

Asymptotic analysis of propagation of a signal with finite rise time in a dispersive, lossy medium

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PROPAGATION of an electromagnetic high frequency modulated signal with a finite rise time through a dispersive medium described by the Lorentz model is considered. Asymptotic approximations, based on uniform asymptotic methods, are found for the Sommerfeld and Brillouin precursors, and for the steady state contribution to the propagated field.

1. Introduction

THIS PAPER is concerned with the analysis of propagation of a plane electromagnetic wave in a linear dispersive medium with absorption. The medium occupying the half-space $z > 0$ is described by the Lorentz (single resonance) model, otherwise it is homogeneous and isotropic. The wave propagating in the z direction has a finite rise time on the medium interface $z = 0$. Fundamental works on an electromagnetic signal evolution as it propagates through a dispersive medium are due to SOMMERFELD [1] and BRILLOUIN [2, 3]. On the grounds of asymptotic considerations, the authors showed that the main change in the form of an electromagnetic signal propagating in a dispersive medium takes place at the initial stage of propagation, at higher penetration depths the pulse form being almost unchanged. They revealed that two different precursors contribute to the signal. The precursors took their names from the names of the aforementioned authors. Those early results are not, however, fully satisfactory. They were obtained with classical (non-uniform) asymptotics and as such, they break down at some space-time points in the field.

Recently, significant research in this area has been done by OUGHSTUN and SHERMAN, see [4–10], based on the use of modern (uniform) asymptotic techniques. In those works the classical results have been reexamined and enriched by removing the obstacles characteristic of non-uniform asymptotic methods, and by providing deeper insight into the dynamics of propagation of waves of various forms. The works by Oughstun and Sherman gave motivation for this paper which depends strongly on basic results obtained in those works.

In the analysis of a signal evolution in dispersive media, asymptotic techniques are particularly appealing for their ability to generate results readily interpreted in physical terms. It is worth mentioning, however, that alternative approach may here be used. It consists in the examination of interaction of various spectral components of the incident signal with the medium, and then summing up

the results. Such an approach was successfully used by BLASHAK and FRANZEN in [12]. The authors studied pulse propagation in dispersive media described by both the Lorentz and Debye models. By assuming oblique incidence of the incoming signal on the media interface they were able, among others, to determine propagation directions of both precursors.

In this paper we examine, using uniform asymptotic apparatus, the propagation of an electromagnetic signal with finite rise time in a dispersive lossy medium described by the Lorentz model. The signal is zero for $t < 0$ and is hyperbolic tangent modulated for $t \geq 0$. Here and throughout t stands for time. In [8] the hyperbolic tangent was used as the signal envelope for time ranging from minus to plus infinity, i.e. the signal was switched on at $t \rightarrow -\infty$. As a consequence, the wave studied here differs in form from that used in [8] and is more realistic as a model for possible applications.

The problem studied here is of much interest from both the applications and scientific point of view. The renewed interest in dispersion phenomena was recently stimulated by investigation concerning interaction of electromagnetic fields with organ tissue. Dispersion is also important in many instances of propagation of electromagnetic high-frequency fields through dielectric media, since all dielectrics are less or more dispersive. On the other hand, thorough investigation of the problem on asymptotic grounds requires application of modern asymptotic techniques, which can be employed to evaluate contour integrals with such special cases as coalescent saddle points, saddle points tending to infinity or interacting saddle points with poles in the integrands.

2. Plane wave description in the dispersive medium

Consider the problem of an electromagnetic plane wave propagation in a linear, homogeneous and isotropic medium whose dispersive properties are described by the Lorentz model of resonance polarization. The complex index of refraction in the medium is given by the following, frequency-dependent function

$$(2.1) \quad n(\omega) = \left(1 - \frac{b^2}{\omega^2 - \omega_0^2 + 2i\delta\omega} \right)^{1/2},$$

where $b^2 = 4\pi N e^2 / m$, N , e and m standing, respectively, for the number of electrons per unit volume, electron charge and its mass, δ is a damping constant and ω_0 is a characteristic frequency. It is assumed that the medium occupies the half-space $z \geq 0$ and that the wave propagates perpendicularly to the plane $z = 0$ in the direction of increasing z . Arbitrary component of the wave itself or of a corresponding Hertz vector can be represented in the medium by the scalar function

$$(2.2) \quad A(z, t) = \frac{1}{2\pi} \int_C \tilde{f}(\omega) \exp \left[\frac{z}{c} \phi(\omega, \theta) \right] d\omega.$$

Here, the complex phase function $\phi(\omega, \theta)$ is given by

$$(2.3) \quad \phi(\omega, \theta) = i \frac{c}{z} [\bar{k}(\omega)z - \omega t] = i\omega[n(\omega) - \theta],$$

where

$$(2.4) \quad \theta = \frac{ct}{z}$$

is a dimensionless parameter that characterizes a space-time point (z, t) in the field. The function $\bar{f}(\omega)$ is a temporal Fourier spectrum of the initial pulse $f(t) = A(0, t)$ at the plane $z = 0$. The contour C is the line $\omega = \omega' + ia$, a being a constant greater than the abscissa of absolute convergence ([13]) for $f(t)$ and ω' ranges from negative to positive infinity.

If the incident signal is a sine wave of fixed real frequency ω_c with its envelope described by a real function $u(t)$ that vanishes for $t < 0$, i.e.

$$(2.5) \quad f(t) = \begin{cases} 0 & t < 0, \\ u(t) \sin(\omega_c t) & t \geq 0, \end{cases}$$

then (2.2) can be represented in the alternative form

$$(2.6) \quad A(z, t) = \frac{1}{2\pi} \operatorname{Re} \left\{ i \int_{ia-\infty}^{ia+\infty} \tilde{u}(\omega - \omega_c) \exp \left[\frac{z}{c} \phi(\omega, \theta) \right] d\omega \right\},$$

where $\tilde{u}(\omega)$ is the Laplace transform of $u(t)$.

It can be proved that if $A(0, t)$ is zero for $t < 0$ and if the model of the material dispersion is casual, then the field $A(z, t)$ vanishes for all $\theta = ct/z < 1$, with $z \geq 0$. Therefore, with these conditions fulfilled one can restrict the study to the case $\theta \geq 1$.

In this paper we specify the envelope of the incident pulse to be a product of a unit step function and a hyperbolic tangent function, i.e.

$$(2.7) \quad u_\beta(t) = \begin{cases} 0 & t < 0, \\ \tanh \beta t & t \geq 0, \end{cases}$$

where the parameter $\beta \geq 0$ determines the rapidity of the pulse growth.

In order to find its Laplace transform we take advantage of ([16])

$$(2.8) \quad \int_0^\infty \frac{e^{-px}}{1 + e^{-qx}} dx = \frac{1}{q} B\left(\frac{p}{q}\right), \quad \operatorname{Re} p > 0, \quad q > 0,$$

where $\mathcal{B}(\cdot)$ is the beta function. The latter function is defined by the psi function as

$$(2.9) \quad \mathcal{B}(x) = \frac{1}{2} \left[\psi \left(\frac{x+1}{2} \right) - \psi \left(\frac{x}{2} \right) \right]$$

and alternatively can be expressed in terms of the series

$$(2.10) \quad \mathcal{B}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{x+k}.$$

It follows that the Laplace transform of $u(t)$ is

$$(2.11) \quad \tilde{u}_\beta(\omega) = \frac{1}{\beta} \mathcal{B} \left(-\frac{i\omega}{2\beta} \right) - \frac{i}{\omega}, \quad \text{Im } \omega > 0, \quad \beta > 0,$$

or, by (2.10),

$$(2.12) \quad \tilde{u}_\beta(\omega) = \frac{i}{\omega} - 2i \left(\frac{1}{\omega + 2i\beta} - \frac{1}{\omega + 4i\beta} + \dots \right).$$

One can see from this formula that in the limit as $\beta \rightarrow \infty$ the function tends to i/ω , which is the Laplace transform of the Heaviside unit step function, corresponding to the pulse with zero rise time.

With (2.11) used in (2.6) $A(z, t)$ specifies to

$$(2.13) \quad A(z, t) = \frac{1}{2\pi} \text{Re} \left\{ i \int_{ia-\infty}^{ia+\infty} \left[\frac{1}{\beta} \mathcal{B} \left(-\frac{i(\omega - \omega_c)}{2\beta} \right) - \frac{i}{\omega - \omega_c} \right] e^{\frac{z}{c}\phi(\omega, \theta)} d\omega \right\}.$$

This integral formula describes the dynamics of propagation of the initial signal with envelope given by (2.7), oscillating with angular frequency ω_c .

3. Asymptotic analysis

As seen from (2.13), construction of an asymptotic approximation to $A(z, t)$ in the mature dispersion regime, i.e. as $z \rightarrow \infty$, is closely related to asymptotic evaluation of the integral describing the field. In general, asymptotic behavior of an integral depends strongly on analytic properties of its integrand [14]. Therefore our first step is to establish all the critical points of the integral in (2.13) in the complex ω plane which contribute to the asymptotic expansion of $A(z, t)$. The critical points associated with the phase function $\phi(\omega, \theta)$ are the saddle points. The first derivative (and possibly higher derivatives) of the phase function with respect to the variable of integration vanishes at those points. In the present case

the phase $\phi(\omega, \theta)$ is an analytic function in the complex ω plane except along the branch cuts $\omega'_-\omega_-$ and $\omega_+\omega'_+$, where

$$(3.1) \quad \begin{aligned} \omega'_\pm &= \pm(\omega_1^2 - \delta^2)^{1/2} - i\delta, \\ \omega_\pm &= \pm(\omega_0^2 - \delta^2)^{1/2} - i\delta \end{aligned}$$

are the branch points of $n(\omega)$ and $\omega_1^2 = \omega_0^2 + b^2$. The requirement $\phi'_\omega(\omega, \theta) = 0$ leads to the equation

$$(3.2) \quad n(\omega) + \omega n'(\omega) - \theta = 0,$$

or

$$(3.3) \quad \left[\omega^2 - \omega_1^2 + 2id\omega + \frac{b^2\omega(\omega + i\delta)}{\omega^2 - \omega_0^2 + 2id\omega} \right]^2 = \theta^2(\omega^2 - \omega_1^2 + 2id\omega)(\omega^2 - \omega_0^2 + 2id\omega).$$

This equation determines exact locations of the relevant saddle-points. It does not seem possible to solve (3.3) exactly. However, from numeric investigation of $\phi(\omega, \theta)$ it follows that there are two kinds of saddle-points: the distant and the near saddle-points. Each kind contains at most two points. The distant saddle-points, to be denoted by SP_D^\pm , are located symmetrically about the imaginary axis in the lower ω half-plane. As θ varies from 1 to ∞ they move in the region $|\omega| > \omega_1$, and take the limiting values $\pm\infty - 2i\delta$ for $\theta = 1$ and ω'_\pm for $\theta \rightarrow \infty$. The near saddle-points, denoted by SP_N^\pm , vary in the region $\omega < |\omega_0|$. As θ increases from 1 to θ_1 , they approach each other along the imaginary ω -axis and meet at $\theta = \theta_1$ to produce one saddle-point of the second order. Next, as θ varies from θ_1 to ∞ , there are two first-order saddle points which detach from the imaginary axis and tend symmetrically about this axis to $\omega = \omega_\pm$ (see Fig. 1).

Equation (3.3) was being solved approximately to find analytic description of the location of the saddle-points. Apparently the best approximation obtained so far is due to OUGHSTUN and SHERMAN (see [4]). According to their results, the distant saddle-point locations are given by

$$(3.4) \quad \omega_{SP_D^\pm} \cong \pm\xi(\theta) - i\delta[1 + \eta(\theta)],$$

where

$$(3.5) \quad \begin{aligned} \xi(\theta) &= \left(\omega_0^2 - \delta^2 + \frac{b^2\theta^2}{\theta^2 - 1} \right)^{1/2}, \\ \eta(\theta) &= \frac{\delta^2/27 + b^2/(\theta^2 - 1)}{\xi^2(\theta)}. \end{aligned}$$

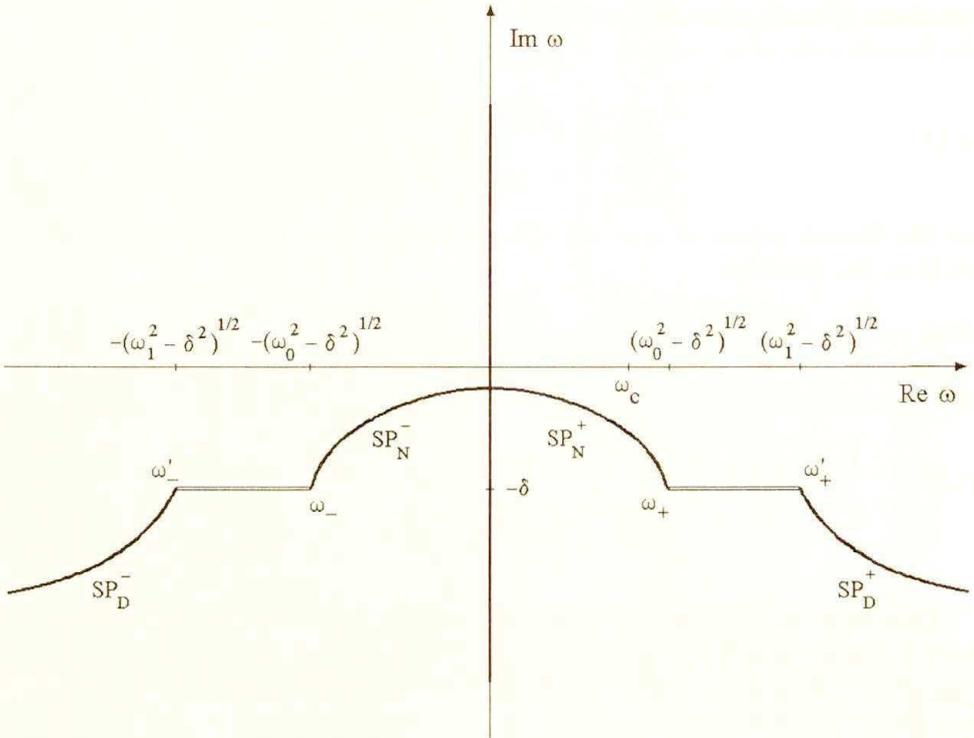


FIG. 1. Trajectories followed by the saddle-points SP_D^\pm and SP_N^\pm in the complex ω -plane as θ varies from 1 to ∞ .

The locations of near saddle-points are described by

$$(3.6) \quad \omega_{SP_N^\pm} \cong \begin{cases} i \left[\pm |\psi(\theta)| - \frac{2}{3} \delta \zeta(\theta) \right], & 1 \leq \theta < \theta_1, \\ -i \frac{2\delta}{3\alpha}, & \theta = \theta_1, \\ \pm \psi(\theta) - i \frac{2}{3} \delta \zeta(\theta), & \theta > \theta_1, \end{cases}$$

where

$$(3.7) \quad \begin{aligned} \psi(\theta) &= \left[\frac{\omega_0^2(\theta^2 - \theta_0^2)}{\theta^2 - \theta_0^2 + 3\alpha b^2/\omega_0^2} - \delta^2 \left(\frac{\theta^2 - \theta_0^2 + 2b^2/\omega_0^2}{\theta^2 - \theta_0^2 + 3\alpha b^2/\omega_0^2} \right)^2 \right]^{1/2}, \\ \zeta(\theta) &= \frac{3(\theta^2 - \theta_0^2 + 2b^2/\omega_0^2)}{2(\theta^2 - \theta_0^2 + 3\alpha b^2/\omega_0^2)}, \\ \alpha &= 1 - \frac{\delta^2}{3\omega_0^2\omega_1^2}(4\omega_1^2 + b^2). \end{aligned}$$

The special values of θ are

$$(3.8) \quad \begin{aligned} \theta_0 &= n(0) = (1 + b^2/\omega_0^2)^{1/2}, \\ \theta_1 &\cong \theta_0 + \frac{2\delta^2 b^2}{\theta_0 \omega_0^2 (3\alpha \omega_0^2 - 4\delta^2)}. \end{aligned}$$

As seen from (2.12), the amplitude factor under integral sign in (2.13) is a meromorphic function with infinite number of simple poles at

$$(3.9) \quad \omega = -i2k\beta + \omega_c, \quad k = 0, 1, 2, \dots$$

Adjacent poles are equally separated by the quantity $i2\beta$. If $\beta \rightarrow \infty$, only the pole at $\omega = \omega_c$ is of importance.

Having established the critical points of the integral in (2.6), one can set about the asymptotic evaluation of $A(z, t)$. The first step is to change the original contour of integration to a new one, to be denoted by $P(\theta)$, which is chosen such that it passes through the saddle points along a path consisting of paths of descent between adjacent saddle points (see Fig. 2 a, b). It was shown in [4] that such a change is possible (in the case of $1 \leq \theta < \theta_1$ the lower saddle point is not included because of the branch cut $\omega_+ \omega'_+$ that makes the contour deformation to the contour through that point forbidden). By using this procedure, together with the Cauchy theorem, it follows that $A(z, t)$ can be represented as

$$(3.10) \quad A(z, t) = I(z, \theta) - \text{Re} [2i\pi \Lambda(\theta)],$$

where

$$(3.11) \quad \Lambda(\theta) = \sum_p \text{Res}_{\omega=\omega_p} \left\{ \frac{i}{2\pi} \tilde{u}_\beta(\omega - \omega_c) e^{\frac{z}{c} \phi(\omega, \theta)} \right\}$$

is the sum of the residues at the poles that were intercepted in the course of contour deformation, and

$$(3.12) \quad I(z, \theta) = \frac{1}{2\pi} \text{Re} \left\{ i \int_{P(\theta)} \tilde{u}_\beta(\omega - \omega_c) \exp \left[\frac{z}{c} \phi(\omega, \theta) \right] d\omega \right\}.$$

The problem thus reduces to the asymptotic evaluation of the integral $I(z, \theta)$ as z tends to infinity.

Results obtained with classical asymptotic methods, often referred to as non-uniform, fail for some special configurations of the critical points in the complex ω -plane (comp. [5]). In the present context these configurations are: (i) the pair of the distant saddle points tend to infinity, (ii) the near simple saddle points coalesce into one saddle point of the second order, and, (iii) the contour $P(\theta)$ crosses a pole of $\tilde{u}_\beta(\omega - \omega_c)$ as θ evolves. The first and the second case occur when θ is close, respectively, to 1 and to $\theta \cong \theta_1$. In order to obtain asymptotic representation of $A(z, t)$ which remains valid for all $\theta \geq 1$ including all three cases, uniform asymptotic techniques will here be used.

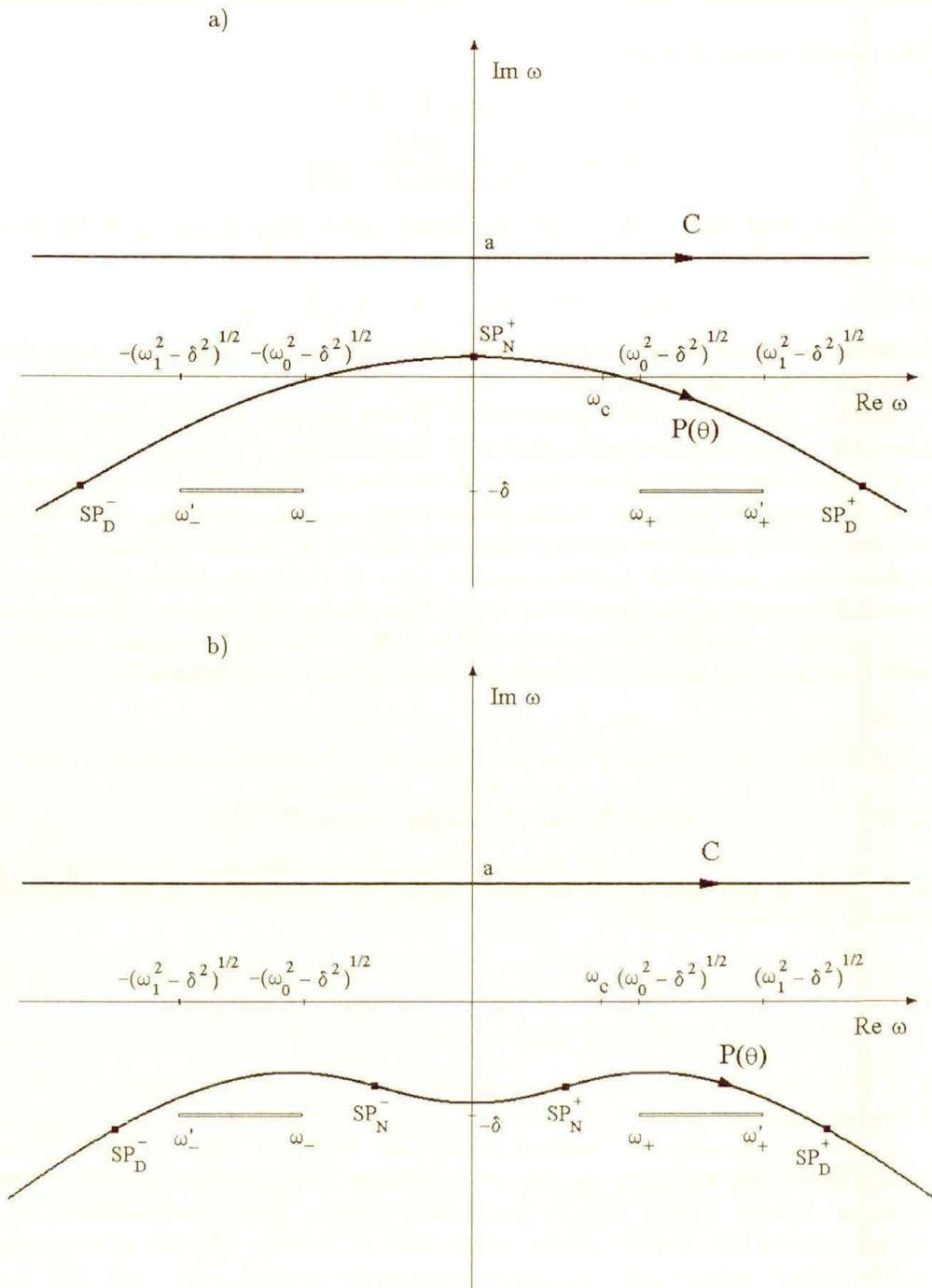


FIG. 2. a. The original contour of integration C and the deformed contour $P(\theta)$ in the case of $1 < \theta \leq \theta_1$. b. The original contour of integration C and the deformed contour $P(\theta)$ in the case of $\theta > \theta_1$.

3.1. Asymptotic representation of the Sommerfeld precursor

First, consider the contribution to the asymptotic expansion of $A(z, t)$ which is due to the pair of the distant saddle points SP_D^\pm . These points are dominant⁽¹⁾ over the near saddle points in the interval $1 \leq \theta < \theta_{SB} < \theta_1$, where θ_{SB} is given by (see [4])

$$(3.13) \quad \theta_{SB} \cong \theta_0 - \frac{4\delta^2 b^2}{3\theta_0 \omega_0^4} - \left[\frac{27\delta^2 b^2 (\theta_0 - 1)^2}{4\theta_0 \omega^4} \right]^{1/3} \\ \times \left(\left\{ \left[1 + \frac{\delta^2 b^2}{27\theta_0 (\theta_0 - 1) \omega_0^4} \right]^{1/2} + 1 \right\}^{1/3} \right. \\ \left. - \left\{ \left[1 + \frac{\delta^2 b^2}{27\theta_0 (\theta_0 - 1) \omega_0^4} \right]^{1/2} - 1 \right\}^{1/3} \right).$$

For θ close to 1 the distant saddle points tend symmetrically about the imaginary axis to $\pm\infty$ and transform in the limit as $\theta = 1$ into a saddle point of infinite order. Classical asymptotic methods fail to describe such a situation; instead, a uniform asymptotic approach is here required. Procedure appropriate for this case was proposed by BLEISTEIN and HANDELSMAN [14]. It was adapted by OUGHSTUN and SHERMAN to integrals of the form of (3.12) to yield [5]

$$(3.14) \quad A_S(z, t) \sim \frac{\xi(\theta)}{2b} \left\{ \theta - 1 + \frac{b^2/2}{\xi^2(\theta) + \delta^2[1 - \eta(\theta)]^2} \right\}^{1/2} \\ \times \exp \left(-\delta \frac{z}{c} \left\{ [1 + \eta(\theta)](\theta - 1) + \frac{(1/2)b^2[1 - \eta(\theta)]}{\xi^2(\theta) + \delta^2[1 - \eta(\theta)]^2} \right\} \right) \\ \times \text{Re} \left\{ \exp \left(-i \frac{\pi}{2} \nu \right) \left[(\tilde{u}(\omega_{SP_D^+} - \omega_c) \{ \xi(\theta) + (3/2)\delta i [1 - \eta(\theta)] \} \right. \right. \\ \left. \left. + (-1)^{1+\nu} \tilde{u}(\omega_{SP_D^-} - \omega_c) \{ \xi(\theta) - (3/2)\delta i [1 - \eta(\theta)] \} \right) \right. \\ \left. \times J_\nu \left(\frac{z}{c} \xi(\theta) \left\{ \theta - 1 + \frac{b^2/2}{\xi^2(\theta) + \delta^2[1 - \eta(\theta)]^2} \right\} \right) \right\} \\ + \exp \left(-i \frac{\pi}{2} \right) \left(\tilde{u}(\omega_{SP_D^+} - \omega_c) \{ \xi(\theta) + (3/2)\delta i [1 - \eta(\theta)] \} \right. \\ \left. - (-1)^{1+\nu} \tilde{u}(\omega_{SP_D^-} - \omega_c) \{ \xi(\theta) - (3/2)\delta i [1 - \eta(\theta)] \} \right) \\ \left. \times J_{\nu+1} \left(\frac{z}{c} \xi(\theta) \left\{ \theta - 1 + \frac{b^2/2}{\xi^2(\theta) + \delta^2[1 - \eta(\theta)]^2} \right\} \right) \right\}$$

⁽¹⁾ A critical point is dominant over other critical points if $\text{Re}[\phi(\omega, \theta)]$ at this point attains its maximum value, thus making the point least attenuated.

as $z \rightarrow \infty$, where J_ν is a Bessel function of the first kind. The parameter ν determines the behavior of the amplitude function at infinity (\tilde{u} behaves like $\omega^{-(1+\nu)}$ as $|\omega| \rightarrow \infty$).

In the Bleistein and Handelsman method it is assumed that the amplitude factor in the integrand has a convergent Laurent series expansion in some neighborhood of infinity. In the case considered here this condition is not satisfied as the function $\tilde{u}(\omega - \omega_c)$ has poles along the line $\omega = -i2k\beta + \omega_c$. Those poles, however, do not affect the procedure of asymptotic expansion construction. It is so because the amplitude function is regular in the region which is the intersection of the region $|\omega| > R$ for some positive R , and a domain where all deformed integration contours appear. Apart from the line where $\tilde{u}(\omega - \omega_c)$ has pole singularities, this function behaves like

$$(3.15) \quad \tilde{u}(\omega - \omega_c) \sim -\frac{\beta}{\omega^2} + O(\omega^{-3}),$$

so that $\nu = 1$. As a result, the asymptotic expansion of $A_S(z, t)$, as $z \rightarrow \infty$, for the signal envelope given by (2.7), becomes

$$(3.16) \quad A_S(z, t) \sim \frac{\xi(\theta)}{2b} \left\{ \theta - 1 + \frac{b^2/2}{\xi^2(\theta) + \delta^2[1 - \eta(\theta)]^2} \right\}^{1/2} \\ \times \exp \left(-\delta \frac{z}{c} \left\{ [1 + \eta(\theta)](\theta - 1) + \frac{(1/2)b^2[1 - \eta(\theta)]}{\xi^2(\theta) + \delta^2[1 - \eta(\theta)]^2} \right\} \right) \\ \times \text{Im} \left[(\tilde{u}_\beta(\omega_{SP_D^+} - \omega_c) \{ \xi(\theta) + (3/2)\delta i [1 - \eta(\theta)] \} \right. \\ \left. + \tilde{u}_\beta(\omega_{SP_D^-} - \omega_c) \{ \xi(\theta) - (3/2)\delta i [1 - \eta(\theta)] \} \right) \\ \times J_1 \left(\frac{z}{c} \xi(\theta) \left\{ \theta - 1 + \frac{b^2/2}{\xi^2(\theta) + \delta^2[1 - \eta(\theta)]^2} \right\} \right) \\ - i \left(\tilde{u}_\beta(\omega_{SP_D^+} - \omega_c) \{ \xi(\theta) + (3/2)\delta i [1 - \eta(\theta)] \} \right. \\ \left. - \tilde{u}_\beta(\omega_{SP_D^-} - \omega_c) \{ \xi(\theta) - (3/2)\delta i [1 - \eta(\theta)] \} \right) \\ \times J_2 \left(\frac{z}{c} \xi(\theta) \left\{ \theta - 1 + \frac{b^2/2}{\xi^2(\theta) + \delta^2[1 - \eta(\theta)]^2} \right\} \right) \Big],$$

where

$$(3.17) \quad \tilde{u}_\beta(\omega_{SP_D^\pm} - \omega_c) \\ \cong \frac{1}{\beta} \mathcal{B} \left(\frac{\pm \xi(\theta) - \omega_c - \delta i [1 + \eta(\theta)]}{2i\beta} \right) - \frac{i}{\pm \xi(\theta) - \omega_c - \delta i [1 + \eta(\theta)]}.$$

This expansion is uniform with respect to $\theta \geq 1$. It represents the Sommerfeld precursor, for it is related to the pair of distant saddle points which are dominant for small θ , and hence for small t .

3.2. Asymptotic representation of the Brillouin precursor

We now consider the contribution to the asymptotic expansion of $A(z, t)$ due to the near saddle points SP_N^\pm . Their contribution is descriptive of the Brillouin precursor and dominates over the Sommerfeld precursor as $\theta > \theta_{SB}$. If θ approaches θ_1 then the near saddle points coalesce and produce one saddle point of the second order. Since classical asymptotic methods fail to describe the field $A(z, t)$ in this case, a uniform approach should then be used. Such an approach was first proposed by CHESTER, FRIEDMAN and URSELL [17]. It is also derivable by using BLEISTEIN and HANDELSMAN theory [14], and was adapted by OUGHSTUN and SHERMAN [5] to integrals of the form of (3.12). Here, we employ the latter result.

Since there are two different descriptions of the locations of the near saddle points, depending on whether $1 \leq \theta < \theta_1$ or $\theta > \theta_1$, (comp. (3.6)), the asymptotic procedure is to be carried out for each of these cases separately. First, consider the case $1 \leq \theta < \theta_1$. Using the Oughstun and Sherman result one obtains

$$(3.18) \quad A_B(z, t) \sim \exp \left[\frac{z}{c} a_0(\theta) \right] \left(\frac{1}{2} \left(\frac{c}{z} \right)^{1/3} \operatorname{Re} \{ i [\tilde{u}_\beta(\omega_{SP_1} - \omega_c) |h_1(\theta)| \right. \right. \\ \left. \left. + \tilde{u}_\beta(\omega_{SP_2} - \omega_c) |h_2(\theta)|] \right\} \operatorname{Ai} \left[|\alpha_1(\theta)| \left(\frac{z}{c} \right)^{2/3} \right] \right. \\ \left. - \frac{1}{2|\alpha_1(\theta)|^{1/2}} \left(\frac{c}{z} \right)^{2/3} \operatorname{Re} \{ i [\tilde{u}_\beta(\omega_{SP_1} - \omega_c) |h_1(\theta)| \right. \right. \\ \left. \left. - \tilde{u}_\beta(\omega_{SP_2} - \omega_c) |h_2(\theta)|] \right\} \operatorname{Ai}^{(1)} \left[|\alpha_1(\theta)| \left(\frac{z}{c} \right)^{2/3} \right] \right)$$

as $z \rightarrow \infty$, where

$$(3.19) \quad \alpha_0(\theta) \cong -\delta \left(\frac{2}{3} \zeta(\theta)(\theta - \theta_0) + \frac{b^2}{\theta_0 \omega_0^4} \left\{ |\psi(\theta)|^2 [\alpha \zeta(\theta) - 1] \right. \right. \\ \left. \left. - \frac{4}{9} \delta^2 \zeta^2(\theta) \left[\frac{1}{3} \alpha \zeta(\theta) - 1 \right] \right\} \right), \\ \alpha_1 \cong |\psi(\theta)| \left(\frac{3}{2} \left\{ \theta - \theta_0 + \frac{b^2}{\theta_0 \omega_0^4} \right. \right. \\ \left. \left. \times \left[\frac{3}{4} \alpha |\psi(\theta)|^2 + \alpha \delta^2 \zeta^2(\theta) - 2\delta^2 \zeta(\theta) \right] \right\} \right)^{2/3}, \\ h_{1,2}(\theta) \cong \left(\left\{ \frac{2\theta_0 \omega^4}{3\alpha b^2 |\psi(\theta)| \pm 2\delta [1 - \alpha \zeta(\theta)]} \right\}^3 |\psi(\theta)| \left\{ \frac{3}{2} (\theta - \theta_0) \right. \right. \\ \left. \left. + \frac{b^2}{\theta_0 \omega_0^4} \left[\frac{3}{4} \alpha |\psi(\theta)|^2 + \alpha \delta^2 \zeta^2(\theta) - 2\delta^2 \zeta(\theta) \right] \right\} \right)^{1/6}$$

and Ai is the Airy function. The plus sign in $(3.19)_3$ corresponds to the index 1 and the minus sign corresponds to the index 2. The functions $\tilde{u}_\beta(\omega_{SP_N^\pm} - \omega_c)$ are given by

$$(3.20) \quad \tilde{u}_\beta(\omega_{SP_{1,2}} - \omega_c) \cong \frac{1}{\beta} \mathcal{B} \left(\frac{\pm |\psi(\theta)| - \frac{2}{3} \delta \zeta(\theta) + i\omega_c}{2\beta} \right) - \frac{1}{\pm |\psi(\theta)| - \frac{2}{3} \delta \zeta(\theta) + i\omega_c}.$$

Since the argument of the Airy function and its derivative is real and nonnegative for $\theta \leq \theta_1$, the Brillouin precursor is described by nonoscillating function in this domain.

In the case of $\theta > \theta_1$ the asymptotic description of the Brillouin precursor takes on the form

$$(3.21) \quad A_B(z, t) \sim \exp \left[\frac{z}{c} a_0(\theta) \right] \left(\frac{1}{2} \left(\frac{c}{z} \right)^{1/3} \text{Re} \{ i [\tilde{u}_\beta(\omega_{SP_N^+} - \omega_c) |h^+(\theta)| + \tilde{u}_\beta(\omega_{SP_N^-} - \omega_c) |h^-(\theta)|] \} \text{Ai} \left[-|\alpha_1(\theta)| \left(\frac{z}{c} \right)^{2/3} \right] + \frac{1}{2|\alpha_1(\theta)|^{1/2}} \left(\frac{c}{z} \right)^{2/3} \text{Re} [\tilde{u}_\beta(\omega_{SP_N^+} - \omega_c) |h^+(\theta)| - \tilde{u}_\beta(\omega_{SP_N^-} - \omega_c) |h^-(\theta)|] \text{Ai}^{(1)} \left[-|\alpha_1(\theta)| \left(\frac{z}{c} \right)^{2/3} \right] \right)$$

as $z \rightarrow \infty$, where

$$(3.22) \quad \alpha_0(\theta) \cong -\delta \left(\frac{2}{3} \zeta(\theta)(\theta_0) + \frac{b^2}{\theta_0 \omega_0^4} \left\{ [1 - \alpha \zeta(\theta)] \psi^2(\theta) + \frac{4}{9} \delta^2 \zeta^2(\theta) \left[\frac{1}{3} \alpha \zeta(\theta) - 1 \right] \right\} \right),$$

$$\alpha_1^{1/2} \cong \left[-\frac{2}{3} i \psi(\theta) \left(\theta - \theta_0 - \frac{b^2}{2\theta_0 \omega_0^4} \times \left\{ \frac{4}{3} \delta^2 \zeta(\theta) [2 - \alpha \zeta(\theta)] + \alpha \psi^2(\theta) \right\} \right) \right]^{1/3},$$

$$h^\pm(\theta) \cong \left[\left[i \frac{2\theta_0 \omega_0^4}{3\alpha b^2 \psi(\theta)} \right]^3 \left[-\frac{3}{2} i \psi(\theta) \right] \left(\theta - \theta_0 - \frac{b^2}{2\theta_0 \omega_0^4} \times \left\{ \frac{4}{3} \delta^2 \zeta(\theta) [2 - \alpha \zeta(\theta)] + \alpha \psi^2(\theta) \right\} \right) \right]^{1/6}.$$

Here,

$$(3.23) \quad \bar{u}_\beta(\omega_{SP_N^\pm} - \omega_c) \cong \frac{1}{\beta} \mathcal{B} \left(\frac{\mp i\psi(\theta) - \frac{2}{3}\delta\zeta(\theta) + i\omega_c}{2\beta} \right) - \frac{1}{\mp i\psi(\theta) - \frac{2}{3}\delta\zeta(\theta) + i\omega_c}.$$

Since the argument of the Airy function and its derivative is real and nonpositive for $\theta \geq \theta_1$, the Brillouin precursor is oscillating in this domain.

It can be shown that the formulas (3.18) and (3.21) represent a continuous function of θ . Moreover, these formulas provide a smooth transition in algebraic order of z , as the argument of the Airy function and its derivative tends to zero. Indeed, the algebraic order of $z^{-1/3} \text{Ai}[-|\alpha_1(\theta)|(z/c)^{2/3}]$ and $z^{-2/3} \text{Ai}^{(1)}[-|\alpha_1(\theta)|(z/c)^{2/3}]$ is $z^{-1/2}$, while the order of both $\text{Ai}(0)$ and $\text{Ai}^{(1)}(0)$ is $0(1)$. Hence, the resultant field is of the order of $z^{-1/2}$ when the near saddle points are separated, and of the order of $z^{-1/3}$ if they coalesce into one saddle point of the second order. This agrees with known results obtainable with non-uniform asymptotic approach.

The Brillouin precursor is insignificant for θ close to 1, but becomes of importance at $\theta > \theta_{SB}$, when it begins to dominate over the Sommerfeld precursor. In particular, at $\theta = \theta_0$ it suffers no exponential attenuation.

3.3. Interaction of pole singularity with the saddle point

As θ increases from 1 to ∞ , singular points associated with the spectral function $\bar{u}_\beta(\omega_{SP_N^\pm} - \omega_c)$ are intercepted while the contour $P(\theta)$ evolves and their contribution is represented by the function $A(\theta)$, as defined by (3.11). This contribution introduces a clearly discontinuous term on the rhs of (3.10), while the lhs is a continuous function of θ . Thus the problem at hand is to find asymptotic evaluation of $I(z, \theta)$ such, that the rhs of (3.10) is also continuous and equal asymptotically to $A(z, t)$.

A suitable tool, appropriate for this task, is that proposed by Bleistein and Handelsman. Their method allows for asymptotic evaluation of a contour integral with simple saddle point coalescing on an algebraic singularity of the integrand, [14]. In case the singularity is a simple pole, their procedure is equivalent to the VAN DER WAERDEN method, [18]. General results for this case have been adopted to integrals considered here by OUGHSTUN and SHERMAN [5] and will be employed in this paper.

Here, it is assumed that β is large enough so that only one pole, equal to $\omega = \omega_c$, is crossed by the contour $P(\theta)$. Additionally, the carrier frequency ω_c is assumed to lie above the dielectric absorption band, i.e. $\omega_c > (\omega_1^2 - \delta^2)^{1/2}$, but otherwise is finite. Under these assumptions the pole at $\omega = \omega_c$ interacts with the distant saddle point SP_D^+ . According to the results obtained in [5],

the asymptotic approximation to $A(z, t)$ depends on the value of $\Delta(\theta)$, which is defined as

$$(3.24) \quad \Delta(\theta) = \left[\phi(\omega_{SP_D^+}, \theta) - \phi(\omega_c, \theta) \right]^{1/2}.$$

As θ increases from 1 to ∞ , the saddle point SP_D^+ moves leftwards in the complex θ plane, and the path $P(\theta)$ through that point crosses the pole at $\theta = \theta_s$. With the help of the Bleistein and Handelsman method and its Oughstun and Sherman adaptation, the following asymptotic contribution to $A(z, t)$ is obtained. If $1 \leq \theta < \theta_s$, then the distance between the origin and the intersection of $P(\theta)$ with the real ω axis is larger than the distance between the origin and the pole, $\text{Im}[\Delta(\theta)] > 0$, and

$$(3.25) \quad A_c(z, t) \sim \frac{1}{2\pi} \left\{ -i\pi \operatorname{erfc} \left[-i\Delta(\theta) \left(\frac{z}{c} \right)^{1/2} \right] \exp \left[\frac{z}{c} \phi(\omega_p, \theta) \right] + \frac{1}{\Delta(\theta)} \left(\frac{\pi c}{z} \right)^{1/2} \exp \left[\frac{z}{c} \phi(\omega_{SP_D^+}, \theta) \right] \right\}$$

as $z \rightarrow \infty$. If $\theta = \theta_s$, i.e. the path $P(\theta)$ crosses the simple pole singularity at $\omega = \omega_c$, then $\text{Im}[\Delta(\theta)] = 0$, $\Delta(\theta) \neq 0$, and

$$(3.26) \quad A_c(z, t) \sim \frac{1}{2\pi} \left\{ -i\pi \operatorname{erfc} \left[-i\Delta(\theta) \left(\frac{z}{c} \right)^{1/2} \right] \exp \left[\frac{z}{c} \phi(\omega_p, \theta_s) \right] + \frac{1}{\Delta(\theta_s)} \left(\frac{\pi c}{z} \right)^{1/2} \exp \left[\frac{z}{c} \phi(\omega_{SP_D^+}, \theta_s) \right] \right\} + \operatorname{Re} \left\{ i \exp \left[\frac{z}{c} \phi(\omega_p, \theta_s) \right] \right\}$$

as $z \rightarrow \infty$. In the remaining case, i.e. when $\theta > \theta_s$, or equivalently, when the distance between the origin and the intersection of $P(\theta)$ with the real ω axis is smaller than the distance between the origin and the pole, one has $\text{Im}[\Delta(\theta)] < 0$, and

$$(3.27) \quad A_c(z, t) \sim \frac{1}{2\pi} \left\{ i\pi \operatorname{erfc} \left[i\Delta(\theta) \left(\frac{z}{c} \right)^{1/2} \right] \exp \left[\frac{z}{c} \phi(\omega_p, \theta) \right] + \frac{1}{\Delta(\theta)} \left(\frac{\pi c}{z} \right)^{1/2} \exp \left[\frac{z}{c} \phi(\omega_{SP_D^+}, \theta) \right] \right\} + \operatorname{Re} \left\{ i \exp \left[\frac{z}{c} \phi(\omega_p, \theta) \right] \right\}$$

as $z \rightarrow \infty$.

Here,

$$(3.28) \quad \operatorname{erfc}(\xi) = \frac{2}{\pi^{1/2}} \int_{\xi}^{\infty} \exp(-y^2) dy.$$

In (3.25) through (3.27) we have taken advantage of

$$(3.29) \quad \lim_{\omega \rightarrow \omega_c} [(\omega - \omega_c)\tilde{u}(\omega - \omega_c)] = i,$$

and of (3.12). The asymptotic expansion of $A_c(z, t)$, as given by (3.25)–(3.27), is a continuous function of θ , and hence yields a uniform asymptotic contribution to $A(z, t)$. As noted in [5], if the absolute value of the argument in erfc function is large enough, then this function can be replaced in (3.25)–(3.27) by its asymptotic representation, thus leading to the non-uniform asymptotic approximation to $A_c(z, t)$. It then follows that for the pole and the distant saddle point bounded away and z large, (3.25) introduces asymptotically no modification to the field; it is important only in the case of moderate values of the erfc argument. On the other hand, if the absolute value of this argument in (3.27) is large, $A_c(z, t)$ contribution to the field is, as expected, due to the residue of $\tilde{u}(\omega - \omega_c)$ at $\omega = \omega_c$. Note, that $A_c(z, t)$ is independent of β , and is the same as in the case of unit step envelope function [5].

To collect the results of the previous sections we note that contributions stemming from various critical points of an integral appear in the asymptotic expansion of the integral in the form of uncoupled components (comp. [14, 5, 19]). Accordingly, the asymptotic approximation to $A(z, t)$ is the sum consisting of the Sommerfeld and the Brillouin precursors, and the steady state contribution due to the pole singularity, i.e.

$$(3.30) \quad A(z, t) \sim A_S(z, t) + A_B(z, t) + A_c(z, t)$$

as $z \rightarrow \infty$.

4. Conclusion

The propagation of an electromagnetic signal in a dispersive medium described by the Lorentz model has been considered. The initial signal was chosen to be a sine wave of high real frequency modulated with the envelope described by the product of hyperbolic tangent and unit step function. A uniform asymptotic expansion of the propagating pulse in the medium in the mature regime was obtained with the help of modern asymptotic techniques.

Although the asymptotic representation for the field $A(z, t)$ was obtained under the restriction that the carrier frequency lies above the medium absorption band, a similar reasoning can be applied if this frequency occurs below that band.

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