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Steady non-uniform extensional motions as applied to kinematic description of polymer fibre formation

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IT IS SHOWN that the concept of steady non-uniform extensional motions (NUEM) can be used for kinematic description of polymer fibre formation, taking into account the variable geometry and shearing effects. To this end, pretty general, materially non-uniform constitutive equations, depending on temperature distributions, structure formations, etc., are applied and the linearized perturbation procedure is developed. Especially simple expressions describing the additional velocity fields are obtained for the first order approximation.

1. Introduction

IN OUR PREVIOUS PAPERS [1, 2], we discussed the concept of steady non-uniform extensional motions (called briefly NUEM) of materially non-uniform (non-homogeneous) fluids and solids. We also mentioned possible applicability of the above concept to various fibre-forming processes and certain flows realized in extensometers. An example of application to the case of cold drawing of polymer fibres was presented in [3].

In this paper, we use the concept of steady NUEM to describe many realistic fibre-forming processes, assuming that the fundamental motions are quasi-elongational and the shearing effects, resulting from the axial variability of fibre geometry, are taken into account. A motivation for the present description arises from the following requirements.

1. We want to apply relatively general constitutive equations describing various fundamental quasi-elongational motions. An assumption of particular rheological models, frequently made for description of fibre-forming processes, is not necessary. Such an approach to the problem enables effective application either of experimental data or numerical results calculated for simpler models (Newtonian, Maxwellian, etc.).

2. Material properties of fibres in the processes considered essentially depend on temperature distributions, crystallization effects, structure orientation etc. (cf. [4]). The concept of steady NUEM of materially non-uniform materials replaces, in some sense, arbitrary distributions of mechanical properties varying from position to position in media which are homogeneous in reality. Moreover, there exists some possibility of smooth transitions from viscous to elastic materials or from fluid-like to solid-like behaviour.

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3. We try to apply a consequent linearization process throught the corresponding perturbation procedure. To this end, an asssumption of thin-thread (layer) approximation, usually satisfied in fibre processing, is very useful.

The concept considered generalizes, to some extent, that of steady flows with dominating extension (briefly called FDE) developed previously in [5] and applied to melt-spinning processes in [6]. We must emphasize, however, that the concept of steady FDE does not satisfy the requirement 1 and 3. The requirement 2 remains valid only for the properly defined viscosity function.

In Sec. 2 the general quasi-elongational motions and the corresponding constitutive equations are considered. Section 3 is entirely devoted to the additional superposed motions describing the variability of fibre geometry and the related shearing effects. Moreover, we introduce the auxiliary concept of thin-tread (layer) approximation. The continuity conditions in local and global forms are discussed in Sec. 4. Sections 5 and 6 contain the equilibrium equations and the boundary conditions presented for the first and second order approximations. In Sec. 7 the corresponding solutions of the previously derived governing equations are obtained for viscoelastic isotropic materials. Certain particular cases are discussed in greater detail. The main results are quoted in Sec. 8 in a form of final remarks.

2. Quasi-elongational motions treated as steady non-uniform extensional motions (NUEM)

Consider the isochoric, quasi-elongational motion for which the deformation gradient at the current time t, relative to a configuration at time 0, is of the diagonal form in cylindrical coordinates:

$$[\mathbf{F}(\mathbf{X},t)] = \begin{bmatrix} \lambda^{1/2} & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \qquad \det \mathbf{F} = 1, \tag{2.1}$$

where the non-uniform stretch ratio $\lambda(\mathbf{X},t)$ depends on time t as well as on the position $\mathbf{X}$ of a particle in the reference configuration $\boldsymbol{\kappa}$ at time 0. We use the following definitions:

$$\lambda = V/V_0\,, \qquad \epsilon = \ln\lambda, \tag{2.2}$$

where ϵ is the Hencky measure of strain; V and V_0 denote the variable axial velocity and the velocity at the exit (feeding velocity), respectively. The above quasi-elongational motion is consistent with the definition of NUEM introduced in [2].

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On the basis of Eq. (2.1), the velocity gradient (strain rate) can be written as

$$(2.3)\qquad [\mathbf{L}(\mathbf{X},t)] = [\dot{\mathbf{F}}\,\mathbf{F}^{-1}] = \begin{bmatrix} -\frac{1}{2}V' & 0 & 0 \\ 0 & -\frac{1}{2}V' & 0 \\ 0 & 0 & V' \end{bmatrix},$$

where V' denotes the axial component of the velocity gradient and the primes denote derivatives with respect to the axial coordinate z.

Equations (2.1) and (2.3) lead to the following forms of the left Cauchy–Green deformation tensor $\mathbf{B}$ and the first Rivlin–Ericksen kinematic tensor $\mathbf{A}_1$ (cf. [7]):

$$(2.4)\qquad [\mathbf{B}(\mathbf{X},t)] = [\mathbf{F}\,\mathbf{F}^T] = \begin{bmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix} = \begin{bmatrix} \frac{V_0}{V} & 0 & 0 \\ 0 & \frac{V_0}{V} & 0 \\ 0 & 0 & \frac{V^2}{V_0^2} \end{bmatrix},$$

$$(2.5)\qquad [\mathbf{A}_1(\mathbf{X},t)] = \begin{bmatrix} -\frac{\dot{\lambda}}{\lambda} & 0 & 0 \\ 0 & -\frac{\dot{\lambda}}{\lambda} & 0 \\ 0 & 0 & 2\frac{\dot{\lambda}}{\lambda} \end{bmatrix} = \begin{bmatrix} -V' & 0 & 0 \\ 0 & -V' & 0 \\ 0 & 0 & 2V' \end{bmatrix},$$

respectively. In the above expressions we have used the relations:

$$(2.6)\qquad \dot{\lambda} = V'\lambda, \qquad \dot{\epsilon} = V',$$

where the dots denote differentiation with respect to time.

For steady NUEM the gradient $\mathbf{L}$ as well as the kinematic tensor $\mathbf{A}_1$ do not depend on time. Thus, according to our previous considerations [2], the constitutive equations of materially non-uniform, simple, locally isotropic materials can be expressed in the form:

$$(2.7)\qquad \mathbf{T}(\mathbf{X},t) = \mathbf{h}(\mathbf{A}_1(\mathbf{X}), \mathbf{B}(\mathbf{X},\mathbf{t}), \varrho(\mathbf{X}); \mathbf{X}),$$

where $\mathbf{T}$ is the non-uniform stress tensor, and $\mathbf{h}$ denotes the non-uniform isotropic function, depending on the reference configuration $\boldsymbol{\kappa}$. In the case of incompressible materials $\mathbf{T}$ should be replaced by the extra-stress tensor $\mathbf{T}_E$ and the dependence on the scalar density ϱ should be disregarded. The question whether Eq. (2.7) describes a fluid or solid can be answered having known the corresponding isotropy (internal symmetry) group (cf. [7]).

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For steady quasi-elongational motions, describing the majority of fibre forming processes, in which material properties depend solely on the coordinate z, there exists a unique correspondence between the material Z (in the reference configuration $\boldsymbol{\kappa}$) and the spatial coordinate z. In particular, we may assume that

$$z = \frac{V}{V_0} Z, \qquad z = Vt, \qquad Z = V_0 t. \tag{2.8}$$

Thus, Eq. (2.7) can be written in the particular form:

$$\mathbf{T}(z) = \mathbf{k}(V'(z), V(z), \varrho(z); z), \tag{2.9}$$

where $\mathbf{k}$ is the tensor function of the indicated scalar arguments. If necessary, the pairs of arguments λ', λ or $\dot{\epsilon}$, ϵ can be used instead of V', V.

For our present purposes the way of reasoning leading to the constitutive equation (2.7) has not been presented with all details (to this end cf. [1, 2]); the simplified Eq. (2.9) can also be taken as a constitutive postulate. Therefore, we assume that the stress (or extra-stress) components in the motions considered depend on the velocity gradient V', the velocity V, the density ϱ and the coordinate z characterizing an explicit dependence of the material properties on the position along the axis.

Since for axisymmetric, quasi-elongational motions only normal components of stresses are meaningful, we can also write

$$\begin{aligned} T^{11} &= T^{22} = \sigma_1(V', V, \varrho; z), \\ T^{33} &= \sigma_3(V', V, \varrho; z), \\ T^{13} &= 0, \\ T^{33} - T^{11} &= \sigma_3 - \sigma_1 = \sigma(V', V, \varrho; z). \end{aligned} \tag{2.10}$$

3. Additional motion and shearing effects

In the motions considered, the inclination of fibre surface is usually a small quantity, i.e. $R' = 0(\varepsilon)$, $\varepsilon = R_0/L \ll 1$, where R, R_0 and L denote the outer radius of the filament, the outer radius at the exit (or the orifice radius) and the total length, respectively.

In what follows, we assume that some small additional velocity field, viz.

$$\mathbf{w}(r, z) = \mathbf{0}(\varepsilon) \tag{3.1}$$

is superposed on the fundamental, quasi-elongational motion described by the axial velocity $V(z)$. Under the above assumption all the quantities relevant for the motions considered undergo some small linear increments denoted by Δ. We have, in particular,

$$\mathbf{L}^* = \mathbf{L} + \Delta\mathbf{L}, \qquad \mathbf{F}^* = \mathbf{F} + \Delta\mathbf{F}, \quad \text{etc.} \tag{3.2}$$

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For the deformation gradients, velocity gradients, deformation tensors and kinematic tensors, we obtain the following matrices:

$$
(3.3)\qquad
\begin{gathered}
[\Delta \mathbf{F}] = \begin{bmatrix} -\dfrac{1}{2}\dfrac{V_0}{V^2}w & 0 & \dfrac{u}{V_0} \\ 0 & -\dfrac{1}{2}\dfrac{V_0}{V^2}w & 0 \\ 0 & 0 & \dfrac{V}{V_0^2}w \end{bmatrix}, \\
[\Delta \mathbf{B}] = \begin{bmatrix} -\dfrac{V_0}{V^2}w & 0 & \left(\dfrac{V_0}{V}\right)^{1/2}\dfrac{u}{V_0} \\ 0 & -\dfrac{V_0}{V^2}w & 0 \\ \left(\dfrac{V_0}{V}\right)^{1/2}\dfrac{u}{V_0} & 0 & 2\dfrac{V}{V_0^2}w \end{bmatrix},
\end{gathered}
$$

and

$$
(3.4)\qquad [\Delta \mathbf{L}] = \begin{bmatrix} \dfrac{\partial u}{\partial r} & 0 & \dfrac{\partial u}{\partial z} \\ 0 & \dfrac{u}{r} & 0 \\ \dfrac{\partial w}{\partial r} & 0 & \dfrac{\partial w}{\partial z} \end{bmatrix}, \qquad [\Delta \mathbf{A}_1] = \begin{bmatrix} 2\dfrac{\partial u}{\partial r} & 0 & \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r} \\ 0 & 2\dfrac{u}{r} & 0 \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r} & 0 & 2\dfrac{\partial w}{\partial z} \end{bmatrix},
$$

where u and w denote the radial and axial components of the additional velocity $\mathbf{w}$, respectively. In the above formulae we have used the simple relations:

$$
(3.5)\qquad \Delta\lambda = \frac{w}{V_0}, \qquad \Delta\lambda' = \frac{1}{V_0}\frac{\partial w}{\partial z}, \qquad \Delta\lambda^{-1} = -\frac{V_0}{V^2}w.
$$

The constitutive equations (2.10), after taking into account the increments resulting from the additional velocity field (3.1), can be presented in the following general form, linear with respect to u and w:

$$
(3.6)\qquad
\begin{aligned}
T^{*11} &= \sigma_1 + \frac{\partial \sigma_1}{\partial V}w + \frac{\partial \sigma_1}{\partial V'}\frac{\partial w}{\partial z} + \frac{\partial \sigma_1}{\partial \varrho}\Delta\varrho + \alpha\frac{\partial u}{\partial r}, \\
T^{*22} &= \sigma_1 + \frac{\partial \sigma_1}{\partial V}w + \frac{\partial \sigma_1}{\partial V'}\frac{\partial w}{\partial z} + \frac{\partial \sigma_1}{\partial \varrho}\Delta\varrho + \beta\frac{u}{r}, \\
T^{*33} &= \sigma_3 + \frac{\partial \sigma_3}{\partial V}w + \frac{\partial \sigma_3}{\partial V'}\frac{\partial w}{\partial z} + \frac{\partial \sigma_3}{\partial \varrho}\Delta\varrho, \\
T^{*13} &= \eta\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) + \gamma u, \\
T^{*33} - T^{*11} &= \sigma + \frac{\partial \sigma}{\partial V}w + \frac{\partial \sigma}{\partial V'}\frac{\partial w}{\partial z} + \frac{\partial \sigma}{\partial \varrho}\Delta\varrho - \alpha\frac{\partial u}{\partial r},
\end{aligned}
$$

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where α, β, γ and η are new, additional material functions, depending on the same arguments as σ, e.g.

$$\eta = \eta(V', V, \varrho; z). \tag{3.7}$$

The functions η, α and β have dimension of viscosity ($\mathrm{Ns/m^2}$), while γ, characterizing the shearing deformations of a material, has dimension of shear modulus divided by velocity ($\mathrm{Ns/m^3}$).

It is worth noting that the representations of constitutive equations in the form (3.6) can also be obtained in a different way. An application of linear perturbation procedure to the Rivlin–Ericksen constitutive equations (cf. [7]), involving 8 material functions (α_i, $i = 1, \ldots, 8$), leads to exactly the same result.

The next step in our perturbation procedure is connected with the so-called thin-thread (layer) approximation (cf. [6]). To this end, we assume again that $\varepsilon = R_0/L$ is a small quantity. Introducing the following dimensionless variables marked with overbars:

$$r = \overline{r} R_0\,, \qquad z = \overline{z} L, \qquad w = U\overline{w}, \qquad u = \varepsilon U \overline{u}, \tag{3.8}$$

where the characteristic velocity $U = V'(0)R_0$, or $U = V'_{\max} R_0$, we arrive at the following increments:

$$[\Delta B] = \begin{bmatrix} -\dfrac{\overline{V}_0}{\overline{V}^2}\overline{w} & 0 & \varepsilon \dfrac{\overline{u}}{\overline{V}_0}\left(\dfrac{\overline{V}_0}{\overline{V}}\right)^{1/2} \\ 0 & -\dfrac{\overline{V}_0}{\overline{V}^2}\overline{w} & 0 \\ \varepsilon \dfrac{\overline{u}_0}{\overline{V}_0}\left(\dfrac{\overline{V}_0}{\overline{V}}\right)^{1/2} & 0 & \dfrac{\overline{V}}{\overline{V}_0^2}\overline{w} \end{bmatrix}, \tag{3.9}$$

$$[\Delta A_1] = \begin{bmatrix} \varepsilon 2\dfrac{\partial \overline{u}}{\partial \overline{r}} & 0 & \dfrac{\partial \overline{w}}{\partial \overline{r}} + \varepsilon^2 \dfrac{\partial \overline{u}}{\partial \overline{z}} \\ 0 & \varepsilon 2 \dfrac{\overline{u}}{\overline{r}} & 0 \\ \dfrac{\partial \overline{w}}{\partial \overline{r}} + \varepsilon^2 \dfrac{\partial \overline{u}}{\partial \overline{z}} & 0 & \varepsilon 2 \dfrac{\partial \overline{w}}{\partial \overline{z}} \end{bmatrix} \dfrac{U}{R_0}. \tag{3.10}$$

Since by assumption, the axial component w of the additional velocity field is of order ε, the radial component u under a thin-thread approximation is of order ε^2. This fact is taken into account when dealing with various terms in the corresponding governing equations (Sec. 5).

4. Continuity conditions

So far, we have not discussed any continuity conditions, assuming tacitly that they are satisfied in a local as well as in a global sense.

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In the local form the continuity condition, valid for the fundamental, quasi-elongational motion, viz.

$$\dot{\varrho} + \varrho \operatorname{div} \mathbf{V} = 0, \tag{4.1}$$

implies that $\varrho = \text{const}$, if $\operatorname{div} \mathbf{V} = 0$ and the motion is steady ($\partial\varrho/\partial t = 0$). The same equation for quantities involving the corresponding increments ($\mathbf{V}^* = \mathbf{V}+\mathbf{w}$, $\varrho^* = \varrho + \Delta\varrho$) amounts to

$$V\frac{\partial \Delta\varrho}{\partial z} + \varrho \operatorname{div} \mathbf{w} = 0, \tag{4.2}$$

and after integration to

$$\Delta\varrho = -\int \frac{\varrho}{V}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z}\right) dz + C(r), \tag{4.3}$$

where $C(r)$ is an arbitrary function of r only.

Since for our thin-thread approximation the radial components u are of order of magnitude less than the axial components w, we may use the approximate formula:

$$\Delta\varrho \cong -\int \frac{\varrho}{V}\frac{\partial w}{\partial z} dz + C(r). \tag{4.4}$$

On the other hand, in the global form the mass output W in a fundamental motion must be constant along the filament, viz.

$$W = \varrho\pi R^2 V = \text{const}. \tag{4.5}$$

Taking into account the corresponding mass output with small increments w and $\Delta\varrho$, we arrive at

$$2\pi\int_0^R (V + w)(\varrho + \Delta\varrho) r\, dr = \text{const}, \tag{4.6}$$

what leads to

$$V\int_0^R \Delta\varrho r\, dr + \varrho\int_0^R wr\, dr = 0. \tag{4.7}$$

If the additional velocity field is such that the second integral is identically equal to zero (cf. Sec. 6), the radial distribution of the density increment $\Delta\varrho$ is determined only by the vanishing first integral (4.7).

Another implication of the condition (4.5) are the following formulae:

$$\frac{R'}{R} = -\frac{1}{2}\frac{V'}{V} \qquad \text{or} \qquad \frac{d}{dz}\ln R = -\frac{1}{2}\frac{d}{dz}\ln V, \tag{4.8}$$

expressing useful relations between the fibre radius and the fundamental velocity V.

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5. Equations of equilibrium

For axisymmetric deformations the inertialess equations of equilibrium, expressed in cylindrical coordinates, viz.

$$
\begin{aligned}
&\frac{\partial T^{*11}}{\partial r} + \frac{1}{r}(T^{*11} - T^{*22}) + \frac{\partial T^{*13}}{\partial z} = 0, \\
&\frac{\partial T^{*13}}{\partial r} + \frac{1}{r}T^{*13} + \frac{\partial T^{*33}}{\partial z} = 0,
\end{aligned} \tag{5.1}
$$

after taking into account Eqs. (3.6), lead to

$$
\begin{aligned}
&\frac{\partial \sigma_1}{\partial r} + \frac{\partial}{\partial r}\left(\frac{\partial \sigma_1}{\partial V}w + \frac{\partial \sigma_1}{\partial V'}\frac{\partial w}{\partial z} + \partial\sigma_1\partial\varrho\Delta\varrho\right) + \frac{\partial}{\partial z}\left(\eta\frac{\partial w}{\partial r}\right) + \frac{\partial}{\partial r}\left(\alpha\frac{\partial u}{\partial r}\right) \\
&\qquad + \frac{1}{r}\left(\alpha\frac{\partial u}{\partial r} - \beta\frac{u}{r}\right) + \frac{\partial}{\partial z}\left(\eta\frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial z}(\gamma u) = 0, \\
&\frac{\partial \sigma_3}{\partial z} + \frac{\partial}{\partial z}\left(\frac{\partial \sigma_3}{\partial V}w + \frac{\partial \sigma_3}{\partial V'}\frac{\partial w}{\partial z} + \frac{\partial \sigma_3}{\partial \varrho}\Delta\varrho\right) \\
&\qquad + \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\eta\frac{\partial w}{\partial r}\right) + \frac{\partial}{\partial r}\left(\eta\frac{\partial u}{\partial z}\right) + \frac{\eta}{\gamma}\frac{\partial u}{\partial z} + \frac{\gamma}{r}\frac{\partial}{\partial r}(ru) = 0.
\end{aligned} \tag{5.2}
$$

Differentiating the first Eq. (5.2) with respect to z and the second one with respect to r, and subtracting (this procedure also eliminates the hydrostatic pressure, if necessary), we arrive at

$$
\begin{aligned}
&\eta\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{d}{dz}\left(\sigma + \frac{\partial \sigma}{\partial V}w + \frac{\partial \sigma}{\partial V'}\frac{\partial w}{\partial z} + \frac{\partial \sigma}{\partial \varrho}\Delta\varrho\right)\right\} \\
&\quad - \frac{\partial^2}{\partial z^2}\left(\eta\frac{\partial w}{\partial r}\right) + \frac{\partial}{\partial r}\left[\frac{\gamma}{r}\frac{\partial}{\partial r}(ru)\right] - \frac{\partial^2}{\partial z^2}(\gamma u) - \frac{\partial^2}{\partial r\partial z}\left(\alpha\frac{\partial u}{\partial r}\right) \\
&\quad - \frac{\partial}{\partial z}\left[\frac{1}{r}\left(\alpha\frac{\partial u}{\partial r} - \beta\frac{u}{r}\right)\right] + \frac{\partial^2}{\partial r\partial z}\left(\eta\frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{\eta}{r}\frac{\partial u}{\partial z}\right) \\
&\quad + \eta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial z}\right) - \frac{\partial^2}{\partial z^2}\left(\eta\frac{\partial u}{\partial z}\right) = 0.
\end{aligned} \tag{5.3}
$$

The corresponding analysis of orders of magnitude determined by the powers of ε leads, after integration with respect to r, to the following governing equation:

$$
\eta\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + r\frac{d}{dz}\left(\sigma + \frac{\partial \sigma}{\partial V}w + \frac{\partial w}{\partial V'}\frac{\partial w}{\partial z} + \frac{\partial \sigma}{\partial \varrho}\Delta\varrho\right) = Cr - \gamma\frac{\partial}{\partial r}(ru), \tag{5.4}
$$

where only terms up to ϱ^2 have been retained and C is an integration constant. In the above equations as well as in our further considerations the symbol d/dz denotes the total derivative with respect to z.

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Let us assume, in agreement with Eq. (3.1), that the additional velocity field can be written as

$$w = \varepsilon w_1 + \varepsilon^2 w_2 . \tag{5.5}$$

Under the above assumption the governing equations resulting from Eq. (5.4) take the following forms:

$$\eta \frac{\partial}{\partial r}\left(r\frac{\partial w_1}{\partial r}\right) = C_1 r - r\frac{d\sigma}{dz}, \tag{5.6}$$

for the first order approximation with respect to ε, and

$$\eta \frac{\partial}{\partial r}\left(r\frac{\partial w_2}{\partial r}\right) = C_2 r - r\frac{d}{dz}\left(\frac{\partial \sigma}{\partial V}w_1 + \frac{\partial \sigma}{\partial V'}\frac{\partial w_1}{\partial z} + \frac{\partial \sigma}{\partial \varrho}\Delta\varrho\right) - \gamma\frac{\partial}{\partial r}(ru) \tag{5.7}$$

for the second order approximation containing terms of order ε^2.

6. Boundary conditions

The governing equations (5.6), (5.7) are the second order partial differential equations which can be integrated with respect to r. To this end, at least two boundary conditions for the additional motions are necessary.

Because of Eq. (4.5) and the boundary conditions satisfied at the exit (feeding velocity) and the end of filament (take-up velocity, cf. [4]):

$$V(0) = V_0, \qquad V(L) = V_L, \tag{6.1}$$

respectively, it is reasonable to assume that the additional velocity field only modifies the uniform velocity profile resulting from the fundamental motion. This means that we assume

$$\int_0^R wr\,dr = 0. \tag{6.2}$$

The above assumption can be justified *a posteriori* by the fact that the solutions for w_1 (cf. Sec. 7) are proportional to R'. Usually this latter quantity is small but finite at the exit and tends to zero for $z = L$ (cf. [4]).

On the free surface of the fibre all the forces acting have to be mutually balanced. Neglecting surface-tension effects, we arrive at the following condition (cf. [6]), earlier derived by KASE [8]:

$$R'\left(T^{*33} - T^{*11}\right)_{r=R} = T^{*13}|_{r=R} . \tag{6.3}$$

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Introducing the corresponding stresses from Eqs. (3.6), we obtain

$$R'\left[\sigma + \frac{\partial\sigma}{\partial V}w + \frac{\partial\sigma}{\partial V'}\frac{\partial w}{\partial z} + \frac{\partial\sigma}{\partial\varrho}\Delta\varrho\right]_{r=R} = \eta\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right)_{r=R} + \gamma u|_{r=R}\,. \tag{6.4}$$

Bearing in mind Eq. (5.5), we can write the following conditions:

$$R'\sigma = \eta\frac{\partial w_1}{\partial r}\bigg|_{r=R} \tag{6.5}$$

for the first order approximation, and

$$R'\left[\frac{\partial\sigma}{\partial V}w_1 + \frac{\partial\sigma}{\partial V'}\frac{\partial w_1}{\partial z} + \frac{\partial\sigma}{\partial\varrho}\Delta\varrho\right]_{r=R} = \eta\frac{\partial w_2}{\partial r}\bigg|_{r=R} + \gamma u_1|_{r=R}\,, \tag{6.6}$$

for the second order approximation, respectively.

7. Solutions for isotropic viscoelastic materials

The governing equation (5.6) together with the boundary conditions (6.2) and (6.5) leads to the solution

$$w_1 = \frac{\sigma}{2\eta}\frac{R'}{R}\left(r^2 - \frac{R^2}{2}\right), \tag{7.1}$$

depending on the fibre geometry ($R'/R = d/dz(\ln R)$) as well as on two material functions σ and η.

Integrating the expression for u, resulting from the continuity condition (4.2), we arrive at

$$u_1 = -\frac{V}{\varrho}\frac{d}{dz}\int\Delta\varrho\, r\, dr - \frac{1}{8}\frac{d}{dz}\left(\frac{\sigma}{\eta}\frac{R'}{R}(r^2 - R^2)r\right), \tag{7.2}$$

where we have taken into account Eq. (7.1) and the requirement that $u_1|_{r=0} = 0$. Moreover, we have

$$u_1|_{r=R} = -\frac{V}{\varrho}\frac{d}{dz}\int_0^R\Delta\varrho\, r\, dr = 0, \tag{7.3}$$

where Eq. (4.7) has been used.

The second order governing equation (5.7) together with the boundary conditions (6.2) and (6.6) leads to the following expression, more complex than that

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for the first approximation:

$$w_2 = \frac{1}{8}RR'\left\{\frac{\partial\sigma}{\partial V}\frac{\sigma}{\eta^2}\frac{R'}{R} + \frac{1}{\eta}\frac{\partial\sigma}{\partial V'}\left[\frac{d}{dz}\left(\frac{\sigma}{\eta}\frac{R'}{R}\right)\right] + 2\frac{\sigma}{\eta}\frac{R'^2}{R^2}\right\}\left(r^2 - \frac{R^2}{2}\right)$$
$$- \frac{1}{32\eta}\frac{d}{dz}\left\{\frac{\partial\sigma}{\partial V}\frac{\sigma}{\eta}\frac{R'}{R} + \frac{\partial\sigma}{\partial V'}\frac{d}{dz}\left(\frac{\sigma}{\eta}\frac{R'}{R}\right)\right\}\left(r^4 - 2R^2r^2 + 3R^4\right)$$
$$+ \frac{1}{2\eta}\frac{R'}{R}\frac{\partial\sigma}{\partial\varrho}\Delta\varrho\left(r^2 - \frac{R^2}{2}\right) + \frac{\gamma}{4\eta}\left(V\frac{\partial}{\partial z}(\ln\Delta\varrho) - \frac{\sigma}{2\eta}R'^2\right)\left(r^2 - \frac{R^2}{2}\right)$$
$$+ \frac{\gamma}{32\eta}\frac{d}{dz}\left(\frac{\sigma}{\eta}\frac{R'}{R}\right)\left(r^4 - 2R^2r^2 + \frac{2}{3}R^4\right). \tag{7.4}$$

The solutions (7.1) and (7.4) are valid for isotropic viscoelastic materials (fluids or solids) described by the constitutive equations of the type (2.10) and (3.6) with three material functions σ, η, γ and the variable temperature-dependent increment of the density $\Delta\varrho$. For purely inviscid or viscous materials, we may disregard in Eq. (7.4) all the terms containing partial derivatives with respect to V' or V, respectively.

For the frequently applied case of viscous, generalized Newtonian fluids, for which

$$\frac{\partial\sigma}{\partial V} = 0, \qquad \sigma = 3\eta V', \qquad \gamma = 0, \tag{7.5}$$

we arrive at

$$w_1 = \frac{3}{2}V'\frac{R'}{R}\left(r^2 - \frac{R^2}{R}\right), \tag{7.6}$$

and at

$$w_2 = \frac{9}{8}RR'\left\{\frac{d}{dz}\left(V'\frac{R'}{R}\right) + 2V'\frac{R'^2}{R^2}\right\}\left(r^2 - \frac{R^2}{2}\right)$$
$$- \frac{9}{32\eta}\frac{d}{dz}\left[\eta\frac{d}{dz}\left(V'\frac{R'}{R}\right)\right]\left(r^4 - 2R^2r^2 + 3R^4\right)$$
$$+ \frac{1}{2\eta}\frac{R'}{R}\frac{\partial\sigma}{\partial\varrho}\Delta\varrho\left(r^2 - \frac{R^2}{2}\right). \tag{7.7}$$

A realistic shape of the additional velocity profiles can easily be predicted, assuming that the outer radius $R(z)$ of the filament may be approximated by the exponential function:

$$R(z) = R_0\exp(-zb), \qquad b = -\frac{1}{L}\ln\frac{R_L}{R_0} = \text{const.}$$

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In such a case: $R'/R = -b$, and the velocity profiles (7.1) or (7.6) are proportional only to σ/η or V', respectively. It is well known from the experiments (cf. [4]) that V' takes small values for $z = 0$, increases rapidly reaching a maximum for $z = 0.15 \div 0.3L$, and tends to zero for $z = L$. It may be expected that possible σ/η profiles along the fibre axis are of the character similar to V'.

8. Final remarks

The linearized perturbation procedure developed in the paper enables determination of the realistic velocity fields taking into account the variable geometry of the elongated fibres as well as the appropriate shearing effects.

To this end some information on the material behaviour in steady quasi-elongational motions is necessary either on the basis of experimental data (measured radii, stresses, forces, etc.) or using various numerical results calculated for particular models of fluids or solids. The constitutive equations used in the paper are sufficiently general; the corresponding material functions (normal stresses, viscosity, etc.) all depend on the strain, strain rate (velocity gradient), variable density and explicitly on the axial coordinate. The latter dependence replaces distributions of temperature, crystallization effects, structure formation, etc. There exists a possibility for simultaneous description of fluid-like or solid-like behaviour along the same fibre-line.

The solutions corresponding to the first order approximation depend on two material functions only: the normal stress and viscosity functions, and the radius variable along the thread.

The additional velocity fields are simply expressed in the case of viscous, generalized Newtonian fluids. Then, a knowledge of such kinematic quantities as the variable radius and the velocity gradient is entirely sufficient.

An example of numerical and experimental results which could, in principle, be used in determining the additional velocity fields and the relevant shearing effects may be found in the paper by PAPANASTASIOU *et al.* [10]. They applied the so-called PSM model and the Newtonian model to calculate the properties of polypropylene, polystyrene and PET and to compare the results with available experiments.

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