf{n}_k = (\det \mathbf{Q}) \sum_{k=1}^{3} \hat{w}_k \hat{\mathbf{n}}_k\,. \tag{A.58}$$

By using the same procedure as that used in deriving (A.48), from (A.46) and (A.58) we can derive (A.48) again. Thus (A.30) and (A.48) yield (A.49). From (A.49) and (A.58) we infer that $\mathbf{A} = c\mathbf{I} + d\mathbf{u} \otimes \mathbf{u}$ for each $\mathbf{Q}$ obeying (A.56) and (A.30) or $\mathbf{A} = c\mathbf{I}$ for each $\mathbf{Q}$ satisfying (A.57) and (A.30). Finally, using the results for Eqs. (A.54)–(A.55) just derived and noticing the fact that for an orthogonal tensor $\mathbf{Q} \in \mathrm{Orth}$, if there are $\mathbf{u}, \mathbf{v} \in \{\mathbf{r}_1, \ldots, \mathbf{r}_4\}$, $\mathbf{u} \neq \mathbf{v}$, such that

$$\mathbf{Qu}, \mathbf{Qv} \in \{(\det \mathbf{Q})\mathbf{r}_1, \ldots, (\det \mathbf{Q})\mathbf{r}_4\},$$

then $\mathbf{Q} \in T_h$, we conclude that Theorem A.7 also holds for each $\mathbf{Q} \in \mathrm{Orth} \setminus T_h$ satisfying (A.30). *Q.E.D.*

A.3. Polynomial tensor equations: the icosahedral group I_h

THEOREM A.8. *Let $\mathbf{I}_h^1$, $\mathbf{I}_h^2$ and $\mathbf{I}_h^3$ be the three tensors given by (5.4), which are invariant under I_h. Then for each $\mathbf{Q} \in \mathrm{Orth}\backslash I_h$, the system of polynomial tensor equations*

$$(\mathbf{Q} * \mathbf{I}_h^r) \odot (\overset{2r+2}{\otimes}\mathbf{x}) = \mathbf{I}_h^r \odot (\overset{2r+2}{\otimes}\mathbf{x}), \qquad r = 1, 2, 3; \tag{A.59}$$

$$(\mathbf{Q} * \mathbf{I}_h^r) \odot (\overset{2r+2}{\otimes}(\mathbf{E} : \mathbf{W})) = \mathbf{I}_h^r \odot (\overset{2r+2}{\otimes}(\mathbf{E} : \mathbf{W})), \qquad r = 1, 2, 3; \tag{A.60}$$

$$(\mathbf{Q} * \mathbf{I}_h^r) \odot (\overset{r+1}{\otimes} \mathbf{A}) = \mathbf{I}_h^r \odot (\overset{r+1}{\otimes} \mathbf{A}), \qquad r = 1, 2, 3; \tag{A.61}$$

has the following solutions:

CASE 1. $\mathbf{x} = a\mathbf{u}$, $\mathbf{W} = b\mathbf{E}\mathbf{u}$, $\mathbf{A} = c\mathbf{I} + d\mathbf{u} \otimes \mathbf{u}$, $\forall\, a, b, c, d \in R$, *if*

$$\exists\, \mathbf{u}, \mathbf{v} \in \{\mathbf{n}_1, \ldots, \mathbf{n}_6\} \;\; or \;\; \mathbf{u}, \mathbf{v} \in \{\mathbf{r}_1, \ldots, \mathbf{r}_{10}\} \;:\; \overset{2}{\otimes} (\mathbf{Q}^{\mathrm{T}}\mathbf{u}) = \overset{2}{\otimes} \mathbf{v}; \tag{A.62}$$

CASE 2. $\mathbf{x} = \mathbf{0}$, $\mathbf{W} = \mathbf{O}$, $\mathbf{A} = c\mathbf{I}$, *if*

$$\forall\, \mathbf{u}, \mathbf{v} \in \{\mathbf{n}_1, \ldots, \mathbf{n}_6\} \;\; and \;\; \mathbf{u}, \mathbf{v} \in \{\mathbf{r}_1, \ldots, \mathbf{r}_{10}\} \;:\; \overset{2}{\otimes} (\mathbf{Q}^{\mathrm{T}}\mathbf{u}) \neq \overset{2}{\otimes} \mathbf{v}\,. \tag{A.63}$$

To prove the above theorem, some facts concerning the symmetry axes of the icosahedral group I_h are needed.

LEMMA A.1. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \{\mathbf{n}_1, \ldots, \mathbf{n}_6\}$ be any four different five-fold axes of the group I_h. Then for $\mathbf{G} \in \mathrm{Orth}$ and any $p, q, r \in R$, the conditions

$$\begin{aligned}
\mathbf{G} * (\mathbf{a} \otimes \mathbf{a} + p\mathbf{d} \otimes \mathbf{d}) &= \mathbf{a} \otimes \mathbf{a} + p\mathbf{d} \otimes \mathbf{d},\\
\mathbf{G} * (\mathbf{b} \otimes \mathbf{b} + q\mathbf{d} \otimes \mathbf{d}) &= \mathbf{b} \otimes \mathbf{b} + q\mathbf{d} \otimes \mathbf{d},\\
\mathbf{G} * (\mathbf{c} \otimes \mathbf{c} + r\mathbf{d} \otimes \mathbf{d}) &= \mathbf{c} \otimes \mathbf{c} + r\mathbf{d} \otimes \mathbf{d},
\end{aligned}$$

imply $\mathbf{G} \in I_h$.

P r o o f. Consider two cases. First, let at least two of p, q and r, say $p = q = 0$, be zero. Then by means of the conditions

$$\mathbf{G} * (\mathbf{a} \otimes \mathbf{a}) = \mathbf{a} \otimes \mathbf{a}, \qquad \mathbf{G} * (\mathbf{b} \otimes \mathbf{b}) = \mathbf{b} \otimes \mathbf{b}, \qquad \mathbf{a} \cdot \mathbf{b} \neq 0$$

we infer

$$\mathbf{G} \in \{\pm\mathbf{I}, \pm\mathbf{R}^{\pi}_{\mathbf{a}\times\mathbf{b}}\} \subset I_h\,,$$

where $\mathbf{a} \times \mathbf{b}$ be a two-fold axis of I_h (cf. Proposition 7.2 in [53]).

Next, let two of p, q and r be nonvanishing, e.g. $pq \neq 0$. Then the two tensors $\mathbf{a} \otimes \mathbf{a} + p\mathbf{d} \otimes \mathbf{d}$ and $\mathbf{b} \otimes \mathbf{b} + q\mathbf{d} \otimes \mathbf{d}$ have no eigenline in common and therefore the first two conditions in the above lemma imply $\mathbf{G} = \pm\mathbf{I} \in I_h$ (see Lemma 3.1.1 given in [48]). In reality, $\mathbf{a} \times \mathbf{d}$ and $\mathbf{b} \times \mathbf{d}$ offer two eigenlines of the just-mentioned two tensors, respectively, and the other eigenlines of the two tensors lie on the two planes perpendicular to these two eigenlines, respectively. Hence, the intersecting line of the two planes is the only possible commom eigenline of the aforementioned two tensors. The former is just $\mathbf{d}$ and can not be an eigenline of any of the aforementioned tensors. *Q.E.D.*

LEMMA A.2. Let $\mathbf{n}_i, \mathbf{n}_j$ and $\mathbf{n}_k$ be any three noncoplanar five-fold axes of the group I_h. Then the following equality holds.

$$\mathbf{n}_i \otimes \mathbf{n}_i + \mathbf{n}_j \otimes \mathbf{n}_j + \mathbf{n}_k \otimes \mathbf{n}_k = x\mathbf{I} + y\mathbf{u} \otimes \mathbf{u}, \qquad y \neq 0\,, \tag{A.64}$$

where

$$\mathbf{u} = (\mathbf{n}_j \cdot \mathbf{n}_k)\mathbf{n}_i + (\mathbf{n}_k \cdot \mathbf{n}_i)\mathbf{n}_j + (\mathbf{n}_i \cdot \mathbf{n}_j)\mathbf{n}_k \tag{A.65}$$

represents a three-fold axis of the group I_h.

P r o o f. In terms of any three noncoplaner three-fold axes $(\mathbf{n}_i, \mathbf{n}_j, \mathbf{n}_k)$ of I_h, the second order identity tensor $\mathbf{I}$ is expressible as (cf. the formula (7.11) in [53])

$$\mathbf{I} = f(\mathbf{n}_i \otimes \mathbf{n}_i + \mathbf{n}_j \otimes \mathbf{n}_j + \mathbf{n}_k \otimes \mathbf{n}_k) + g \overset{2}{\otimes} ((\mathbf{n}_j \cdot \mathbf{n}_k)\mathbf{n}_i + (\mathbf{n}_k \cdot \mathbf{n}_i)\mathbf{n}_j + (\mathbf{n}_i \cdot \mathbf{n}_j)\mathbf{n}_k), \quad fg \neq 0 .$$

From the above equality we derive (A.64). Moreover, from Proposition 7.1 in [53] we know that the vector $\mathbf{u}$ given by (A.65) represents a three-fold axis of I_h. *Q.E.D.*

The proof for Theorem A.8 is as follows. By using (5.3) we deduce that $\det((\mathbf{n}_i \cdot \mathbf{n}_j)^2) = 2(\frac{4}{5})^5 \neq 0$ and hence that $\{\mathbf{n}_i \otimes \mathbf{n}_i\}$ offers a basis of the space Sym. In terms of this basis each $\mathbf{A} \in$ Sym is expressible as (cf. Proposition 7.4 given in [53])

$$\mathbf{A} = \frac{5}{4} \sum_{k=1}^{6} A_k \mathbf{N}_k - \frac{1}{2}(\mathrm{tr}\mathbf{A})\mathbf{I}, \tag{A.66}$$

where

$$\mathbf{N}_k = \mathbf{n}_k \otimes \mathbf{n}_k , \qquad A_k = \mathbf{n}_k \cdot \mathbf{A}\mathbf{n}_k , \qquad k = 1, \ldots, 6.$$

Utilizing (A.66) we infer that the following identities hold.

$$\sum_{k=1}^{6} \mathbf{N}_k = \sum_{k=1}^{6} \mathbf{Q} * \mathbf{N}_k \ (= 2\mathbf{I}),$$

$$\sum_{k=1}^{6} A_k \mathbf{N}_k = \sum_{k=1}^{6} A'_k \mathbf{Q} * \mathbf{N}_k ,$$

for any $\mathbf{Q} \in$ Orth, where $A'_k = (\mathbf{Q}\mathbf{n}_k) \cdot \mathbf{A}(\mathbf{Q}\mathbf{n}_k)$. The above two identities and the equations (A.61) may be combined into

$$\sum_{k=1}^{6} (A_k)^r \mathbf{N}_k = \sum_{k=1}^{6} (A'_k)^r \mathbf{Q} * \mathbf{N}_k , \qquad r = 0, 1, 2, 3, 4 . \tag{A.67}$$

Let

$\mathbf{A}_r \equiv$ the left-hand side of (A.67); $\qquad \mathbf{A}'_r \equiv$ the right-hand side of (A.67).

Then $\mathrm{tr}\mathbf{A}_r = \mathrm{tr}\mathbf{A}'_r$, $r = 0, 1, 2, 3, 4$, yield

$$\sum_{k=1}^{6}(A_k)^r = \sum_{k=1}^{6}(A'_k)^r, \qquad r = 0, 1, 2, 3, 4.$$

Here and hereafter $\mathrm{tr}\mathbf{B}$ is used to represent the trace of the tensor $\mathbf{B} \in T_2$. Furthermore, from (5.3) and the following equalities

$$(\mathrm{tr}\mathbf{A}_s)(\mathrm{tr}\mathbf{A}_t) = (\mathrm{tr}\mathbf{A}'_s)(\mathrm{tr}\mathbf{A}'_t), \qquad \mathrm{tr}(\mathbf{A}_s\mathbf{A}_t) = \mathrm{tr}(\mathbf{A}'_s\mathbf{A}'_t),$$

we derive

$$\sum_{k=1}^{6}(A_k)^{s+t} = \sum_{k=1}^{6}(A'_k)^{s+t}. \tag{A.68}$$

Let P_6 be the symmetric group on six letters. Then (A.68) yields

$$A'_k = A_{\sigma(k)}, \qquad k = 1, \ldots, 6; \qquad \sigma \in P_6. \tag{A.69}$$

Hence the five equations for $\mathbf{A} \in \mathrm{Sym}$ given by (A.67) can be recast in the form

$$\sum_{k=1}^{6}(A_{\sigma(k)})^r\mathbf{Q} * \mathbf{N}_k = \sum_{k=1}^{6}(A_k)^r\mathbf{N}_k, \qquad r = 0, 1, 2, 3, 4.$$

Since for any given $\sigma \in P_6$ there is $\mathbf{R} \in I_h$ such that

$$\mathbf{R}^{\mathrm{T}} * \mathbf{N}_k = \mathbf{N}_k, \qquad k = 1, \ldots, 6,$$

the above system of equations for $\mathbf{A}$ can be rewritten as

$$\sum_{k=1}^{6} A_k^r\mathbf{G}*\mathbf{N}_k = \sum_{k=1}^{6} A_k^r\mathbf{N}_k, \qquad \mathbf{G} = \mathbf{QR}, \qquad \mathbf{R} \in I_h, \quad r = 0, 1, 2, 3, 4. \tag{A.70}$$

Suppose that all A_k are pairwise distinct. Reformulating (A.70) in matrix notation as follows:

$$VX' + Y' = VX + Y,$$

where V is the 5×5 Vandermonde matrix of $A_1 \cdots A_5$, the sth row of which is given by $(A_1^{s-1} \cdots A_5^{s-1})$, and moreover, X, Y, X', and Y' are the following 5×1 column matrices:

$$X = (\mathbf{N}_1 \cdots \mathbf{N}_5)^{\mathrm{T}}, \qquad X' = (\mathbf{G} * \mathbf{N}_1 \cdots \mathbf{G} * \mathbf{N}_5)^{\mathrm{T}},$$

$$Y = (\mathbf{N}_6\ A_6\mathbf{N}_6 \cdots (A_6)^4\mathbf{N}_6)^{\mathrm{T}}, \qquad Y' = (\mathbf{G} * \mathbf{N}_6\ A_6\mathbf{G} * \mathbf{N}_6 \cdots (A_6)^4\mathbf{G} * \mathbf{N}_6)^{\mathrm{T}}.$$

Since the matrix V is invertible, we obtain

$$X' + V^{-1}Y' = X + V^{-1}Y,$$

i.e.

$$\mathbf{G} * (\mathbf{N}_k + x_k\mathbf{N}_6) = \mathbf{N}_k + x_k\mathbf{N}_6\,, \qquad k = 1,2,3,4,5.$$

Then by Lemma A.1 we infer that $\mathbf{G} \in I_h$ and therefore that $\mathbf{Q} = \mathbf{G}\mathbf{R}^{\mathrm{T}} \in I_h$, which violates the condition $\mathbf{Q} \notin I_h$.

Suppose that some of $A_1, \ldots, A_6$ coincide. By means of the similar procedure as that just used, we infer that the following facts hold.

(i) If there are $i, j \in \{1, \ldots, 6\}$, $i \neq j$, such that $A_i \neq A_j$ and $A_k \neq A_i$, A_j for all $k \in \{1, \ldots, 6\}$, $k \neq i, j$, then

$$\mathbf{G} * \mathbf{N}_i = \mathbf{N}_i\,, \qquad \mathbf{G} * \mathbf{N}_j = \mathbf{N}_j.$$

(ii) If there are $i, j \in \{1, \ldots, 6\}$, $i \neq j$, such that $A_i = A_j$ and $A_k \neq A_i$ for all $k \in \{1, \ldots, 6\}$, $k \neq i, j$, then

$$\mathbf{G} * (\mathbf{N}_i + \mathbf{N}_j) = \mathbf{N}_i + \mathbf{N}_j.$$

(iii) If $A_i = A_j = A_k \neq A_l = A_m = A_n$, where $(i, \ldots, n)$ is a permutation of $1, \cdots, 6$, then

$$\mathbf{A} = c'\mathbf{I} + d'(\mathbf{N}_i + \mathbf{N}_j + \mathbf{N}_k), \qquad d' \neq 0,$$
$$\mathbf{G} * (\mathbf{N}_i + \mathbf{N}_j + \mathbf{N}_k) = \mathbf{N}_i + \mathbf{N}_j + \mathbf{N}_k\,,$$

i.e. (cf. Lemma A.2)

$$\mathbf{A} = c\mathbf{I} + d\mathbf{u} \otimes \mathbf{u}, \qquad d \neq 0\,,$$
$$\overset{2}{\otimes}\,(\mathbf{Q}^{\mathrm{T}}\mathbf{u}) = \overset{2}{\otimes}\,\mathbf{v}\,,$$

where $\mathbf{v} = \mathbf{R}\mathbf{u}$ represents a three-fold axis of I_h, since $\mathbf{R} \in I_h$ and $\mathbf{u}$ represents a three-fold axis of I_h;

(iv) If $A_i \neq A_j = A_k = A_l = A_m = A_n$, then

$$\mathbf{A} = c\mathbf{I} + d\mathbf{N}_i\,, \qquad d \neq 0\,,$$
$$\mathbf{Q}^{\mathrm{T}} * \mathbf{N}_i = \mathbf{R} * \mathbf{N}_i\,, \qquad \mathbf{R} \in I_h\,.$$

(v) If $A_1 = \cdots = A_6 = c$, then $\mathbf{A} = c\mathbf{I}$.

In the last two cases, the identity (A.66) for $\mathbf{A} = \mathbf{I}$ has been used.

The cases (i)–(v) exhaust all the cases when $A_1, \ldots, A_6$ are not pairwise distinct. For the first two cases, we have

$$\mathbf{G} \in \{\pm\mathbf{I}, \pm\mathbf{R}^{\pi}_{\mathbf{n}_i+\mathbf{n}_j}, \pm\mathbf{R}^{\pi}_{\mathbf{n}_i-\mathbf{n}_j}, \pm\mathbf{R}^{\pi}_{\mathbf{n}_i\times\mathbf{n}_j}\}.$$

Since the vectors $\mathbf{n}_i + \mathbf{n}_j$, $\mathbf{n}_i - \mathbf{n}_j$ and $\mathbf{n}_i \times \mathbf{n}_j$ give three two-fold axes of I_h (cf. Proposition 7.2 in [53]), we infer that $\mathbf{G} \in I_h$ and hence $\mathbf{Q} = \mathbf{G}\mathbf{R}^{\mathrm{T}} \in I_h$ for the first two cases, which violates the condition $\mathbf{Q} \notin I_h$. Thus, the first two cases are excluded. On the other hand, the latter three cases yield three kinds of solutions to the polynomial tensor equations (A.61) for $\mathbf{A} \in \mathrm{Sym}$, and from them the solutions to the polynomial tensor equations (A.59) and (A.60) for $\mathbf{x} \in V$ and $\ddot{\mathbf{W}} \in \mathrm{Skw}$ can be derived immediately, since both $\mathbf{x} \otimes \mathbf{x}$ and $\overset{2}{\otimes}(\mathbf{E} : \mathbf{W})$ can be visualized as two particular forms of the symmetric second order tensor $\mathbf{A}$. It is evident that the solutions thus obtained agree with those given by the two cases in Theorem A.8. *Q.E.D.*

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