Arch. Mech., **49**, 5, pp. 815–829, Warszawa 1997

The velocity of the fluid due to the many-sphere Oseen hydrodynamic interactions

I. PIEŃKOWSKA (WARSZAWA)

WE CONSIDER the velocity field, generated in the incompressible, viscous fluid due to the hydrodynamic interactions between a finite number of solid spheres. The particular properties of the velocity field, due to the convective inertia of the fluid, are examined. The inertia effects are taken into account up to the contributions of the order of 0(Re).

1. Introduction

THE PRESENT PAPER concerns the hydrodynamic interactions of a finite number of solid spheres at small, but finite sphere Reynolds number Re. In the previous paper [1] we have investigated the effects of the hydrodynamic interactions on the friction relations between the spheres. In this paper the respective velocity field of the fluid is studied. In particular, some properties of the velocity field, not to be expected on the basis of the Stokes equation, will be analysed. The analysis is performed in the framework of the Oseen equation of motion of an incompressible fluid. The inertia of the fluid is evaluated up to the contributions of the order of 0(Re), where $\mathrm{Re} = a|\mathbf{U}|/\nu$ (a – the radius of the sphere, $\mathbf{U}$ – the uniform velocity of the fluid at infinity, $\mathbf{U} = (U, \theta, \varphi)$ in spherical polar coordinates, ν – the kinematic viscosity).

Under the condition of vanishingly small Re, the velocity field, generated by the many-sphere hydrodynamic interactions, has been recently considered by DURLOFSKY, BRADY and BOSSIS [12] and by PHILLIPS [13]. In the paper [12], devoted to the dynamic simulation of hydrodynamically interacting particles, it has been shown that the velocity field may be expressed in terms of the propagators, acting on the forces, torques and stresslets, exerted by the particles on the fluid ((2.13), (2.14) in [12]). That representation of the velocity profile is the basis of the dynamic simulation of hydrodynamically interacting spheres in a quiescent second-order fluid, developed in the paper [13], to account for the non-Newtonian behaviour.

The influence of the inertia of the fluid on the hydrodynamic interactions of a cluster of spheres moving in the fluid at small Re has been recently examined both theoretically and experimentally by KUMAGAI [2]. The author has extended the conventional reflection method of the description of the interactions, developed for the Stokes flow regime, to the case of the Oseen flow regime. His numerical results, concerning the inertia effects in the free-fall motion of spheres, show a good agreement with the experimental results. Earlier approaches to the analysis of the nonlinear effects have been quoted in [1].

In the present paper, we use the multiple scattering approach [3] to the analysis of the hydrodynamic interactions and the velocity field. Starting from the integral formulation for the Oseen flow, the interactions and the velocity field are expressed in terms of the following parameters:

(i) $\sigma = a/R$, describing the dependence of the interactions on the radial distribution of the spheres (R is a typical distance between the centres of two spheres),

(ii) $\kappa = a/P_k$, giving the dependence of the velocity field on the radial distance between the centre of the k-th sphere and the point $\mathbf{r}$ in the fluid,

(iii) RU/ν, P_kU/ν – characterizing the regime of the interactions (the role of the convective inertia effects).

Here we consider the regime specified by the following conditions:

$$\sigma < 1, \qquad \kappa < 1, \qquad RU/\nu < 1, \qquad P_kU/\nu < 1.$$

It means, we regard the intermediate sphere spacing and the velocity field in the region near to the assemblage of the spheres. The spheres are held fixed. No lubrication behaviour is included. The hydrodynamic interactions and the velocity profile are regarded up to a given order with respect to σ and κ. The 0(Re) convective inertia effects are taken into account.

2. Governing equations

The presence of the spheres in the fluid is accounted through the induced forces $\mathbf{f}_j$, $j = 1, ..., N$, distributed on the surfaces of the spheres. In an external Cartesian coordinate system, the centres and the surfaces of the spheres are given, respectively, by $\mathbf{R}_j^0$, and $\mathbf{R}_j$. The fluid velocity $\boldsymbol{\nu}(\mathbf{r})$ and pressure $p(\mathbf{r})$ satisfy the Oseen [8] and continuity equations:

$$\begin{gathered} \varrho \mathbf{U}\cdot\nabla\boldsymbol{\nu} - \mu\Delta\boldsymbol{\nu} + \nabla p = \sum_{j=1}^{N}\int d\Omega_j \delta\left[\mathbf{r} - \mathbf{R}_j(\Omega_j)\right]\mathbf{f}_j(\Omega_j), \\ \nabla\cdot\boldsymbol{\nu} = 0, \end{gathered} \tag{2.1}$$

where ϱ and μ are the density and the dynamic viscosity of the fluid, $\delta[\mathbf{r}-\mathbf{R}_j(\Omega_j)]$ indicate the positions of the surfaces of the spheres, $\mathbf{R}_j = \mathbf{R}_j^0 + \mathbf{r}_j$. In the local spherical polar coordinates $\mathbf{r}_j = (a, \Omega_j) = (a, \theta_j, \varphi_j)$. Inside the volumes of the spheres, the respective stress tensors $\mathbf{P}(\mathbf{r}_j)$ satisfy

$$\nabla\cdot\mathbf{P}(\mathbf{r}_j) = 0, \qquad |\mathbf{r}_j| < a. \tag{2.2}$$

On the surfaces of the spheres, we impose the no-slip boundary conditions:

$$\dot{\mathbf{R}}_j(\Omega_j) = \boldsymbol{\nu}(\mathbf{R}_j(\Omega_j)), \qquad \dot{\mathbf{R}}_j(\Omega_j) = 0, \tag{2.3}$$

where $\dot{\mathbf{R}}_j(\Omega_j)$ denotes the velocity of the j-th sphere.

The velocity field in the considered system can be presented in the following form of the convolution integral:

$$\boldsymbol{\nu}(\mathbf{r}) = \mathbf{U} + \int d\mathbf{r}'\mathbf{T}(\mathbf{r}-\mathbf{r}')\cdot\sum_{j=1}^{N}\int d\Omega'_j\delta\left[\mathbf{r}'-\mathbf{R}'_j(\Omega'_j)\right]\mathbf{f}'_j(\Omega'_j), \tag{2.4}$$

where $\mathbf{T}(\mathbf{r}-\mathbf{r}')$ is the free-space Green tensor.

Its space-Fourier transform reads [4]:

$$\mathbf{T}(\mathbf{r}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{\exp(i\mathbf{k}\cdot\mathbf{r})(\mathbf{1}-\hat{\mathbf{k}}\hat{\mathbf{k}})}{\mu(k^2+i\nu^{-1}\mathbf{U}\cdot\mathbf{k})}, \tag{2.5}$$

where $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$, $\mathbf{k}(k,\chi,\eta)$ in spherical polar coordinates.

The second term on the r.h.s. of (2.4) describes the disturbance of the uniform fluid velocity $\mathbf{U}$ due to the hydrodynamic interactions of the spheres. To perform the integrations over the surfaces of the spheres, appearing in this term, we expand the induced forces $\mathbf{f}_j$, $\exp(i\mathbf{k}\cdot\mathbf{P}_k)$ and $\exp(i\mathbf{k}\cdot\mathbf{r}_j)$ in terms of the normalized surface spherical harmonics Y_l^m [5]:

$$\mathbf{f}_j(\mathbf{r}_j) = \frac{1}{\sqrt{4\pi}}\sum_{lm}\mathbf{f}_{j,lm}Y_l^m(\Omega_j), \qquad l \geq 0, \quad |m| \leq l, \tag{2.6}$$

$$\begin{aligned}\exp(i\mathbf{k}\cdot\mathbf{P}_k) &= 4\pi\sum_{lm} i^l j^l(P_k k)Y_l^m(\chi_k,\eta_k)Y_l^{-m}(\chi,\eta),\\ \exp(i\mathbf{k}\cdot\mathbf{r}_j) &= 4\pi\sum_{lm} i^l j^l(ak)Y_l^m(\theta_j,\phi_j)Y_l^{-m}(\chi,\eta),\end{aligned} \tag{2.7}$$

where j_l is the spherical Bessel function, $\mathbf{P}_k = \mathbf{R}_k^0 - \mathbf{r} = (P_k,\chi_k,\eta_k)$ in spherical polar coordinates. Finally, we arrive at the following representation

$$\boldsymbol{\nu}(\mathbf{r}) = \mathbf{U} + \sum_{k=1}^{N}\sum_{l_2m_2}\mathbf{C}^{l_2m_2}(\mathbf{P}_k)\cdot\mathbf{f}_{k,l_2m_2}, \tag{2.8}$$

giving the velocity field in terms of the (l_2m_2) components of the induced forces $\mathbf{f}_k$. The second order tensors $\mathbf{C}^{l_2m_2}(\mathbf{P}_k)$ are called the velocity field tensor. They are introduced to examine the disturbance of the velocity field $\mathbf{U}$ due to the hydrodynamic interactions of the k-th sphere in the presence of the $N-1$ other spheres. For further consideration, the tensors are written down in the following form:

$$\mathbf{C}^{l_2 m_2}(\mathbf{P}_k) = \sum_{l_3 m_3} \mathbf{C}^{l_2 m_2}_{l_3 m_3}(P_k) Y^{m_3}_{l_3}(\chi_k, \eta_k), \tag{2.9}$$

where

$$\mathbf{C}^{l_2 m_2}_{l_3 m_3} = \frac{i^{-l_2-l_3}}{\mu\pi\sqrt{\pi}} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{(\mathbf{1} - \hat{\mathbf{k}}\hat{\mathbf{k}})}{k^2 + i\nu^{-1}\mathbf{U}\cdot\mathbf{k}} Y^{m_2}_{l_2} Y^{-m_3}_{l_3} j_{l_2}(ak) j_{l_3}(P_k k). \tag{2.10}$$

We note that the properties of the velocity field tensors follow the properties of the Green tensor $\mathbf{T}(\mathbf{r})$. In the description of the velocity profile, the role of the velocity tensors is similar to the role of the propagators, introduced by DURLOFSKY, BRADY and BOSSIS [12]. In what follows, the dependence of the tensors on the parameters κ and Re will be discussed.

3. Properties of tensors $\mathbf{C}^{l_2 m_2}_{l_3 m_3}(P_k)$

It is shown in the Appendix that the tensors $\mathbf{C}^{l_2 m_2}_{l_3 m_3}(P_k)$ can be presented in the following form (A.7):

$$\begin{aligned} \mathbf{C}^{l_2 m_2}_{l_3 m_3} = \pm 4\pi \sum_{m=0}^{\infty} \beta_m(l_2, l_3) i^{|l_2+2m-l_3|} \\ \cdot \sum_{m_4, m_5} R(l_2, m_2, m_4) R(l_3, -m_3, -m_5) d^{m_4}_{l_2} d^{-m_5}_{l_3} \\ \cdot \int_0^1 d\xi \left[\frac{2}{3}\delta_{m_4, m_5} - \sqrt{\frac{2\pi}{15}} \sum_{m_6, m_7} \delta_{m_7+m_4, m_5} \mathbf{K}_{m_6} R(2, m_6, m_7) d^{m_7}_2 P^{m_7}_2(\xi) \right] \\ \cdot P^{m_4}_{l_2}(\xi) P^{-m_5}_{l_3}(\xi) I_{\tilde{z}}(P_k \alpha \xi) K_{\tilde{\varrho}}(P_k \alpha \xi), \end{aligned} \tag{3.1}$$

where the signs $\{\pm\}$ refer to the cases $l_2+l_3 = 2n$, $l_2+l_3 = 2n+1$, respectively, the quantity $\beta_m(l_2, l_3)$, depending on P_k, is given by the formula (A.3), the quantities $R(l_i, m_i, m_j)$ describe the rotation of the coordinate system, the functions Y^m_l are written down in the form:

$$Y^m_l = d^m_l P^m_l e^{im\varphi},$$

$I_{\tilde{z}}$, $K_{\tilde{\varrho}}$ are the modified Bessel functions, $\alpha = U/\nu$, $\tilde{z} = \max(l_2+2m+1/2, l_3+1/2)$, $\tilde{\varrho} = \min(l_2 + 2m + 1/2,\ l_3 + 1/2)$.

We note the appearance of the parameter $P_k U/\nu$ (in the arguments of the modified Bessel functions), characterizing the regime of the disturbances of the velocity field $\mathbf{U}$. The above formula is valid for arbitrary values of that parameter.

In what follows we are going to discuss the properties of the above tensors in the range $P_k U/\nu < 1$, referring to the weak inertia effects in the velocity profile. In this range, the products of the modified Bessel functions behave as follows:

$$(3.2)\qquad I_{\tilde{z}}(P_k\alpha\xi)K_{\tilde{\varrho}}(P_k\alpha\xi) = \left(\frac{1}{2}\right)^{\tilde{z}-\tilde{\varrho}+1}\frac{\Gamma(\tilde{\varrho})}{\Gamma(\tilde{z}+1)}(P_k\alpha\xi)^{|l_2+2m-l_3|}+\dots\,.$$

From (3.2) it follows that for the case considered we have two kinds of the velocity field tensors:

(i) the Stokes velocity field tensors (disregarding the role of the inertia of the fluid);

(ii) the 0(Re) Oseen velocity field tensors (taking into account the weak inertia effects).

We see that the leading order contributions to the velocity tensors, which do not depend on Re, are equal to

$$(3.3)\qquad \mathbf{C}^{l_2m_2}_{l_3m_3} = \pm 4\pi\sum_{m=0}^{\infty}\beta_m(l_2,l_3)\sum_{m_4,m_5}R(l_2,m_2,m_4)R(l_3,-m_3,-m_5)d^{m_4}_{l_2}d^{-m_5}_{l_3}$$
$$\cdot\int_0^1 d\xi\left[\frac{2}{3}\delta_{m_4,m_5}-\sqrt{\frac{2\pi}{15}}\sum_{m_6,m_7}\delta_{m_7+m_4,m_5}\mathbf{K}_{m_6}R(2,m_6,m_7)d^{m_7}_2P^{m_7}_2(\xi)\right]$$
$$\cdot\, P^{m_4}_{l_2}(\xi)P^{-m_5}_{l_3}(\xi)\left(\frac{1}{2}\right)^{\tilde{z}-\tilde{\varrho}+1}\frac{\Gamma(\tilde{\varrho})}{\Gamma(\tilde{z}+1)}\,.$$

The integrals over the associated Legendre functions are different from zero for the following sets of the indices l_i [6]:

$$(3.4)\qquad l_2 = l_3 \qquad \text{and} \qquad l_2 = l_3 - 2.$$

Hence the leading order contributions to the considered tensors are characterized by the following parameters:

$$(3.5)\qquad \begin{array}{lll}\text{(i)} & m=0, & l_2=l_3\,;\\ \text{(ii)} & m=1, & l_2=l_3-2.\end{array}$$

The tensors exhibit the characteristic dependence on the inverse powers of the distances P_k:

(3.6)

(i) they are of the leading order of $\left(\dfrac{a}{P_k}\right)^{l_2+1}$;

(ii) the tensors with $m=1$ contain the contributions of the order of $\left(\dfrac{a}{P_k}\right)^{l_2+3}$.

For example, the velocity tensors of low indices assume the following form:

(i) diagonal with respect to l_i $(m = 0)$:

$$\mathbf{C}^{00}_{00}(P_k) = \frac{1}{3\sqrt{\pi}\mu P_k}\mathbf{I}; \tag{3.7}$$

(ii) off-diagonal with respect to l_i $(m = 1)$:

$$\mathbf{C}^{00}_{2m_3}(P_k) = \frac{\sqrt{2}}{8\sqrt{15\pi}\mu P_k}\left[1 - \left(\frac{a}{P_k}\right)^2\right]\mathbf{K}_{m_3}. \tag{3.8}$$

The leading order contributions to the velocity tensors, given by (3.3), will be used to describe the velocity field past N spheres, provided the inertial effects are negligible.

In the considered range $P_k U/\nu < 1$ the second group of the velocity tensors, being of our interest, are the tensors of the order of 0(Re). It follows from (3.2) that they are equal to

$$\begin{aligned}\mathbf{C}^{l_2m_2}_{l_3m_3} = \pm 4\pi i \sum_{m=0}^{\infty} \beta_m(l_2, l_3) \sum_{m_4,m_5} R(l_2, m_2, m_4) R(l_3, -m_3, -m_5) d^{m_4}_{l_2} d^{-m_5}_{l_3} \\ \cdot \int_0^1 d\xi \left[\frac{2}{3}\delta_{m_4,m_5} - \sqrt{\frac{2\pi}{15}} \sum_{m_6,m_7} \delta_{m_7+m_4,m_5} \mathbf{K}_{m_6} R(2, m_6, m_7) d^{m_7}_2 P^{m_7}_2(\xi)\right] \\ \cdot P^{m_4}_{l_2}(\xi) P^{-m_5}_{l_3}(\xi) \left(\frac{1}{2}\right)^{\tilde{z}-\tilde{\varrho}+1} \frac{\Gamma(\tilde{\varrho})}{\Gamma(\tilde{z}+1)} P_k \alpha \xi + \ldots .\end{aligned} \tag{3.9}$$

Taking again into account the properties of the integrals over ξ we deduce that the tensors, belonging in that group, are characterized by the following sets of their indices:

$$\begin{aligned} &\text{(i)} \qquad m = 0, \qquad l_3 = l_2 - 1, \\ &\text{(ii)} \qquad m = 0, 1, \qquad l_3 = l_2 + 1, \\ &\text{(iii)} \qquad m = 1, 2, \qquad l_3 = l_2 + 3. \end{aligned} \tag{3.10}$$

It follows from (3.9) that the above tensors are built up of the contributions of the following orders with respect to (a/P_k):

$$\begin{aligned} \mathbf{C}^{l_2m_2}_{l_2-1m_3} &\sim \left(\frac{a}{P_k}\right)^{l_2}, \\ \mathbf{C}^{l_2m_2}_{l_2+1m_3} &\sim \left(\frac{a}{P_k}\right)^{l_2}, \left(\frac{a}{P_k}\right)^{l_2+2}, \\ \mathbf{C}^{l_2m_2}_{l_2+3m_3} &\sim \left(\frac{a}{P_k}\right)^{l_2}, \left(\frac{a}{P_k}\right)^{l_2+2}, \left(\frac{a}{P_k}\right)^{l_2+4}. \end{aligned} \tag{3.11}$$

Here we list, for example, a few low indices 0(Re) tensors, for the particular case of $\hat{U}(0,0,1)$:

(i) $l_3 = l_2 - 1, \quad m = 0,$

$$\text{(3.12)} \qquad \mathbf{C}_{00}^{1m_2}(P_k) = -\frac{\text{Re}}{9\sqrt{3\pi}\,\mu P_k}(-1)^{(m_2-|m_2|)/2}\Bigg\{\delta_{m_2,0} - \frac{3}{2\sqrt{5}}\sum_{m_6=-2}^{2}\delta_{m_2+m_6,0}(-1)^{(m_6-|m_6|)/2}\begin{pmatrix} 2 & 1 & 1 \\ m_6 & m_2 & 0 \end{pmatrix}\mathbf{K}_{m_6}\Bigg\};$$

(ii) $l_3 = l_2 + 1, \quad m = 0,\ 1,$

$$\text{(3.13)} \qquad \mathbf{C}_{1m_3}^{00}(P_k) = \frac{\text{Re}}{6\sqrt{3\pi}\,a\mu}(-1)^{(-m_3-|m_3|)/2}\Bigg\{-\delta_{m_3,0} + \frac{9}{8\sqrt{5}}\sum_{m_6=-2}^{2}\delta_{m_3,m_6}(-1)^{(m_6-|m_6|)/2}\begin{pmatrix} 2 & 1 & 1 \\ m_6 & -m_3 & 0 \end{pmatrix}\mathbf{K}_{m_6}\Bigg\} + 0\left(\left(\frac{a}{P_k}\right)^2\right);$$

(iii) $l_3 = l_2 + 3, \quad m = 1,\ 2,$

$$\text{(3.14)} \qquad \mathbf{C}_{3m_3}^{00}(P_k) = \frac{\sqrt{5}\text{Re}}{56\sqrt{2\pi}\,a\mu}\sum_{m_6=-2}^{2}\delta_{m_3,m_6}\begin{pmatrix} 2 & 3 & 1 \\ m_6 & -m_3 & 0 \end{pmatrix}\mathbf{K}_{m_6} + 0\left(\left(\frac{a}{P_k}\right)^2\right),$$

where the Wigner 3-j symbols $\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ are given by the formula (3.7.11) from [6].

In view of the properties of the Bessel functions $I_{1/2}$ and $K_{1/2}$, the contributions linear in Re appear also in the series expansion of the tensor $\mathbf{C}_{00}^{00}$ with respect to $P_k U/\nu$. It follows from (A.8) that the tensor $\mathbf{C}_{00}^{00}(P_k)$ can be presented in the following form:

$$\text{(3.15)} \qquad \mathbf{C}_{00}^{00}(P_k) = \mathbf{C}_k + \mathbf{C}_k^1 + \dots ,$$

where $\mathbf{C}_k$ denote the Stokes contributions (3.7), $\mathbf{C}_k^1$ are the 0(Re) contributions, equal to:

$$\text{(3.16)} \qquad \mathbf{C}_k^1 = \frac{\text{Re}}{16\sqrt{6\pi}\,a\mu}\sum_{m_6=-2}^{2} R(2,m_6,0)\mathbf{K}_{m_6},$$

and the quantity $R(2,m_6,0)$ is defined by (A.4).

We note that the leading order contributions to $\mathbf{C}_{1m_3}^{00}(P_k)$, $\mathbf{C}_{3m_3}^{00}(P_k)$ and $\mathbf{C}_k^1$ are independent of P_k. In the paper [7] we have discussed an analogous lack of $|\mathbf{R}_{jk}|$ in the leading order contributions to the mutual interaction tensors

$\mathbf{T}^{l_2m_2}_{l_1m_1}(\mathbf{R}_{jk})$ ($\mathbf{R}_{jk} = \mathbf{R}^0_k - \mathbf{R}^0_j$, the formula (4.20) in [7]). The above contributions to the velocity tensors, acting on the component $\mathbf{f}_{k,00}$ of the induced forces, give rise to the P_k-independent terms in the expression (2.8) for the velocity field.

That type of the independence has been reported, for example, in the paper by PROUDMAN and PEARSON [14], concerning the flow past one sphere. The authors have considered the velocity field in the framework of the Navier–Stokes equations, applying the method of the matched asymptotic expansions. The above contributions to the velocity field, being proportional to Re, vanish at the Stokes conditions.

4. The components $\mathbf{f}_{j,l_2m_2}$ of the induced forces

The hydrodynamic interactions between the spheres are treated as the multiple scattering events, describing the scattering of the disturbances of the velocity field due to the presence of the spheres. The approach leads to the following formula, providing the representation for the components $\mathbf{f}_{j,l_2m_2}$ in terms of the relative velocity of the fluid with respect to the spheres $\mathbf{V}_{j,lm}$:

$$\mathbf{f}_{j,l_1m_1} = \sum_{l_2m_2} \widetilde{\mathbf{T}}^{l_2m_2}_{l_1m_1}(\mathbf{O}_j)\cdot\Bigg[\mathbf{V}_{j,l_2m_2} - \sum_{k\neq j}\sum_{l_im_i}\mathbf{T}^{l_3m_3}_{l_2m_2}(\mathbf{R}_{jk})\cdot\widetilde{\mathbf{T}}^{l_4m_4}_{l_3m_3}(\mathbf{O}_k)\cdot\mathbf{V}_{k,l_4m_4}$$
$$+\sum_{k\neq j}\sum_{k\neq k_1}\sum_{l_im_i}\mathbf{T}^{l_3m_3}_{l_2m_2}(\mathbf{R}_{jk})\cdot\widetilde{\mathbf{T}}^{l_4m_4}_{l_3m_3}(\mathbf{O}_k)\cdot\mathbf{T}^{l_5m_5}_{l_4m_4}(\mathbf{R}_{kk_1})\cdot\widetilde{\mathbf{T}}^{l_6m_6}_{l_5m_5}(\mathbf{O}_{k_1})\cdot\mathbf{V}_{k_1,l_6m_6} - \ldots\Bigg], \tag{4.1}$$

where $i = 2, 3, 4, 5, 6$,

$$\mathbf{V}_{j,lm} = \left\{\begin{matrix} -\mathbf{U}, & l = 0 \\ 0, & l \geq 1 \end{matrix}\right\}. \tag{4.2}$$

$\widetilde{\mathbf{T}}^{l_2m_2}_{l_1m_1}(\mathbf{O}_j)$ and $\mathbf{T}^{l_3m_3}_{l_2m_2}(\mathbf{R}_{jk})$ are respectively the inverse self- and mutual interaction tensors; their properties have been analysed in the author's previous paper [1], under the assumption $R_{jk}U/\nu < 1$. For example, we list below a few hydrodynamic interaction tensors with the lowest indices, including the contributions up to 0(Re):

(i) self-interaction tensors:

$$\mathbf{T}^{00}_{00}(\mathbf{O}_j) = \mathbf{T}_j + \mathbf{T}^1_j + \ldots\ ,$$

where

$$\mathbf{T}_j = \frac{1}{6\pi\mu a}\mathbf{I}, \qquad \mathbf{T}^1_j = \frac{1}{6\pi\mu a}\left[-\frac{3}{16}\mathrm{Re}\left(3\mathbf{I} - \hat{\mathbf{U}}\hat{\mathbf{U}}\right)\right];$$

(ii) inverse self-interaction tensors:

$$\widetilde{\mathbf{T}}^{00}_{00}(\mathbf{O}_j) = \widetilde{\mathbf{T}}_j + \widetilde{\mathbf{T}}^1_j + \ldots\ , \tag{4.3}$$

where

$$\widetilde{\mathbf{T}}_j = 6\pi\mu a\mathbf{I}, \qquad \widetilde{\mathbf{T}}_j^1 = 6\pi\mu a\left[\frac{3}{16}\mathrm{Re}\left(3\mathbf{I} - \hat{\mathbf{U}}\hat{\mathbf{U}}\right)\right];$$

(iii) mutual-interactions tensors:

$$\mathbf{T}_{00}^{00}(\mathbf{R}_{jk}) = \mathbf{T}_{jk} + \mathbf{T}_{jk}^1 + \ldots ,$$

where

$$\mathbf{T}_{jk} = \frac{1}{8\pi\mu R_{jk}}\left[\mathbf{1} + \hat{\mathbf{e}}_{jk}\hat{\mathbf{e}}_{jk} + \frac{2a^2}{R_{jk}^2}\left(\frac{1}{3}\mathbf{I} - \hat{\mathbf{e}}_{jk}\hat{\mathbf{e}}_{jk}\right)\right],$$

$$\hat{\mathbf{e}}_{jk} = \frac{\mathbf{R}_{jk}}{|\mathbf{R}_{jk}|},$$

$$\mathbf{T}_{jk}^1 = -\frac{\mathrm{Re}}{32\pi\mu a}\left(3\mathbf{I} - \widehat{\mathbf{U}}\widehat{\mathbf{U}}\right) + \sum_{m_3}\mathbf{T}_{00,1m_3}^{00}Y_1^{m_3} + \sum_{m_3}\mathbf{T}_{00,3m_3}^{00}Y_3^{m_3} + \ldots ,$$

$$\mathbf{T}_{00,1m_3}^{00} \sim 0(\mathrm{Re}), \qquad \mathbf{T}_{00,3\mathrm{m}_3}^{00} \sim 0(\mathrm{Re}), \qquad \mathrm{R_{jk}U}/\nu < 1.$$

The first contributions to the above tensors describe the Stokes interactions, the second terms, respectively, the 0(Re) Oseen interactions. Taking into account the properties of the tensors $\mathbf{T}_{l_1m_1}^{l_2m_2}$, the formula (4.1) yields the series expansion of the f_{j,l_1m_1} with respect to σ and Re. For example, the components $\mathbf{f}_{j,00}$ are equal to:

(i) for the case of the flow past one sphere:

$$\mathrm{f}_{j,00} = -\left[\widetilde{\mathbf{T}}_j + \widetilde{\mathbf{T}}_j^1 + \ldots\right]\cdot\mathbf{U} = \mathrm{f}_j + \mathrm{f}_j^1 + \ldots , \tag{4.4}$$

where f_j denotes the Stokes drag force, f_j^1 – the 0(Re) Oseen force;

(ii) for the case of the flow past N spheres:

$$\begin{aligned}\mathrm{f}_{j,00} = -\widetilde{\mathbf{T}}_{00}^{00}(\mathbf{O}_j)\Bigg\{1 - \sum_{k\neq j}\Bigg[\mathbf{T}_{00}^{00}(\mathbf{R}_{jk})\cdot\widetilde{\mathbf{T}}_{00}^{00}(\mathbf{O}_k) + \sum_m \mathbf{T}_{00}^{1m}(\mathbf{R}_{jk})\cdot\widetilde{\mathbf{T}}_{1m}^{00}(\mathbf{O}_k)\Bigg] \\ + \sum_{k\neq j}\sum_{l\neq k}\mathbf{T}_{00}^{00}(\mathbf{R}_{jk})\cdot\widetilde{\mathbf{T}}_{00}^{00}(\mathbf{O}_k)\cdot\mathbf{T}_{00}^{00}(\mathbf{R}_{kl})\cdot\widetilde{\mathbf{T}}_{00}^{00}(\mathbf{O}_l) \\ - \sum_{k\neq j}\sum_{l\neq k}\sum_{n\neq l}\mathbf{T}_{00}^{00}(\mathbf{R}_{jk})\cdot\widetilde{\mathbf{T}}_{00}^{00}(\mathbf{O}_k)\cdot\mathbf{T}_{00}^{00}(\mathbf{R}_{kl})\cdot\widetilde{\mathbf{T}}_{00}^{00}(\mathbf{O}_l)\cdot\mathbf{T}_{00}^{00}(\mathbf{R}_{ln})\cdot\widetilde{\mathbf{T}}_{00}^{00}(\mathbf{O}_n)\Bigg\}\cdot\mathbf{U} \\ + \sum_{k\neq j}\sum_m \widetilde{\mathbf{T}}_{00}^{1m}(\mathbf{O}_j)\cdot\mathbf{T}_{1m}^{00}(\mathbf{R}_{jk})\cdot\widetilde{\mathbf{T}}_{00}^{00}(\mathbf{O}_k)\cdot\mathbf{U} + \ldots ,\end{aligned} \tag{4.5}$$

where, taking into account (4.3), the Stokes f_j and the 0(Re) f_j^1 contributions can be separated. The above expression is written down up to the terms of the order of $0(\sigma^2)$ in inverse sphere spacing. In that approximation, the four body effects enter the formula (4.5). Hence, to analyse that range of the interactions, a pairwise additivity assumption cannot be used. The more detailed discussion of the properties of $\mathbf{f}_{j,lm}$ can be found in the paper [1].

5. The velocity field in the range $\mathrm{Re} < 1$, $(\mathrm{Re}/\kappa) < 1$

It follows from (2.8) that the velocity field can be presented as the sum of the contributions, generated by each sphere in the presence of $(N-1)$ other spheres. In view of the properties of the velocity tensors, the contributions exhibit different features in the regions near to and far from the assemblage of the spheres. The velocity of the fluid in the region, characterized by $P_k U/\nu < 1$, assumes the form of the sum of the Stokes (linear in $\mathbf{U}$) and the Oseen (quadratic with respect to $\mathbf{U}$) terms.

Within the considered approximation (i.e. including the contributions up to $0(\kappa^2)$), the Stokes terms can be expressed by means of the four Stokes velocity tensors ($\mathbf{C}^{00}_{00}$, $\mathbf{C}^{00}_{2m_3}$, $\mathbf{C}^{1m_2}_{1m_3}$, $\mathbf{C}^{1m_2}_{3m_3}$), whereas the description of the 0(Re) terms requires, in addition, the tensors $\mathbf{C}^{00}_{1m_3}$, $\mathbf{C}^{00}_{3m_3}$, $\mathbf{C}^{1m_2}_{00}$, $\mathbf{C}^{1m_2}_{2m_3}$ and $\mathbf{C}^{1m_2}_{4m_3}$. Below we continue the list of the relevant tensors (comp. (3.7), (3.8), (3.12), (3.13), (3.14)):

(i) the Stokes velocity tensors:

$$(5.1)\qquad \mathbf{C}^{1m_2}_{1m_3}(P_k) = \frac{1}{6\sqrt{\pi}a\mu}\left(\frac{a}{P_k}\right)^2(-1)^{(m_2-m_3-|m_2|-|m_3|)/2}\left\{-\frac{2}{3}(-1)^{m_3}\delta_{m_2,m_3} + \sqrt{\frac{1}{5}}\sum_{m_6=-2}^{2}\delta_{m_2+m_6,m_3}(-1)^{(m_6-|m_6|)/2}\begin{pmatrix} 2 & 1 & 1 \\ m_6 & m_2 & -m_3\end{pmatrix}\mathbf{K}_{m_6}\right\},$$

$$(5.2)\qquad \mathbf{C}^{1m_2}_{3m_3}(P_k) = -\frac{\sqrt{3}}{8\sqrt{10\pi}a\mu}\left(\frac{a}{P_k}\right)^2(-1)^{(m_2-m_3-|m_2|-|m_3|)/2}\sum_{m_6=-2}^{2}\delta_{m_2+m_6,m_3} \cdot(-1)^{(m_6-|m_6|)/2}\begin{pmatrix} 2 & 1 & 3 \\ m_6 & m_2 & -m_3\end{pmatrix}\mathbf{K}_{m_6} + 0\left(\left(\frac{a}{P_k}\right)^4\right);$$

(ii) the 0(Re) velocity tensors (for the case of $\hat{U}(0,0,1)$):

$$(5.3)\qquad \mathbf{C}^{1m_2}_{2m_3}(P_k) = \frac{\sqrt{2}\mathrm{Re}}{18\sqrt{\pi}a\mu}\left(\frac{a}{P_k}\right)(-1)^{(m_2-m_3-|m_2|-|m_3|)/2}\left\{\delta_{m_2,m_3} \cdot\begin{pmatrix} 1 & 2 & 1 \\ m_2 & -m_3 & 0\end{pmatrix} - \frac{\sqrt{3}}{5}\sum_{m_6}\delta_{m_2+m_6,m_3}(-1)^{(m_6-|m_6|)/2}\left[-\frac{9}{4\sqrt{7}} \cdot\begin{pmatrix} 1 & 2 & 3 \\ m_2 & -m_3 & m_6\end{pmatrix} + \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 2 & 1 \\ m_2 & -m_3 & m_6\end{pmatrix}\right]\mathbf{K}_{m_6}\right\} + 0\left(\left(\frac{a}{P_k}\right)^3\right),$$

$$\mathbf{C}^{1m_2}_{4m_3}(P_k) = \frac{\sqrt{7}\mathrm{Re}}{60\sqrt{6\pi}a\mu}\left(\frac{a}{P_k}\right)(-1)^{(m_2-m_3-|m_2|-|m_3|)/2}\sum_{m_6}\delta_{m_6+m_2,m_3} \cdot (-1)^{(m_6-|m_6|)/2}\begin{pmatrix} 3 & 1 & 4 \\ m_6 & m_2 & -m_3 \end{pmatrix}\mathbf{K}_{m_6} + 0\left(\left(\frac{a}{P_k}\right)^3\right). \tag{5.4}$$

The Stokes and the 0(Re) contributions to the velocity field are expressed in terms of the listed velocity tensors, acting on the respective components of the induced forces. The components are the results of the many-sphere hydrodynamic, non-additive interactions. Up to the contributions of the order of $0(\sigma^2)$, the non-additive interactions of three and four spheres enter the description of the velocity of the fluid. In Table 1 we have written down the admissible (from

Table 1. The velocity field ($\boldsymbol{\nu}(\mathbf{r}) - \mathbf{U}$, cf. (2.8)) near to N spheres ($P_k U/\nu < 1$), including terms up to $0(\kappa^2)$ and $0(\sigma^2)$, $j = 1, ..., N$.

	Stokes contributions	Oseen contributions
κ^0	–	$\sum_j \mathbf{C}^1_j Y^0_0 \cdot \mathbf{f}_j$ $\sum_j \sum_{m_3} \mathbf{C}^{00}_{1m_3} Y^{m_3}_1 \cdot \mathbf{f}_j$ $\sum_j \sum_{m_3} \mathbf{C}^{00}_{3m_3} Y^{m_3}_3 \cdot \mathbf{f}_j$
κ^1	$\sum_j \mathbf{C}_j Y^0_0 \cdot \mathbf{f}_j$ $\sum_j \sum_{m_3} \mathbf{C}^{00}_{2m_3} Y^{m_3}_2 \cdot \mathbf{f}_j$	$\sum_j \mathbf{C}_j Y^0_0 \cdot \mathbf{f}^1_j$ $\sum_j \sum_{m_3} \mathbf{C}^{00}_{2m_3} Y^{m_3}_2 \cdot \mathbf{f}^1_j$ $\sum_j \sum_{m_2} \mathbf{C}^{1m_2}_{00} Y^0_0 \cdot \mathbf{f}_{j,1m_2}$ $\sum_j \sum_{m_2,m_3} \mathbf{C}^{1m_2}_{2m_3} Y^{m_3}_2 \cdot \mathbf{f}_{j,1m_2}$ $\sum_j \sum_{m_2,m_3} \mathbf{C}^{1m_2}_{4m_3} Y^{m_3}_4 \cdot \mathbf{f}_{j,1m_2}$
κ^2	$\sum_j \sum_{m_2,m_3} \mathbf{C}^{1m_2}_{1m_3} Y^{m_3}_1 \cdot \mathbf{f}_{j,1m_2}$ $\sum_j \sum_{m_2,m_3} \mathbf{C}^{1m_2}_{3m_3} Y^{m_3}_3 \cdot \mathbf{f}_{j,1m_2}$	$\sum_j \sum_{m_3} \mathbf{C}^{00}_{1m_3} Y^{m_3}_1 \cdot \mathbf{f}_j$ $\sum_j \sum_{m_3} \mathbf{C}^{00}_{3m_3} Y^{m_3}_3 \cdot \mathbf{f}_j$ $\sum_j \sum_{m_2,m_3} \mathbf{C}^{1m_2}_{1m_3} Y^{m_3}_1 \cdot \mathbf{f}_{j,1m_2}$ $\sum_j \sum_{m_2,m_3} \mathbf{C}^{1m_2}_{3m_3} Y^{m_3}_3 \cdot \mathbf{f}_{j,1m_2}$

the point of view of the properties of the velocity tensors and of the components $\mathbf{f}_{j,lm}$) sequences of the hydrodynamic interactions.

We note the following qualitative properties of the velocity profile, due to the inertia of the fluid:

(i) the velocity exhibits the stronger, than under the Stokes conditions, dependence on the non-additivity of the interactions (at the Stokes regime the non-additive interactions of three spheres enter);

(ii) the components $\mathbf{f}_{j,00}$ generate the contributions to the velocity field starting from the terms of the order of $0(\kappa^0)$ (at the Stokes regime, respectively, from the terms of $0(\kappa^1)$);

(iii) the components $\mathbf{f}_{j,1m}$ generate the analogous contributions, starting from the terms of the order of $0(\kappa^1)$ (at the Stokes conditions, respectively, from the terms of $0(\kappa^2)$);

(iv) there appear the contributions, expressed in terms of the functions Y_1^m and Y_3^m, which describe the lack of the fore-aft symmetry.

We note that for the particular case of one sphere, the tensor $\mathbf{C}_j$ and $\mathbf{C}_{2m_3}^{00}$, acting on the component $\mathbf{f}_j = -6\pi\mu a\mathbf{U}$, give the classical Stokes velocity profile. To our knowledge, the description of the velocity field past N spheres, involving the Oseen hydrodynamic interactions between more than two spheres, is at present not available in the literature. Summing up, in the present paper it has been investigated to what extent the weak convective inertia of the fluid increases the role of the hydrodynamic interactions and modifies the symmetry properties of the generated velocity field.

Appendix. Series expansion of the tensors $\mathbf{C}_{l_3m_3}^{l_2m_2}$ with respect to P_kU/ν

The tensors $\mathbf{C}^{l_2m_2}(\mathbf{P}_k)$ describe the effect of the component $\mathbf{f}_{k,l_2m_2}$ of the force, induced on the surface of the k-th sphere, on the velocity field of the fluid. The tensors $\mathbf{C}_{l_3m_3}^{l_2m_2}(P_k)$ concern the respective radial properties. To examine these properties, we start with the integration over $|\mathbf{k}|$ in the expression (2.10). To this end we use the properties of the Bessel functions $J_{l+1/2}$, expressed by the formula (7) on the page 45, and by the formula (7) on the page 99 of [9], and we apply the expansion of $\mathbf{1} - \hat{\mathbf{k}}\hat{\mathbf{k}}$ in terms of Y_2^m [5]:

$$\mathbf{1} - \hat{\mathbf{k}}\hat{\mathbf{k}} = \frac{2}{3}I - \sqrt{\frac{2\pi}{15}} \sum_{m_6=-2}^{2} \mathbf{K}_{m_6} Y_2^{m_6}, \tag{A.1}$$

where

$$\begin{aligned}
\mathbf{K}_0 &= \sqrt{\frac{2}{3}}\left(-\mathbf{e}_x\mathbf{e}_x - \mathbf{e}_y\mathbf{e}_y + 2\mathbf{e}_z\mathbf{e}_z\right),\\
\mathbf{K}_{\pm 1} &= \mathbf{e}_x\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_x \mp i\mathbf{e}_y\mathbf{e}_z \mp i\mathbf{e}_z\mathbf{e}_y\,,\\
\mathbf{K}_{\pm 2} &= \mathbf{e}_x\mathbf{e}_x - \mathbf{e}_y\mathbf{e}_y \mp i\mathbf{e}_x\mathbf{e}_y \mp i\mathbf{e}_y\mathbf{e}_x\,.
\end{aligned}$$

As a result, we obtain the following expression:

$$\text{(A.2)}\qquad \mathbf{C}^{l_2 m_2}_{l_3 m_3} = \sum_{m=0}^{\infty} \beta_m(l_2, l_3) \int d\Omega_k \left[\frac{2}{3} I - \sqrt{\frac{2\pi}{15}} \sum_{m_6=-2}^{2} \mathbf{K}_{m_6} Y_2^{m_6} \right] Y_{l_2}^{m_2} Y_{l_3}^{-m_3} \cdot \int d\tilde{k} \frac{\tilde{k} - iP_k\alpha\xi}{\tilde{k}^2 + (P_k\alpha\xi)^2} J_{l_2+1/2+2m}(\tilde{k}) J_{l_3+1/2}(\tilde{k}),$$

where

$$\text{(A.3)}\qquad \beta_m(l_2, l_3) = \frac{i^{l_2-l_3}}{2a\mu} \left(\frac{a}{P_k} \right)^{l_2+1} \cdot (l_2)! \sum_{n=0}^{l_2} \frac{(-1)^n (2l_2 - 2n + 1) \left(l_2 + \frac{1}{2} + 2m \right) \Gamma\left(l_2 + \frac{1}{2} + m \right)}{n!(2l_2 - n + 1)! \Gamma\left(l_2 - n + \frac{3}{2} \right) \Gamma\left(-l_2 + n + \frac{1}{2} \right) m!} \cdot F_4 \left[-m, l_2 + \frac{1}{2} + m; l_2 - n + \frac{3}{2},\ -l_2 + n + \frac{1}{2}; \left(\frac{a}{2P_k} \right)^2, \left(\frac{a}{2P_k} \right)^2 \right],$$

F_4 is the hypergeometric series,

$$\alpha = U/\nu, \qquad \hat{U} = \mathbf{U}/U, \qquad \xi = \cos(\hat{U}, \hat{k}), \qquad \tilde{k} = P_k k\,.$$

To perform the $\tilde{k}$ integration, we apply a rotation of the coordinate system by the linear transformation, $\hat{\mathbf{k}} = \mathbf{A} \cdot \hat{\mathbf{y}}$, letting the new axis 3 coincide with $\mathbf{U}$. Then, taking into account the properties of the functions Y_l^m:

$$\text{(A.4)}\qquad Y_l^q(\hat{k}) = \sum_{|n| \le l} R(l, q, n) Y_l^n(\hat{y}),$$

and using the formula (6.577) from [10], we obtain:

$$\text{(A.5)}\qquad \mathbf{C}^{l_2 m_2}_{l_3 m_3} = \pm \sum_{m=0} \beta_m(l_2, l_3) \int_{\xi>0} d\Omega_y \{\ldots\} i^{|l_2+2m-l_3|} I_{\tilde{z}}(P_k\alpha\xi) K_{\tilde{\varrho}}(P_k\alpha\xi) + \sum_{m=0}^{\infty} \beta_m(l_2, l_3) \int_{\xi\le 0} d\Omega_y \{\ldots\} i^{|l_2+2m-l_3|} I_{\tilde{z}}(P_k\alpha|\xi|) K_{\tilde{\varrho}}(P_k\alpha|\xi|),$$

where $(+)$ refers to the cases $l_2 + l_3 = 2n$, and $(-)$ – to the cases $l_2 + l_3 = 2n + 1$, $I_{\tilde{z}}$ and $K_{\tilde{\varrho}}$ denote the modified Bessel functions, $\tilde{z} = \max(l_2 + 1/2 + 2m,\ l_3 + 1/2)$, $\tilde{\varrho} = \min(l_2 + 1/2 + 2m,\ l_3 + 1/2)$, and the expression in the parentheses reads:

$$\text{(A.6)}\qquad \{\ldots\} = \sum_{m_4, m_5} R(l_2, m_2, m_4) R(l_3, -m_3, -m_5) \left[\frac{2}{3} I - \sqrt{\frac{2\pi}{15}} \sum_{m_6, m_7} \mathbf{K}_{m_6} R(2, m_6, m_7) \cdot Y_2^{m_7}(\hat{y}) \right] Y_{l_2}^{m_4}(\hat{y}) Y_{l_3}^{-m_5}(\hat{y}).$$

Carrying out the integrations over the meridional angle, we arrive at:

$$
\text{(A.7)}\qquad \mathbf{C}^{l_2 m_2}_{l_3 m_3} = \pm 4\pi \sum_{m}^{\infty} \beta_m(l_2, l_3) i^{|l_2+2m-l_3|} \cdot \sum_{m_4, m_5} R(l_2, m_2, m_4) R(l_3, -m_3, -m_5) d^{m_4}_{l_2} d^{-m_5}_{l_3}
$$
$$
\cdot \int_0^1 d\xi \left[\frac{2}{3}\delta_{m_4,m_5} - \sqrt{\frac{2\pi}{15}} \sum_{m_6,m_7} \delta_{m_7+m_4,m_5} \mathbf{K}_{m_6} R(2, m_6, m_7) d^{m_7}_{2} P^{m_7}_{2}(\xi) \right] \cdot P^{m_4}_{l_2}(\xi) P^{-m_5}_{l_3}(\xi) I_{\tilde{z}}(P_k \alpha \xi) K_{\tilde{\varrho}}(P_k \alpha \xi),
$$

where the coefficients d^m_l read:

$$
d^m_l = (-1)^{(m-|m|)/2} \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}\,.
$$

$P^m_l(\xi)$ are the associated Legendre functions [cf. [6], formulae (2.5.17) and (2.5.18)].

Taking into account the properties of $P^m_l(\xi)$, we can accomplish the integration with the help of the formula 1.11.3.2 from [11]:

$$
\text{(A.8)}\qquad \int_0^1 d\xi\, \xi^{\lambda} I_{\tilde{z}}(P_k \alpha \xi) K_{\tilde{\varrho}}(P_k \alpha \xi)
$$
$$
= \frac{\pi}{2 \sin \tilde{\varrho}\pi} \Bigg[\frac{(P_k\alpha/2)^{\tilde{z}-\tilde{\varrho}}}{(\lambda + \tilde{z} - \tilde{\varrho} + 1)\Gamma(\tilde{z}+1)\Gamma(-\tilde{\varrho}+1)} \cdot {}_3F_4 \left[\frac{\lambda + \tilde{z} - \tilde{\varrho} + 1}{2}, \frac{\tilde{z} - \tilde{\varrho} + 1}{2}, \frac{\tilde{z} - \tilde{\varrho} + 2}{2}; \tilde{z}+1,\ -\tilde{\varrho}+1,\ \tilde{z} - \tilde{\varrho} + 1,\ \frac{\lambda + \tilde{z} - \tilde{\varrho} + 3}{2};\ (P_k\alpha)^2 \right]
$$
$$
- \frac{(P_k\alpha/2)^{\tilde{z}+\tilde{\varrho}}}{(\lambda + \tilde{z} + \tilde{\varrho} + 1)\Gamma(\tilde{z}+1)\Gamma(\tilde{\varrho}+1)} \cdot {}_3F_4 \left[\frac{\lambda + \tilde{z} + \tilde{\varrho} + 1}{2}, \frac{\tilde{z} + \tilde{\varrho} + 1}{2}, \frac{\tilde{z} + \tilde{\varrho} + 2}{2}; \tilde{z}+1,\ \tilde{\varrho}+1,\ \tilde{z} + \tilde{\varrho} + 1,\ \frac{\lambda + \tilde{z} + \tilde{\varrho} + 3}{2};\ (P_k\alpha)^2 \right] \Bigg],
$$

where the parameter λ depends on the $P^m_l(\xi)$ involved, and the following condition should be fulfilled:

$$
\lambda + \tilde{z} + \tilde{\varrho} > -1.
$$

The last integration leads to the representation of the velocity field tensors in terms of the linear combinations of the ${}_3F_4$ functions. That representation is valid for the arbitrary values of Re. We are going to examine the particular regime of the hydrodynamic interactions, which is described by the velocity tensors, having arguments $P_k\alpha < 1$.

References

1. I. PIEŃKOWSKA, Arch. Mech., **48**, 231, 1996.
2. T. KUMAGAI, JSME Int. J., Ser. B, **38**, 563, 1995.
3. P. MAZUR and W. VAN SAARLOOS, Physica A, **115**, 21, 1982.
4. P. MAZUR and A.J. WEISENBORN, Physica A, **123**, 209, 1984.
5. T. YOSHIZAKI and H. YAMAKAWA, J. Chem. Phys., **73**, 578, 1980.
6. A.R. EDMONDS, *Angular momentum in quantum mechanics*, Princeton University Press, 1974.
7. I. PIEŃKOWSKA, Arch. Mech., **45**, 665, 1993.
8. R. BERKER, *Encyclopedia of Physics*, vol. VIII/2, S. FLÜGGE, Springer, 1963.
9. A. ERDÉLYI, *Higher transcendental functions*, vol. II, New York 1953.
10. I.S. GRADŠTEIN and I.M. RYZHIK, *Tables of integrals, series and products* [in Russian], Moskva 1962.
11. A.P. PRUDNIKOV, YU.A. BRYČKOV and O.I. MARIČEV, *Integrals and series, special functions* [in Russian], Nauka, 1983.
12. L. DURLOFSKY, J.F. BRADY and G. BOSSIS, J. Fluid Mech., **180**, 21, 1987.
13. R.J. PHILLIPS, J. Fluid Mech., **315**, 345, 1996.
14. I. PROUDMAN and J.R.A. PEARSON, J. Fluid Mech., **2**, 237, 1957.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH
e-mail: ipienk@ippt.gov.pl

Received March 5, 1997.