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Symmetrization of systems of conservation equations and the converse to the condition of Friedrichs and Lax

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THE RESULT OF FRIEDRICHS AND LAX [Proc. Nat. Acad. Sci. U.S.A., **68**, 8, 1686–1688, 1971] concluding that if the system of conservation equations implies the additional conservation equation (balance of entropy) then it can be symmetrized by premultiplication by the Hessian matrix of the entropy, is well known. Basic ingredients of the proof of the converse to this result can be found in the paper by BOILLAT [C.R. Acad. Sci. Paris, **278 A**, 909–914 1974], however this converse has not been explicitly formulated there and, as a consequence, it seems to be overlooked. Therefore, an explicit formulation and the detailed proof of the converse to the condition of Friedrichs and Lax is given in this paper. Due to this result, the restrictions imposed on the system of conservation equations by consistency with the additional conservation equations can be alternatively derived from requirement that it admits Hessian matrix as a symmetrizer while the corresponding entropies can be determined by direct integration of the admissible Hessian symmetrizers. As an illustrative example, the system of conservation equations given in [DOMAŃSKI, JABŁOŃSKI and KOSIŃSKI, Arch. Mech., **48**, 541–550, 1996] is analysed. It is shown that this system can be brought into equivalent symmetric hyperbolic form without appealing to the existence of the additional conservation equation and the whole family of symmetric symmetrizers is determined. Then, the condition that the system admits the additional conservation equation reduces to the requirement that the family of symmetric symmetrizers contains at least one Hessian matrix. This requirement is, in turn, equivalent to the integrability condition for the overdetermined system of second order partial differential equations for the scalar entropy function. Finally, the family of entropies is obtained as a result of integration of this system.

1. Introduction

IN [1], FRIEDRICHS AND LAX have shown that if the system of N conservation equations implies the additional conservation equation (usually called balance of "entropy"), then premultiplication (left multiplication) of this system by Hessian matrix of "entropy" (matrix of second-order partial derivatives) makes it symmetric.

In the paper [2] by DOMAŃSKI, JABŁOŃSKI and KOSIŃSKI, this result has been explicitly quoted as a means for symmetrization of considered particular system of conservation equations, but the employed procedure of symmetrization and interpretation of the results should be rather referred to the *converse* to the result of FRIEDRICHS and LAX [1]. Clearly, in [2], it has been required from the system of conservation equations to be symmetrizable (by premultiplication) by prescribed Hessian matrix. As a result of this requirement, the algebraic relation between the entries of the prescribed Hessian matrix and the entries of the matrices corresponding to the considered system of conservation equations (written in a matrix form) has been obtained. This relation has been called "symmetrizability condition" for the considered system of conservation equations. Since the system of conservation equations treated in [2] implies the additional conservation (balance of entropy) and prescribed Hessian matrix corresponds precisely to one of the entropies admitted by this system, the "symmetrizability condition", of course, corresponds to "the model compatibility condition which, on the other hand, can be obtained from the second law of thermodynamics", as it has been concluded in [2].

Apparently, the procedure performed in [2] in nothing else but checking that the *converse* to the well known result of FRIEDRICHS and LAX [1] is true for the particular system of conservation equations. Unfortunately, this aspect of the performed calculations has not been noticed in [2].

The converse to the result of FRIEDRICHS and LAX [1], of course, holds for the general case of the system of conservation equations and its proof can be easily deduced from the paper by BOILLAT [3]. Since this result has not been explicitly formulated in [3] as a separate "theorem", contrary to the condition of Friedrichs and Lax, it seems to be overlooked (for example, [2, 6]).

The objective of this note is to formulate explicitly this converse and to demonstrate its complete detailed proof, mostly for pedagogical reasons, as well as to show how this result can be directly applied in practice for derivation of the condition that the system of conservation equations implies the additional conservation equation, and then to determine the "entropies" that can be assigned to this system. It should be emphasised that the crucial points of the reasoning employed in the proof presented here have been found in [3] and, therefore, the converse to the condition of Friedrichs and Lax should be attributed to Boillat. The original method proposed here, consisting in application of this converse for derivation of the restrictions on systems of conservation equations imposed by consistency with the additional conservation equation (balance of "entropy"), can be considered as an alternative to the method of Lagrange-Liu multipliers [5] developed in the framework of extended thermodynamics. In this alternative approach, the restrictions on the system of conservation equations as well as the "entropies" are obtained directly, without use of the auxiliary fields of Lagrange-Liu multipliers.

In Sec. 2, we demonstrate that *if the system of N conservation equations has a symmetrizer which is the Hessian matrix of a certain function of the unknowns then this system of conservation equations implies the additional conservation equation, in which this function of the unknowns is a "density"*. The result of FRIEDRICHS and LAX [1] together with the converse leads to the following necessary and sufficient condition for the system of N conservation equations to be symmetrizable (by premultiplication) by a Hessian matrix: *The system of N con-*

servation equations is symmetrizable (by premultiplication) by a Hessian matrix iff it implies the additional conservation equation.

Therefore, the conditions imposed on the system of N conservation equations by the requirement that it implies the additional conservation equation (for example, thermodynamic restrictions implied by consistency with the balance of entropy) can be obtained by requiring that the system admits symmetric symmetrizer which is a Hessian matrix. The respective procedure can be accomplished in the following five steps: 1) to rewrite the system of conservation equations in a matrix form, 2) to derive the condition on the entries of the respective matrices (necessary and sufficient), that those matrices have common left symmetric symmetrizers, 3) to determine the family of common left symmetric symmetrizers (the entries of such family of matrices are related to the entries of the matrices corresponding to the considered system of conservation equations), 4) to derive the condition on this family of symmetrizers that it contains at least one Hessian matrix (this condition corresponds to the condition that the system of $(1/2)N(N+1)$ second-order partial differential equations for entropy function is integrable), 5) finally, to calculate the entropies admitted by the considered system of $(1/2)N(N+1)$ equations for entropy.

In Sec. 3, the example of application of this complex procedure is presented. In this example, we use the system of conservation equations considered in [2] because of its particular simplicity. Since this system is consistent with the additional conservation equation (balance of entropy) without any restrictions on the functions of dependent variables involved in it, the respective matrices admit common left symmetric symmetrizers without any additional relations between their entries. The family of the respective symmetrizers is derived and the step 2) of the above procedure is not needed in this case. To this end, we note that the equivalence result (existence of entropy and symmetrizability by Hessian matrices) together with the described procedure of application enables one to employ the methods of matrix analysis in studying the properties of systems of conservation equations endowed with entropies.

2. Converse to the condition of Friedrichs and Lax

We consider a quasilinear system of N conservation equations for N unknowns in normal (Cauchy) form

$$\partial_t \mathbf{u} + \partial_i \mathbf{f}^i(\mathbf{u}) = \mathbf{b}(\mathbf{u}), \qquad i = 1,\ 2, ..., m \tag{2.1}$$

with the corresponding matrix form

$$\partial_t \mathbf{u} + \mathbf{A}^i(\mathbf{u}) \partial_i \mathbf{u} = \mathbf{b}(\mathbf{u}), \tag{2.2}$$

where

$$
\begin{aligned}
\mathbf{u}^T &= \left[u^1(t,x^i),...,u^N(t,x^i)\right],\\
\mathbf{f}^{iT}(\mathbf{u}) &= \left[f^{i1}(u^K(t,x^i)),...,f^{iN}(u^K(t,x^i)\right],\\
\mathbf{b}^T(\mathbf{u}) &= \left[b^1(u^K(t,x^i)),...,b^N(u^K(t,x^i)\right],\\
\mathbf{A}^i(\mathbf{u}) &= \nabla_{\mathbf{u}}\mathbf{f}^i(\mathbf{u}) = \left[A_L^{iK}(u^M)\right] = \left[\frac{\partial f^{iK}(u^M)}{\partial u^L}\right],\\
i &= 1,\ 2,...,n, \qquad K,L,M = 1,\ 2,...,N,
\end{aligned}
\tag{2.3}
$$

and

$$
\partial_t := \frac{\partial}{\partial_t}, \qquad \partial_i := \frac{\partial}{\partial x^i}.
$$

The usual summation convention over repeated upper and lower indices is understood and $(\cdot)^T$ denotes a transpose.

For the system (2.1), (2.2), (2.3), we consider the following additional conservation equation

$$
\partial_t h^0\mathbf{u} + \partial_i h^i\mathbf{u} = \sigma(\mathbf{u}). \tag{2.4}
$$

For the clarity and completeness, we recall the well-known results of FRIEDRICHS and LAX [1]. In [1], FRIEDRICHS and LAX formulated the *statement* which can be expressed in the following way:

The conservation equation (2.4) *"follows from" (is implied by or is a consequence of, in other words) the system of N conservation equations* (2.1) *if and only if*

$$
\frac{\partial h^0(u^M)}{\partial u^J}\frac{\partial f^{iJ}(u^M)}{\partial u^R} = \frac{\partial h^i(u^M)}{\partial u^R}, \qquad i = 1,2,...,m, \tag{2.5}
$$

$$
\frac{\partial h^0(u^M)}{\partial u^J} b^J(u^M) = \sigma(u^M), \qquad J,M,R = 1,2,...,n. \tag{2.6}
$$

The term "follows from" (is implied by or is a consequence of, in other words), used in this statement, is understood in a sense that there are N functions $l_J(u^M)$, not all identically zero, such that conservation equation (2.4) is a combination of N equations of (2.1) multiplied by respective $l_J(u^M)$; namely, the equality

$$
\partial_t h^0(u^M) + \partial_i h^i(u^M) - \sigma(u^M) = l_J(u^M)\left[\partial_t u^J + \partial_i f^{iJ}(u^M) - b^J(u^M)\right], \tag{2.7}
$$

holds for all functions $u^M(t,x^i)$ (in the domain of such functions). With this interpretation, the following "proof" justifies this *statement*.

Assume that (2.4) "follows from" (2.1). Then, the following identity is implied by (2.7):

$$\left[\frac{\partial h^0(u^M)}{\partial u^J} - l_J(u^M)\right]\partial_t u^J + \left[\frac{\partial h^i(u^M)}{\partial u^R} - l_J(u^M)\frac{\partial f^{iJ}(u^M)}{\partial u^R}\right]\partial_i u^R + \left[l_J(u^M)b^J(u^M) - \sigma(u^M)\right] \equiv 0 \tag{2.8}$$

which holds for all functions $u^M(t,x^i)$. Since in (2.8) the values of $u^M(t,x^i)$, $\partial_t u^J(t,x^i)$ and $\partial_i u^R(t,x^i)$ can be taken arbitrarily at each point (t,x^i), the terms in square brackets must vanish and, as a consequence, we obtain the following system of identities:

$$l_J(u^M) = \frac{\partial h^0(u^M)}{\partial u^J}, \tag{2.9}$$

$$\frac{\partial h^j(u^M)}{\partial u^R} = l_J(u^M)\frac{\partial f^{iJ}(u^M)}{\partial u^R}, \tag{2.10}$$

$$\sigma(u^M) = l_J(u^M)b^J(u^M). \tag{2.11}$$

Substituting (2.9) into (2.10) and (2.11) we obtain (2.5), (2.6). Conversely, assume that (2.5), (2.6) hold. Multiplying both sides of (2.1) by row vector composed of $\partial h^0(u^M)/\partial u^J$ we obtain the following conservation equation

$$\frac{\partial h^0(u^M)}{\partial u^J}\partial_t u^J + \frac{\partial h^0(u^M)}{\partial u^J}\,\frac{\partial f^{iJ}(u^M)}{\partial u^R}\partial_i u^R = \frac{\partial h^0(u^M)}{\partial u^J}b^J(u^R) \tag{2.12}$$

which, in view of (2.5), (2.6), corresponds to (2.4).

Therefore, the equality (2.7) is satisfied for

$$l_J(u^M) = \frac{\partial h^0(u^M)}{\partial u^J}.$$

Then, the following *condition* was proved in [1]:

If the system (2.1) *implies the additional conservation equation* (2.4) *then the system* (2.2) *premultiplied (left-multiplied) by the Hessian matrix of* $h^0(\mathbf{u})$ *is symmetric.*

It was also mentioned in [1] that this symmetric system is equivalent to (2.1) if the Hessian matrix of $h^0(\mathbf{u})$ is non-singular, and it is symmetric hyperbolic if $h^0(\mathbf{u})$ is convex (Hessian of $h^0(\mathbf{u})$ is positive definite).

The proof of this *condition* given in [1] is based on differentiation of (2.10) with respect to the components of $\mathbf{u}$, which yields

$$\frac{\partial^2 h^i(u^M)}{\partial u^P \partial u^R} - \frac{\partial h^0(u^M)}{\partial u^J}\,\frac{\partial^2 f^{iJ}(u^M)}{\partial u^P \partial u^R} = \frac{\partial^2 h^0(u^M)}{\partial u^P \partial u^J}\,\frac{\partial^2 f^{iJ}(u^M)}{\partial u^R}. \tag{2.13}$$

The left-hand side of (2.13) is symmetric in the indices P, R, so is the right-hand side. It therefore follows from $(2.3)_4$ that Hessian matrix of $h^0(\mathbf{u})$ is the left symmetrizer of the matrices $\mathbf{A}^i(\mathbf{u})$.

The converse to this condition can be formulated as the following implication:

If the system of N conservation equations (2.1) *has a left symmetrizer which is a Hessian matrix of a certain function* $h^0(\mathbf{u})$ *then the system* (2.1) *implies the additional conservation equation* (2.4) *with* $h^i(\mathbf{u})$ *and* $\sigma(\mathbf{u})$ *given by* (2.5), (2.6).

P r o o f. Assume that there exists function $h^0(\mathbf{u})$, such that Hessian matrix of $\mathbf{h}^0(\mathbf{u})$ is the left symmetrizer of the matrices $\mathbf{A}^i(\mathbf{u})$ given by $(2.3)_4$. Then,

$$\frac{\partial^2 h^0(u^M)}{\partial u^P \partial u^Q} \frac{\partial f^{iQ}(u^M)}{\partial u^R} = \frac{\partial^2 h^0(u^M)}{\partial u^R \partial u^S} \frac{\partial f^{iS}(u^M)}{\partial u^P}. \tag{2.14}$$

Let us denote

$$h^i_R(u^M) := \frac{\partial h^0(u^M)}{\partial u^S} \frac{\partial f^{iS}(u^M)}{\partial u^R}. \tag{2.15}$$

For each i, functions $h^i_R(u^M)$ can be interpreted as components of the row vector $\mathbf{h}^{iT}(\mathbf{u})$ which, according to $(2.3)_4$, (2.15), is given by the equation

$$\mathbf{h}^{iT}(\mathbf{u}^M) = \mathbf{l}^T(\mathbf{u})\mathbf{A}^i(\mathbf{u}), \qquad i = 1, 2, ..., n, \tag{2.16}$$

where $\mathbf{l}^T(\mathbf{u}^M)$ is a row vector with components $[\partial h^0(u^M)]/\partial u^S$.

Differentiation of (2.15) with respect to u^Q yields

$$\frac{\partial h^i_R(u^M)}{\partial u^Q} = \frac{\partial^2 h^0(u^M)}{\partial u^Q \partial u^S} \frac{\partial f^{iS}(u^M)}{\partial u^R} + \frac{\partial h^0(u^M)}{\partial u^S} \frac{\partial^2 f^{iS}(u^M)}{\partial u^Q \partial u^R} \tag{2.17}$$

and it follows from (2.14), (2.15), (2.17) that

$$\frac{\partial h^i_R(u^M)}{\partial u^Q} = \frac{\partial h^i_Q(u^M)}{\partial u^R}, \qquad i = 1, 2, ..., m, \tag{2.18}$$

what means that the matrix representing gradient of $\mathbf{h}^i$ (with respect to $\mathbf{u}$) is symmetric.

Equalities (2.18) are necessary and sufficient for the following 1 – forms to be closed

$$\Omega^i = h^i_S \, du^S, \tag{2.19}$$

and, for $\mathbf{u}$ from an open convex domain (without loss of generality, it can be assumed that in (2.1) the domain of $\mathbf{u}$ is an open convex set in R^N), it is exact (see, for example [4]). Hence, there exist functions $h^i(u^M)$ such that

$$\Omega^i = dh^i, \tag{2.20}$$

and therefore

$$h^i_R(u^M) = \frac{\partial h^i(u^M)}{\partial u^R}. \tag{2.21}$$

Substituting (2.21) into (2.15) we obtain (2.5). Then, multiplying both sides of the system (2.1) by the row vector composed of $\partial h^0(u^M)/\partial u^J$ and taking into account (2.5) (implied by (2.21), (2.15)), we finally obtain the following conservation equation

$$\partial_t h^0(\mathbf{u}) + \partial_i h^i(\mathbf{u}) = \frac{\partial h^0(u^M)}{\partial u^J} b^J(u^M). \tag{2.22}$$

Hence, the system (2.1) implies the additional conservation equation with the right-hand side term

$$\sigma(u^M) = \frac{\partial h^0(u^M)}{\partial u^J} b^J(u^M).$$

The following *observation* given in [3] is a direct consequence of the *condition* of Friedrichs and Lax and its converse:

The system (2.1), (2.2) *of* **N** *conservation equations implies the additional conservation equation* (2.4) *(with* $h^i(\mathbf{u})$ *and* $\sigma(\mathbf{u})$ *given by* (2.5), (2.6), *respectively) iff there exists a function* $h^0(\mathbf{u})$ *such that its Hessian matrix is the common left symmetrizer of the matrices* $(2.3)_4$.

The necessary and sufficient condition corresponding to this *observation* but expressed in the framework of geometrical (coordinate-free) description of the systems of conservation equations (affine transformations of independent variables and dependent variables interpreted as local coordinate system on the manifold) is given by PIEKARSKI [6].

3. Example of application of the converse to the condition of Friedrichs and Lax

In the Introduction, we have briefly described the details of the general procedure of determination of the conditions that the system of conservation equations is consistent with the additional conservation equation, based on the consequences of the converse to the result of Friedrichs and Lax. In order to illustrate this procedure, we have chosen here, as an example, the system of conservation equations from [2] because of its extreme structural simplicity and because, in [2], the respective calculations related to verification of the converse to the result of Friedrichs and Lax are given in explicit form. Moreover, this system of conservation equations is consistent with the balance of entropy and therefore it admits symmetric symmetrizers. Hence, the procedure considerably simplifies due to those facts and reduces to determination of the family of symmetric left symmetrizers of the system, to exploitation of the condition that this family contains Hessian matrices and, finally, to integration of the respective system of second-order partial differential equations in order to obtain the entropies.

3.1. System of conservation equations considered in [2]

In [2], the following particular case of $N = 5$ conservation equations (2.2) in $i = 3$ spatial dimensions has been considered

$$\mathbf{u}^T(t, x^i) = \left[e(t, x^i), q^1(t, x^i), q^2(t, x^i), q^3(t, x^i), \beta(t, x^i)\right], \tag{3.1}$$

$$\mathbf{A}^1(\mathbf{u}) = \begin{bmatrix} \alpha'(e)q^1 & \alpha(e) & 0 & 0 & 0 \\ f_1'(e) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{A}^2(\mathbf{u}) = \begin{bmatrix} \alpha'(e)q^2 & 0 & \alpha(e) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ f_1'(e) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.2}$$

$$\mathbf{A}^3(\mathbf{u}) = \begin{bmatrix} \alpha'(e)q^3 & 0 & 0 & \alpha(e) & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ f_1'(e) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\alpha'(e) = \frac{d\alpha(e)}{de}, \qquad f_1'(e) = \frac{df_1(e)}{de},$$

$$\mathbf{b}^T(\mathbf{u}) = \left[\varrho r,\ f_2'(\beta)q^1,\ f_2'(\beta)q^2,\ f_2'(\beta)q^3,\ f_1(e) + f_2(\beta)\right]. \tag{3.3}$$

The system of conservation equations (2.2), (3.1), (3.2), (3.3) corresponds to the phenomenological model of a rigid conductor of heat with internal state variable β (called "semi-empirical temperature"). In this model, e is the internal energy density, q_i $(i = 1, 2, 3)$ are the components of gradient of $(-\beta)$, ϱr is the heat source density and $\alpha(e)$, $f_1(e)$ are constitutive functions.

In [2], the condition

$$\mathbf{H}(\mathbf{u})\mathbf{A}^i(\mathbf{u}) = \left[\mathbf{H}(\mathbf{u})\mathbf{A}^i(\mathbf{u})\right]^T, \qquad i = 1, 2, ..., n, \tag{3.4}$$

has been imposed on the matrices (3.2) for *prescribed* postulated matrix $\mathbf{H}(\mathbf{u})$

$$\mathbf{H}(\mathbf{u}) = \operatorname{diag}\left[\eta_e''(e),\ c_1,\ c_1,\ c_1,\ c_2\right], \tag{3.5}$$
$$\eta_e''(e) = \frac{d^2\eta_e(e)}{de^2}, \qquad c_1, c_2 - \text{const},$$

which, in fact, is a *Hessian matrix* of the following function h^0 of the arguments e, q^i, β,

$$h^0(\mathbf{u}) = h^0(e, q^i, \beta) = \eta_e(e) + \frac{1}{2}c_1 q_i q^i + \frac{1}{2}c_2\beta^2. \tag{3.6}$$

As a consequence of this condition, the relation between $\alpha(e)$, $f_1(e)$ and $\eta_e(e)$ has been obtained

$$c_1 f_1'(e) = \alpha(e)\eta_e''(e), \tag{3.7}$$

and called "*symmetrizability condition*" for the system (2.2), (3.1), (3.2), (3.3).

In view of the *observation* given in Sec. 2, the "symmetrizability condition" (3.7) is nothing else but the condition that the system (2.1), (3.1), (3.2), (3.3) implies the balance of entropy (2.4) for the entropy (3.6).

In the following, we show how the condition (3.7) and the family of entropies containing, as a special case, the function (3.6) can be obtained from the requirement that the family of common symmetric left symmetrizers of the matrices (3.2) admits Hessian matrices, in other words, by selecting Hessians from this family.

3.2. Family of the symmetric left symmetrizers of the matrices (3.2)

The term "symmetrizability condition" used in [2] to denote the condition (3.7) (which, in fact, is the condition of consistency with the entropy balance (2.4) for entropy (3.6)) seems to be particularly inadequate in view of the fact that the matrices (3.2) have a family of common left symmetric symmetrizers (for arbitrary $\alpha(e)$, $f_1(e)$)

$$\mathbf{S}(\mathbf{u}) = \operatorname{diag}\left[\chi,\ \chi\frac{\alpha(e)}{f_1'(e)},\ \chi\frac{\alpha(e)}{f_1'(e)},\ \chi\frac{\alpha(e)}{f_1'(e)},\ \lambda\right] \tag{3.8}$$

parametrized by two arbitrary functions $\chi(e, q^i, \beta)$, $\lambda(e, q^i, \beta)$.

To see this, one can simply verify by inspection that

$$\mathbf{S}(\mathbf{u})\mathbf{A}^i(\mathbf{u}) = \left[\mathbf{S}(\mathbf{u})\mathbf{A}^i(\mathbf{u})\right]^T, \qquad i = 1, 2, 3, \tag{3.9}$$

holds for $\mathbf{A}^i(\mathbf{u})$ given by (3.2) and $\mathbf{S}(\mathbf{u})$ given by (3.8). The family of matrices (3.8) represents all symmetric solutions $\mathbf{S}(\mathbf{u})$ ($\mathbf{S}(\mathbf{u}) = \mathbf{S}^T(\mathbf{u})$) of the system of three matrix equations (3.9) with the matrices $\mathbf{A}^i(\mathbf{u})$ given by (3.2).

The fact that the matrices (3.2) have common symmetric left symmetrizers and the family of those symmetrizers takes the form (3.8), is a consequence of a very specific structure of the set of matrices (3.2); namely, except the first entry on the main diagonal, they can be obtained one from the other simply by permuting the respective rows and columns (similarity transformations by the respective permutation matrices).

Thus, the system (2.2), (3.1), (3.2), (3.3) considered in [2] can be symmetrized without appealing to the fact that it admits the additional conservation equation and, as it follows from (3.8), it admits a more general class of symmetrizers than that obtained in [2]. By choosing $\chi(e, q^i, \beta) > 0$ and $\lambda(e, q^i, \beta) > 0$ for all e, q^i, β (from the respective domain), positive definite symmetrizers can be obtained from the family (3.8) provided that either $\alpha(e) > 0$ and $f_1'(e) > 0$ or $\alpha(e) < 0$ and $f_1'(e) < 0$. Hence, the only conditions on $\alpha(e)$ and $f_1(e)$ that ensure symmetric hyperbolicity of symmetric systems obtained by premultiplication of (2.2), (3.2) by $\mathbf{S}(\mathbf{u})$ from (3.1) with $\chi > 0$, $\lambda > 0$ is either $\alpha(e) < 0$, $f_1'(e) < 0$ or $\alpha(e) > 0$, $f_1'(e) > 0$.

3.3. Condition of consistency with balance of entropy and the family of entropies

The condition that the system (2.2), (3.1), (3.2), (3.3) is symmetrizable by Hessian matrix of a certain entropy function $h^0(e, q^i, \beta)$ is equivalent to the condition that at least one of the matrices $\mathbf{S}(\mathbf{u})$ of the family (3.8) is the Hessian of $h^0(e, q^i, \beta)$. This condition is, therefore, the integrability condition of the following system of 15 second-order partial differential equations for $h^0(e, q^i, \beta)$.

$$
\begin{aligned}
&\frac{\partial^2 h^0}{\partial e^2} = \chi(e, q^i, \beta), \\
&\frac{\partial^2 h^0}{\partial e \partial q^i} = 0, \qquad i = 1, 2, 3, \\
&\frac{\partial^2 h^0}{\partial e \partial \beta} = 0, \\
&\frac{\partial^2 h^0}{\partial (q^i)^2} = \chi(e, q^i, \beta)\frac{\alpha(e)}{f_1'(e)}, \qquad i = 1, 2, 3, \\
&\frac{\partial^2 h^0}{\partial q^i \partial q^j} = 0, \qquad i, j = 1, 2, 3, \quad i \neq j, \\
&\frac{\partial^2 h^0}{\partial q^i \partial \beta} = 0, \qquad i = 1, 2, 3, \\
&\frac{\partial^2 h^0}{\partial \beta^2} = \lambda(e, q^i, \beta).
\end{aligned}
\tag{3.10}
$$

It can be easily verified that the system (3.10) is integrable iff

$$
\begin{aligned}
\chi(e, q^i, \beta) &= \overline{\chi}(e), \\
\lambda(e, q^i, \beta) &= \overline{\lambda}(e), \\
\overline{\chi}(e)\frac{\alpha(e)}{f_1'(e)} &= c, \qquad c = \text{const.}
\end{aligned}
\tag{3.11}
$$

With the conditions (3.11), the system (3.10) can be integrated and its solutions (entropies) take the following form:

$$
h^0(e, q^i, \beta) = h_e^0(e) + \frac{1}{2}cq_iq^i + h_\beta^0(\beta),
\tag{3.12}
$$

where

$$
\begin{aligned}
\overline{\chi}(e) &= \frac{d^2h_e^0(e)}{de^2} = h_e^{0\,\prime\prime}(e) = c\frac{f_1'(e)}{\alpha(e)}, \\
\overline{\lambda}(e) &= \frac{d^2h_\beta^0(e)}{d\beta^2} = h_\beta^{0\,\prime\prime}(e),
\end{aligned}
\tag{3.13}
$$

and $h_e^0(e)$, $h_\beta^0(\beta)$ are arbitrary $C^2(R)$ functions, and c is an arbitrary real constant. Identifying $h_e^0(e)$ with $\eta_e(e)$ and c with c_1 and taking into account $(3.13)_1$, we recognize the condition (3.7) in integrability conditions $(3.11)_{1,3}$ and notice that the function (3.6) corresponding to prescribed symmetrizer (3.5) is a particular entropy (3.12) corresponding to $h_\beta^0(\beta) = (1/2)c_2\beta^2$.

It follows from Sec. 2 of [2] that the original system of field equations corresponding to the considered model of a rigid conductor of heat is the system of two partial differential equations of the first order with respect to the temperature Θ, and of the second order with respect to β, and that the second law of thermodynamics (entropy inequality) implies the entropy associated to this system of the form [2. Eq. (2.5)]

$$
\eta^*(\Theta, \nabla\beta) = \eta_e^*(\Theta) - \frac{1}{2}c|\nabla\beta|^2, \qquad c - \text{const.}
\tag{3.14}
$$

It is assumed in [2] that internal energy e is the inevitable function of Θ, so the original system of field equations can be equivalently expressed as the system of two partial differential equations for e and β (first order with respect to e and second order with respect to β), and, according to (3.14), the corresponding entropy must be of the form

$$
\eta(e, \nabla\beta) = \eta_e(e) - \frac{1}{2}c|\nabla\beta|^2.
\tag{3.15}
$$

The system of five conservation equations (2.2), (3.1), (3.2), (3.3) has been obtained in [2] from the original system of two field equations expressed in terms of e, β, supplemented by the additional three equations obtained through spatial differentiation of one suitably chosen member of this original system. For such system of five equations, the variable $\mathbf{q}$ has been introduced through the substitution

$$\mathbf{q} = -\nabla\beta. \tag{3.16}$$

With the substitution (3.16), the entropy (3.15) corresponding to the original system of field equations expressed in terms of e, β (implied by the entropy inequality) takes the form

$$\eta(e, \mathbf{q}) = \eta_e(e) - \frac{1}{2}c|\mathbf{q}|^2, \tag{3.17}$$

while the system (2.2), (3.1), (3.2), (3.3) admits the family of entropies (3.12) implied by the balance of entropy (entropy inequality replaced by the corresponding balance law). The entropy (3.17) is a particular member of the family (3.12) corresponding to $h^0_\beta(\beta) \equiv 0$.

Hence, the entropy (3.17) obtained from thermodynamic restrictions imposed on the original second order system of field equations cannot be employed for symmetrization of the corresponding first order system of conservation equations (2.2), (3.1), (3.2), (3.3) since, when treated as a function of e, q^i, β, it will lead to the singular Hessian matrix.

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