

# The shock wave structure by model equations of capillarity

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IN THIS PAPER we investigate the influence of the capillarity terms on the shock wave structure. To this end we compare the shock structures derived from the viscosity-capillarity model and from the Navier–Stokes equations, i.e. the viscosity model. Let  $A$  and  $\varepsilon$  characterize the values of the capillarity and viscosity effects, respectively. First, we prove that if the ratio  $A/\varepsilon^2 \ll 1$  then the viscosity-capillarity and viscosity shock structures differs but only a little. Secondly, if  $A/\varepsilon^2 \gg 1$  then the viscosity-capillarity shock waves are oscillatory, whereas the viscosity waves are never such. Thirdly, to investigate the intermediate case of  $A/\varepsilon^2 \approx 1$  we study numerically so-called impending shock splitting. This effect consists in that the shock profile has two inflection points, under suitably chosen data, instead of one, what is usual. Our calculations show that the capillarity, if strong enough, kills this effect totally.

## 1. Introduction

AS IT IS WELL KNOWN, the Navier–Stokes equations with the van der Waals equation of state rule out certain experimentally observed phase boundaries. To circumvent this difficulty, SLEMROD [1] for van der Waals fluids and TRUSKINOVSKY [2] for elastic bars introduced higher order gradients to the dispersive equations basing on the Korteweg’s theory of capillarity. Consequently, the viscosity-capillarity equations were used mainly to study various problems concerning phase transitions.

The aim of this paper is an investigation of the influence of the capillarity effects on the shock wave structure. To our knowledge this topic is almost untouched. Exceptions are the papers by AFFOUF and CAFLISCH [3], and by ABEYARATNE and KNOWLES [4] where some results concerning our problem can be found.

We start with the capillarity equations deduced from a kinetic four-velocity model of the Enskog–Vlasov equation [5], and look for solutions in the form of travelling waves.

We define the shock wave as a plane travelling wave which is supersonic with respect to the sound speed in the equilibrium state ahead of it and subsonic with respect to that behind it.

We distinguish between viscosity and viscosity-capillarity shock waves. The shock wave is, by definition, a viscosity wave if its structure is described by the Navier–Stokes equations, and it is a viscosity-capillarity one if its structure is described by the capillarity equations, i.e. the Navier–Stokes equations with additional terms representing capillarity forces.

In Sec. 2 we present equations which we use to describe shock wave structures, and show that the viscosity-capillarity shocks can be oscillatory, provided that the capillarity coefficient is sufficiently large, whereas the viscosity shocks have monotone profiles.

In Sec. 3 we discuss the problem of so-called impending shock splitting. Roughly speaking, the question is that within the Navier–Stokes equations with a non-convex equation of state like the van der Waals one, one can choose such state after the wave and its speed that the shock profile has two inflection points instead of one, what is usual [7]. In this section we reconsider this problem, but within the framework of the capillarity equations. Numerically we show that if the capillarity coefficient is very small, then the viscosity-capillarity shock wave profiles differ but a little from the corresponding viscosity ones. However, our calculations show that the increase of the capillarity coefficient kills the effect of the impending shock splitting, so it should be treated as an artefact introduced by the Navier–Stokes equations.

In Sec. 4 we show how to extend these results to other scalar equations of travelling waves.

We complete the paper with Sec. 5, where we prove rigorously that if the capillarity coefficient is small enough, then the viscosity and capillarity shock wave profiles differ by a little only.

## 2. The model equations of capillarity and the travelling waves

The model equations of capillarity we consider in this paper consist of the following system of two partial differential equations [5]

$$(2.1) \quad \frac{\partial}{\partial t} w - \frac{\partial}{\partial x} u = 0,$$

$$(2.2) \quad \frac{\partial}{\partial t} u + \frac{\partial}{\partial x} p(w, u) = \varepsilon \frac{\partial}{\partial x} \left( \mu \frac{\partial}{\partial x} u \right) + A \frac{\partial}{\partial x} \left[ \frac{5}{w^6} \left( \frac{\partial}{\partial x} w \right)^2 - \frac{2}{w^5} \frac{\partial^2}{\partial x^2} w \right].$$

In (2.1), (2.2),  $t > 0$  is the time,  $x \in \mathbb{R}$  is the Lagrangian mass coordinate,  $u$  is the velocity,  $w$  is the specific volume,  $p$  is the pressure, and  $\varepsilon\mu$  is the coefficient of viscosity.

The pressure formula reads

$$(2.3) \quad p = p(w, u) = \frac{1 - u^2}{2(w - b)} - \frac{a}{w^2},$$

where  $a$  and  $b$  are positive constants;  $a$  is the ratio of the mean value of the potential of the attractive intermolecular forces to the mean kinetic energy of molecules, and  $b$  can be taken to be equal to unity.

Next,  $\varepsilon > 0$  is a parameter, and  $\mu = \mu(w, u)$  is given by

$$(2.4) \quad \mu(w, u) = \frac{1 - u^2 + 2b^2w^2\rho^2(w)}{8w^2\rho(w)}, \quad \rho(w) = \frac{w}{w - b}.$$

Finally,  $A > 0$  is another parameter, the term proportional to it represents the capillarity forces. Therefore  $A$  is called the capillarity coefficient.

We consider Eqs. (2.1), (2.2) in the domain  $\mathcal{D}$  defined by [6]

$$(2.5) \quad \mathcal{D} = \left\{ (w, u) : w > b, \quad u^2 < 1 - \frac{a}{2b}, \quad \frac{a}{2b} < 1 \right\}.$$

For  $(w, u) \in \mathcal{D}$ , the mass density  $1/w$  does not exceed the close-packing density  $1/b$ , and the pressure  $p$  is positive (also the viscosity  $\mu$  is strictly positive).

Two simplified versions of our equations, namely the first one with  $\varepsilon = 0$ ,  $A = 0$ , and the second one with  $\varepsilon > 0$  but  $A = 0$ , are called the Euler and Navier – Stokes model equations of hydrodynamics, respectively. They were analyzed in [6].

A travelling wave solution to (2.1), (2.2) is a solution of the form

$$(2.6) \quad (w, u)(x, t) = (w, u)(z), \quad z = \frac{x - st}{\varepsilon} \in \mathbb{R},$$

where  $s = \text{const}$  is the wave-speed, such that

$$(2.7) \quad \lim_{z \rightarrow -\infty} (w, u)(z) = (w_l, u_l),$$

$$(2.8) \quad \lim_{z \rightarrow \infty} (w, u)(z) = (w_r, u_r),$$

$$(2.9) \quad \lim_{z \rightarrow \pm\infty} (w', u')(z) = (0, 0),$$

$$(2.10) \quad \lim_{z \rightarrow \pm\infty} (w'', u'')(z) = (0, 0),$$

where the dash  $'$  denotes differentiation with respect to  $z$ .

Usually, the left-hand equilibrium state  $(w_l, u_l)$  is treated as given, and the right-hand state  $(w_r, u_r)$  has to be determined. However, we proceed in a different way. Namely, we introduce the notions of the states before and after the wave:

the state before the wave is defined by

$$(2.11) \quad (w_b, u_b) = \begin{cases} (w_r, u_r) & \text{for } s > 0, \\ (w_l, u_l) & \text{for } s < 0, \end{cases}$$

and the state after the wave is by definition

$$(2.12) \quad (w_a, u_a) = \begin{cases} (w_l, u_l) & \text{for } s > 0, \\ (w, u) & \text{for } s < 0. \end{cases}$$

The case of  $s = 0$  is not considered in this paper.

We take the state after the wave as given.

Now, we act in a very standard way. Namely, we substitute (2.6) into Eqs. (2.1), (2.2), perform one integration with respect to  $z$ , and use the limit conditions (2.7) – (2.10). Having done that we find that the states before and after the wave are related algebraically

$$(2.13) \quad \begin{aligned} sw_b + u_b &= sw_a + u_a, \\ -su_b + p(w_b, u_b) &= -su_a + p(w_a, u_a). \end{aligned}$$

These relations are called the Rankine–Hugoniot conditions and were in detail analyzed in [6].

Next, we find the velocity  $u$ . It is given by

$$(2.14) \quad u = u_a - s(w - w_a),$$

where  $w$  is a solution of the following limit value problem

$$(2.15) \quad \alpha^2 \left[ \frac{2}{w^5} w'' - \frac{5}{w^6} w'^2 \right] + s\mu(w)w' + f(w) = 0,$$

where  $\alpha = A/\varepsilon^2$ , and

$$(2.16) \quad \mu = \mu(w) = \mu(w) = \mu(w, u_a - s(w - w_a)) > 0,$$

$$(2.17) \quad f(w) = s^2(w - w_a) + p(w, u_a - s(w - w_a)) - p(w_a, u_a),$$

subject to the conditions

$$(2.18) \quad \lim_{z \rightarrow -\infty} w(z) = \begin{cases} w_a & \text{for } s > 0, \\ w_b & \text{for } s < 0, \end{cases}$$

$$(2.19) \quad \lim_{z \rightarrow \infty} w(z) = \begin{cases} w_b & \text{for } s > 0, \\ w_a & \text{for } s < 0, \end{cases}$$

$$(2.20) \quad \lim_{z \rightarrow \pm\infty} w'(z) = 0, \quad \lim_{z \rightarrow \pm\infty} w''(z) = 0.$$

These conditions must be supplemented by Eqs. (2.13), which we write in the form

$$(2.21) \quad f(w_a) = 0, \quad f(w_b) = 0.$$

We take the following two assumptions:

A1. The equation  $f(w) = 0$  has no solutions between  $w_a$  and  $w_b$ .

A2. The following inequalities hold true

$$(2.22) \quad f'(w_a) < 0,$$

$$(2.23) \quad f'(w_b) > 0.$$

Explicitly (2.22) and (2.23) are equivalent to

$$(2.24) \quad s^2 - sp'_u(w_a, u_a) + p'_w(w_a, u_a) < 0,$$

and

$$(2.25) \quad s^2 - sp'_u(w_b, u_b) + p'_w(w_b, u_b) > 0,$$

respectively.

The characteristic speeds (sound speeds)  $c_{\pm}(w, u)$  are defined as the real solutions of (cf. [6])

$$(2.26) \quad c^2 - sp'_u(w, u) + p'_w(w, u) = 0.$$

Hence, (2.24) is equivalent to

$$(2.27) \quad c_-(w_a, u_a) < s < c_+(w_a, u_a),$$

what means that the wave is subsonic with respect to the state after it.

Next, we notice that (2.25) is equivalent to

$$(2.28) \quad s < c_-(w_b, u_b) \quad \text{or} \quad s > c_+(w_b, u_b),$$

i.e. the wave is supersonic with respect to the state ahead of it.

The solution of (2.15)–(2.21) with  $f(w)$  satisfying A1 and A2 is called the viscosity-capillarity shock wave, and the graph of the solution is called the shock wave structure or profile.

The aim of this paper is to compare the viscosity-capillarity shock waves to the viscosity ones, which are solutions of

$$(2.29) \quad s\mu(w)w' + f(w) = 0,$$

with  $\mu(w)$  and  $f(w)$  given by (2.16), (2.17), respectively, and satisfying the limit conditions (2.18), (2.19), and (2.20)<sub>1</sub>. We assume also that Eqs. (2.21) and A1, A2 hold true.

The first observations concerning the differences between the two descriptions can be derived from *the analysis of the points of equilibrium*. This is a standard procedure. We linearize equations (2.15) or (2.29) around, say,  $w = w_a$ , and find that the characteristic exponents satisfy

$$(2.30) \quad \alpha \frac{2}{w_a^5} \lambda_a^2 + s\mu(w_a)\lambda_a + f'(w_a) = 0,$$

if Eq. (2.15) is concerned, or

$$(2.31) \quad s\mu(w_a)\lambda_a + f'(w_a) = 0,$$

for the case of Eq. (2.29).

Thanks to (2.22), Eq. (2.30) has two real solutions  $\lambda_a^- < 0 < \lambda_a^+$ . Of course, the solution to (2.31) is always real.

If we perform the linearization around  $w = w_b$ , then the characteristic exponents, this time labelled with the subscript  $b$ , will satisfy an equation similar to (2.30) (respectively (2.31)). But this time, owing to (2.23) the characteristic exponents in the viscosity-capillarity case can be complex if  $\alpha$  is sufficiently large. The characteristic exponent in the viscosity case is always real.

Thus we obtain

#### OBSERVATION 1.

If  $\alpha$  is sufficiently large, then the viscosity-capillarity shock wave structure is oscillatory in the downstream part of its profile. The viscosity shock wave structure is always monotonic.

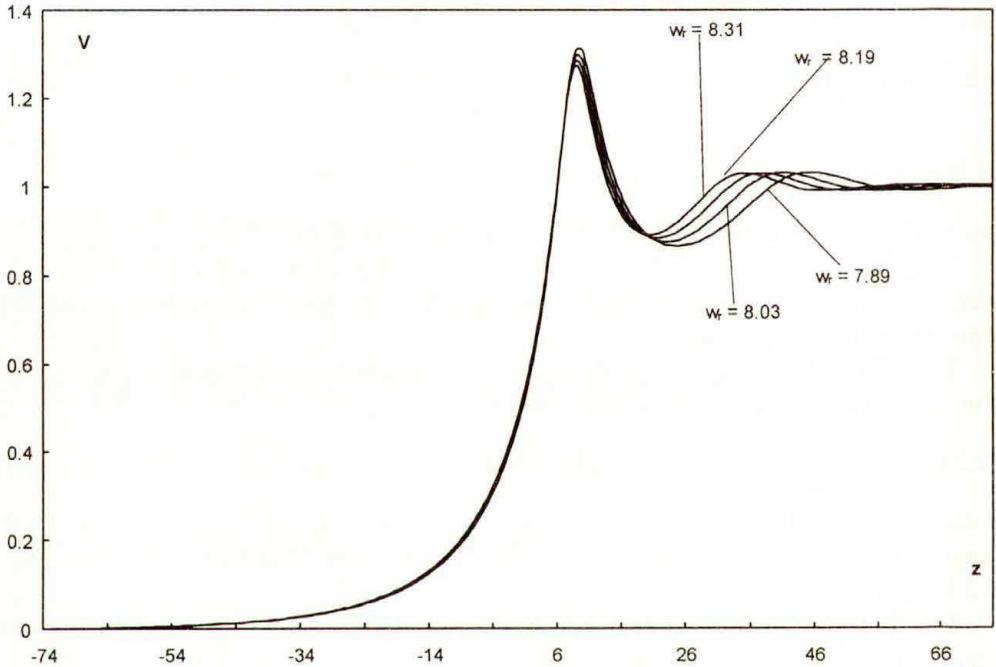


FIG. 1. Normalized oscillating shock structures,  $\alpha = 100$ .

The graphs of oscillatory shock waves are presented in Fig. 1 (see also [3]). Experimentally, oscillatory shock waves were observed in [11].

### 3. The impending shock splitting

The notion of impending shock splitting was introduced by CRAMER and CRICKENBERGER [7] to denote such shock structures which are monotone, but

have two inflection points instead of one, what is usual. If the impending shock wave occurs then, roughly speaking, the shock profile consists of two domains where rapid, shock-like changes take place, separated by a region in which the profile is quite flat.

Cramer and Crickenberger used the nonisothermal Navier–Stokes equations with a realistic equation of state, but they ignored capillarity forces. The profiles with an impending shock splitting are presented in Fig. 10 of their paper [7].

Later, using our model equations of van der Waals fluids with capillarity effects neglected we obtained quite similar results [6].

For the sake of completeness of our arguments we give a series of shock profiles exhibiting the impending shock wave splitting (see Fig. 2). Of course, they resemble the quoted results of [6] and [7].

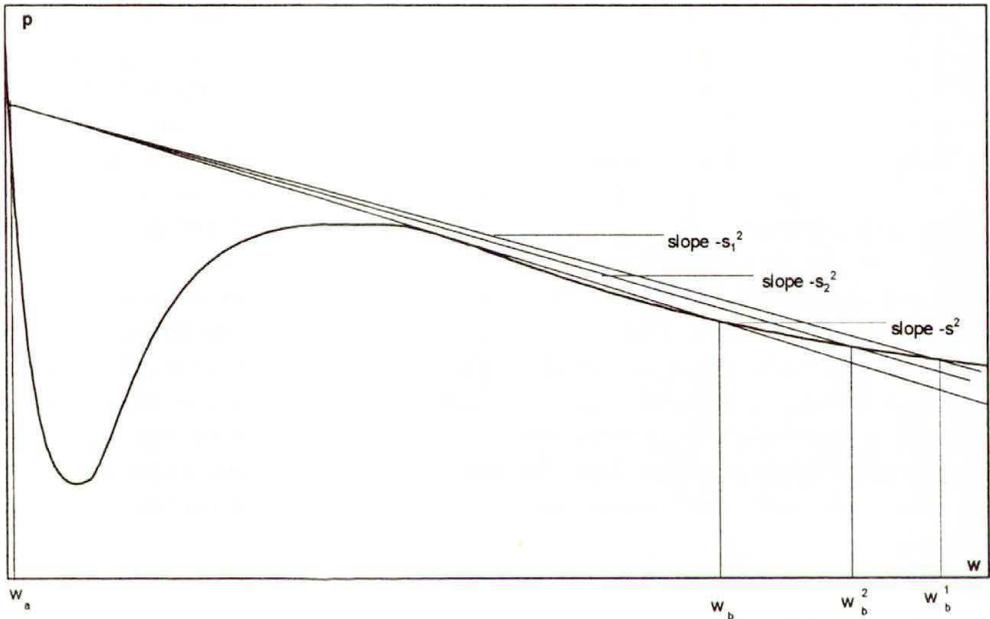


FIG. 2. Schematic diagram of the pressure and the Rayleigh radii.

We explain briefly how to obtain such results (see also [7]). First, we ignore the capillarity terms, i.e. we set  $\alpha = 0$  in Eq. (2.15). Consequently, we consider the problem consisting of Eq. (2.29) subject to (2.18), (2.19), and (2.20)<sub>1</sub>. We start from the simpler case of isothermal gas with the pressure  $p$  given by

$$(3.1) \quad p(w) = \frac{T}{w - b} - \frac{a}{w^2},$$

where  $T = \text{const}$  is the dimensionless temperature.

If  $T$  is such that

$$(3.2) \quad \frac{1}{4} \frac{a}{b} < T < \frac{81}{256} \frac{a}{b},$$

then the pressure is positive for  $w > b$ , but the graph of  $p(w)$  in the  $w-p$  plane is a concave or even nonmonotonic curve. Under (3.2), the curve  $p = p(w)$  has two points of inflection, say, at  $w = w^*$  and at  $w = w^{**}$ , such that  $b < w^* < w^{**}$ .

Let us take the state after the wave  $w_a$  such that  $b < w_a < w^*$ , and let  $\bar{s} > 0$  be such that the Rayleigh radius  $r(w) = -s^2(w - w_a) + p(w_a)$  is tangent to  $p = p(w)$  at  $w = \bar{w}$ ,  $w^* < \bar{w} < w^{**}$ . Now, we take a sequence  $\{s_n\}$  of speeds such that  $s_{n+1} > s_n > 0$ ,  $n = 1, 2, \dots$ , and  $\lim_{n \rightarrow \infty} s_n = \bar{s}$  (see Fig. 2). Then the Rayleigh radii  $r_n = -s_n^2(w - w_a) + p(w_a)$  lie above the graph of  $p = p(w)$  for  $w$  between  $w = w_a$  and  $w = w_b^{(n)}$ , where  $w = w_b^{(n)}$  is the  $w$ -coordinate of the other point of intersection of  $p = p(w)$  with  $p = r_n(w)$ . This guarantees that, for any  $n$ , the functions  $f_n(w) = p(w) - r_n(w)$  are negative for  $w$  between  $w_a$  and  $w_b^{(n)}$ . Hence, our limit value problems have unique solutions. However, for sufficiently large  $n$ , the Rayleigh radii are almost tangent to the graph of  $p = p(w)$ , but lying above it. Therefore, for sufficiently large  $n$ , the function  $f_n(w)$  can be arbitrarily close to zero in a vicinity of  $w = \bar{w}$ . Consequently, in this vicinity the derivative of  $w = w(z)$  with respect to  $z$  is close to zero. This means that the graphs of the solutions in the  $z-p$  plane become flatter and flatter, as  $n$  tends to infinity.

In our case, when the pressure depends not only on  $w$  but on  $u$  as well, the situation is more difficult. This is due to the fact that, when considering the shock wave problem, we have to consider  $p(w, u)$  along the Hugoniot locus (cf. [6]) what results in that the profiles of  $p(w, u)$  depend on  $s$  as well. So they change when changing  $s$ , but the Rayleigh radii remain unchanged. Luckily, by regrouping the parameters present in our problem we can reduce it to a form which resembles the isothermal case, and consequently we can use the construction described above.

Indeed, as it follows from (2.14) we can write

$$(3.3) \quad u = \tilde{u}_a - s(w - b),$$

where

$$(3.4) \quad \tilde{u}_a = u_a + s(w_a - b) = \text{const.}$$

Next, (2.3), (2.17) and the above yield

$$(3.5) \quad f(w) = \tilde{p}(w) - \tilde{p}(w_a) + \tilde{s}^2(w - w_a),$$

where

$$(3.6) \quad \tilde{p}(w) = \frac{T_a}{w - b} - \frac{a}{w^2},$$

with

$$(3.7) \quad T_a = \frac{1 - \tilde{u}_a^2}{2} = \text{const},$$

and

$$(3.8) \quad \tilde{s} = s/\sqrt{2}.$$

Formula (3.6) resembles (3.1) if we identify  $T_a$  with  $T$ .

When preparing the graphs presented in Figs. 1 and 3-7, we kept  $w_a$  and  $T_a$  fixed and changed  $s$ . Consequently we changed  $u_a$ , i.e. we took  $u_a = u_a(s)$ , in agreement with (3.4) and (3.7).

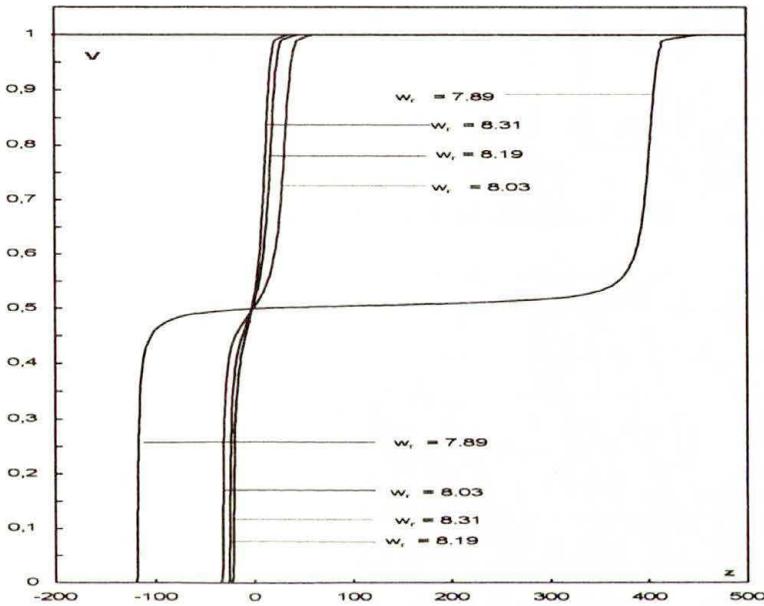


FIG. 3. Normalized shock structures for  $\alpha = 0$ .

Next, we reconsidered the shock wave structure problem using Eq. (2.15) and the same values of  $T_a$ ,  $w_a$ , and  $s_n$ . Let us notice that the Rankine-Hugoniot conditions (2.13) are the same in the viscosity and in the capillarity-viscosity case.

In Figs. 1 and 3-7 the normalized specific volume  $V$  defined by

$$V = (w - w_l)/(w_r - w_l)$$

is shown for  $\alpha = 100, 0, 0.5, 1.0, 5.0$ , and  $10.0$ .

Now, let us take a greater value of  $\alpha$ , i.e.  $\alpha = 1$  (Fig. 5). We see that the impending shock splitting is still observable, but the overall shock thickness has decreased by the factor of 4.

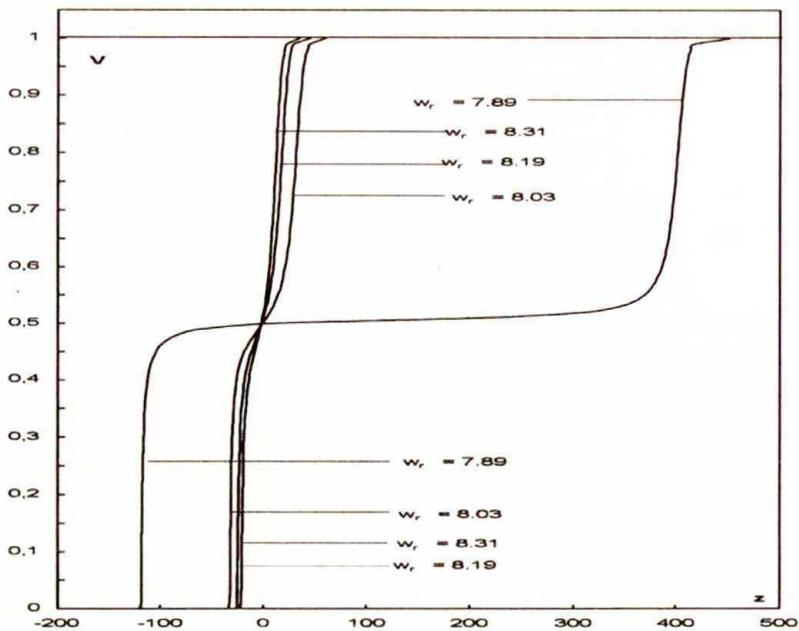


FIG. 4. Normalized shock structures for  $\alpha = 0.5$ .

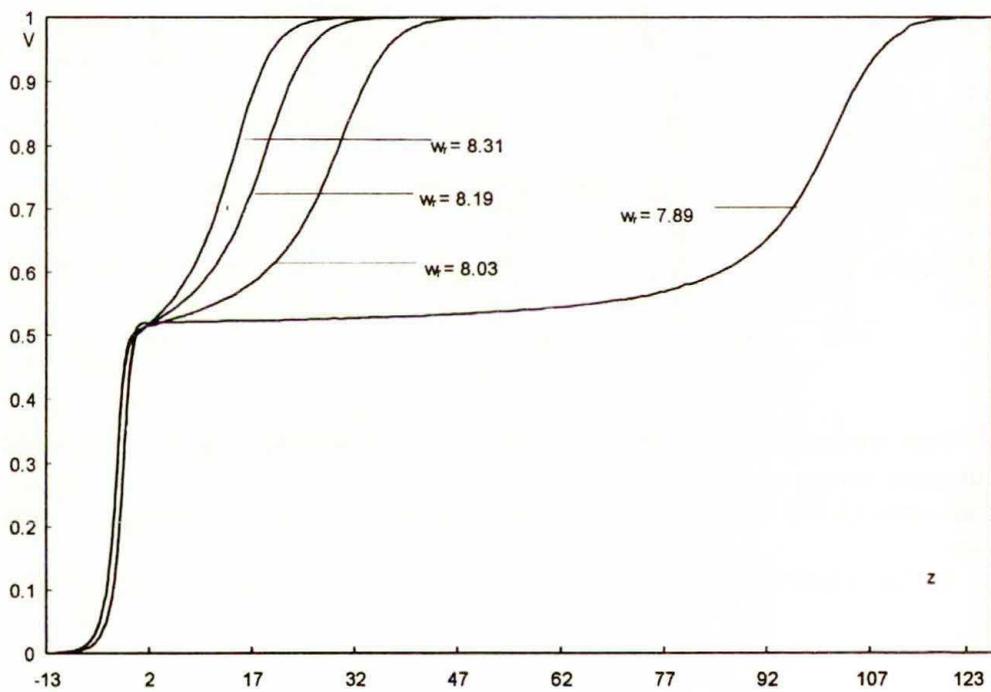


FIG. 5. Normalized shock structures for  $\alpha = 1$ .

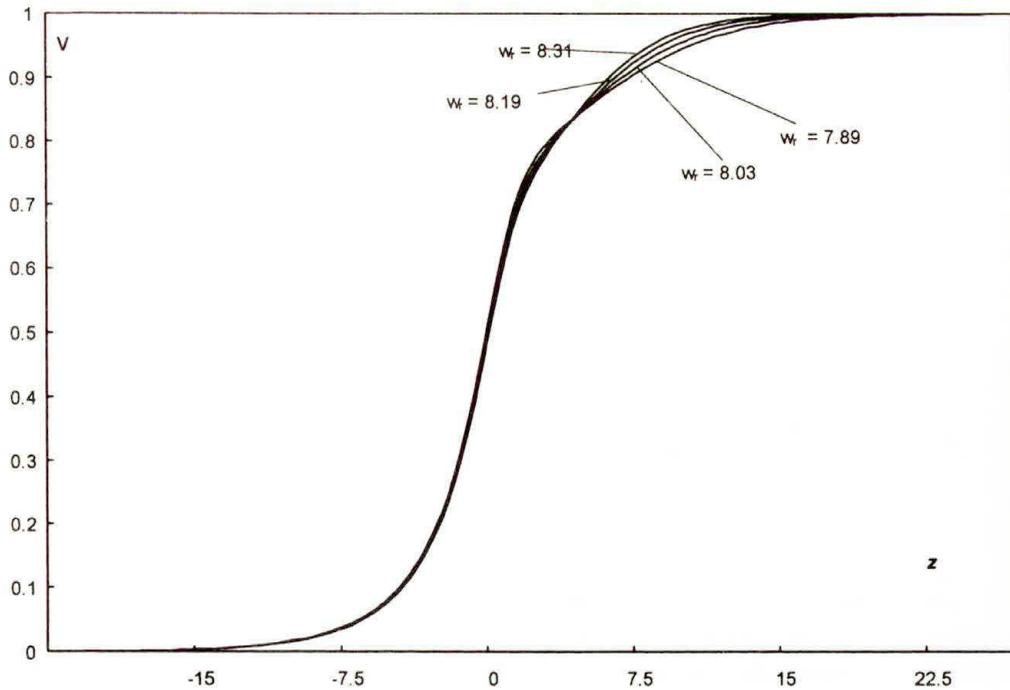


FIG. 6. Normalized shock structures for  $\alpha = 5$ .

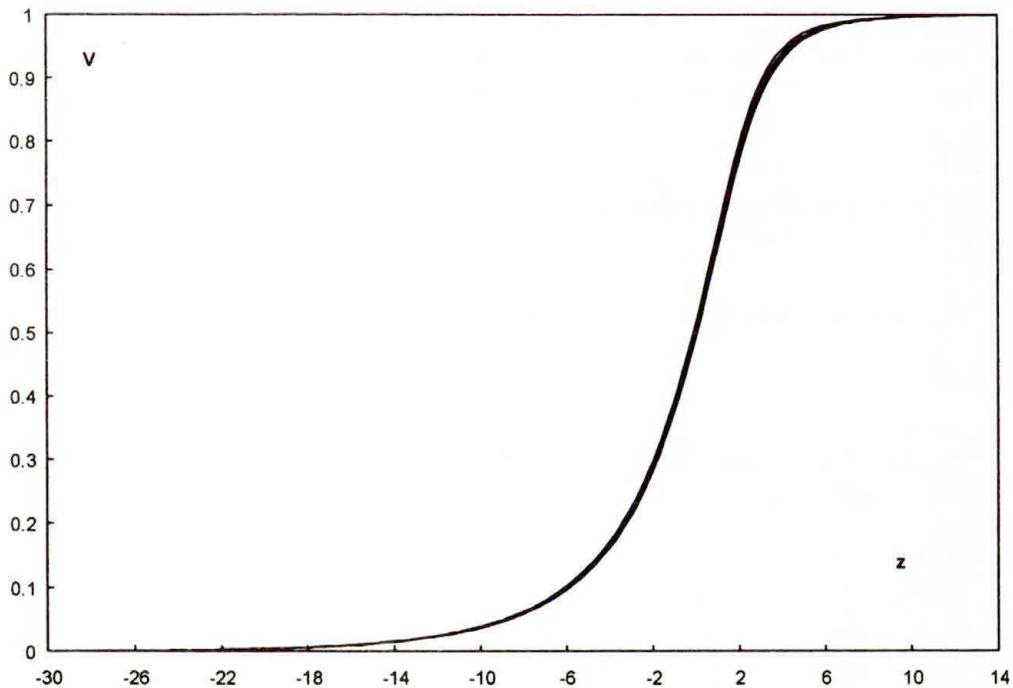


FIG. 7. Normalized shock structures for  $\alpha = 10$ .

If we increase the value of  $\alpha$  even more to,  $\alpha = 5$  and  $\alpha = 10$  like in Figs. 6 and 7, respectively, then it turns out that the splitting disappears. If we continue to increase the value of the capillarity coefficient, then the shock wave can become oscillatory without any splitting (cf. the previous section and Fig. 1).

#### 4. Generalization of these results

We show below that our equation (2.15) for plane travelling waves can be put into a more general and standard scheme, and simplified at the same time. We have not done this up to now because we used Eq. (2.15) for our calculations described in the previous section.

We consider a more general problem than that of (2.15)–(2.20), namely we take

$$(4.1) \quad \alpha \left[ A(w)w'' + B(w)w'^2 \right] + s\mu(w; w_a, s)w' + f(w; w_a, s) = 0,$$

subject to the limit conditions (2.18)–(2.20).

Here,  $\alpha > 0$  is a parameter,  $A(w)$  and  $B(w)$  are given continuous functions of  $w > b$ , where  $b$  can be such as previously or any other fixed real number;  $s$  is the wave speed, and  $w_a > b$  is a fixed quantity. Next, the source function  $f(w; w_a, s)$  and the viscosity coefficient  $\mu(w; w_a, s)$  are given smooth functions of their arguments such that

i)  $\mu(w; w_a, s)$  is strictly positive, i.e. there is a positive constant  $\mu_0$  such that for any  $w, w_a > 0$  and any admissible value of  $s$

$$(4.2) \quad \mu(w; w_a, s) \geq \mu_0 > 0;$$

ii) for any admissible value of  $s$

$$(4.3) \quad f(w_a; w_a, s) = 0;$$

iii) there exists  $w_b > b, w_b \neq w_a$  such that

$$(4.4) \quad f(w_b; w_a, s) = 0,$$

i.e.  $w_b = w_b(w_a, s)$ .

Finally,  $A(w)$  is assumed to be strictly positive as well, that is there exists a positive constant, say  $A_0$ , such that

$$(4.5) \quad A(w) \geq A_0 > 0,$$

for  $w > b$ .

Let

$$(4.6) \quad D(w) = \exp \left[ \int_{w_a}^w \frac{B(\zeta)}{A(\zeta)} d\zeta \right],$$

and let us define the transformation

$$(4.7) \quad w \longrightarrow U(w) = \frac{\int_{w_a}^w D(\zeta) d\zeta}{\int_{w_a}^{w_b} D(\zeta) d\zeta}.$$

Since  $D > 0$ , then  $U'_w > 0$  and this transformation is invertible; i.e. there is a twice differentiable function  $U \rightarrow w(U)$  such that

$$(4.8) \quad w(0) = w_a \quad \text{and} \quad w(1) = w_b.$$

By applying the transformation (4.7) to Eq. (4.1), the latter reduces to

$$(4.9) \quad \alpha U'' + sM(U)U' + F(U) = 0,$$

where

$$(4.10) \quad M(U) = \frac{\mu(w(U); w_a, s)}{A(w(U))},$$

and

$$(4.11) \quad F(U) = F(U; w_a, s) = \frac{D(w(U))}{A(w(U)) \int_{w_a}^{w_b} D(\zeta) d\zeta} f(w(U); w_a, s).$$

The limit conditions (2.18) – (2.20) become

$$(4.12) \quad \lim_{z \rightarrow -\infty} U(z) = \begin{cases} 0 & \text{for } s > 0, \\ 1 & \text{for } s < 0, \end{cases}$$

$$(4.13) \quad \lim_{z \rightarrow \infty} U(z) = \begin{cases} 1 & \text{for } s > 0, \\ 0 & \text{for } s < 0, \end{cases}$$

$$(4.14) \quad \lim_{z \rightarrow \pm\infty} U'(z) = 0, \quad \lim_{z \rightarrow \pm\infty} U''(z) = 0.$$

Equations (4.2), (4.3) take the form:

for any  $w_a > b$ ,  $w_b > b$ , and  $s$

$$(4.15) \quad F(0; w_a, w_b, s) = 0,$$

there is  $w_b = w_b(w_a, s) > b$  such that

$$(4.16) \quad F(1; w_a, w_b, s) = 0.$$

The limit value problem of the type (4.9), (4.12)–(4.14) along with (4.15), (4.16) was considered in many papers on the isothermal travelling waves solutions to conservation principles (an extensive list of references is given in [6]). Also, a similar problem arises in the analysis of travelling waves in reaction-diffusion systems. The difference between the conservation principles and the reaction-diffusion ones is that the source term  $F(U)$  in the latter case does not depend on the wave speed  $s$  and that (4.16) holds for any  $w_a$ ,  $w_b$  and/or eventually other parameters. So, in this case there is nothing like the Rankine–Hugoniot conditions (see [8] and the references therein).

In the case of our original equation (2.15)

$$A(w) = 2/w^5, \quad B(w) = 5/w^6 = \frac{1}{2}A'_w(w).$$

Hence, as the function  $D(w)$  we take  $D(w) = w^{-5/2}$ . Therefore

$$(4.17) \quad U(w) = \frac{w^{-3/2} - w_a^{-3/2}}{w_b^{-3/2} - w_a^{-3/2}}.$$

The inverse to it  $w = w(U)$  is then

$$(4.18) \quad w(U) = \left[ w_a^{-3/2} + U \left( w_b^{-3/2} - w_a^{-3/2} \right) \right]^{-2/3}.$$

Now, we establish the equivalence between the limit value problems (4.1)–(4.3) subject to conditions (2.18)–(2.20) and that of (4.9), (4.12)–(4.16).

First, due to (4.5)–(4.7) and (4.10), (4.11) we have immediately

**PROPOSITION 1.**

- i)  $M(U) > 0$  if and only if  $\mu(w) > 0$ ;
- ii)  $F(U) > 0$  ( $< 0$ ) if and only if  $(w_b - w_a)f(w) > 0$  ( $< 0$ );
- iii)  $F(U_0) = 0$  if and only if  $f(w_0) = 0$ , where  $U_0 = U(w_0)$ .

We have also

**PROPOSITION 2.**

The singular point  $(\bar{w}, 0)$  of Eq. (4.1) in the plane of  $(w, w')$  is of the same type as the singular point  $(\bar{U}, 0)$  of Eq. (4.9) in the  $(U, U')$ -plane.

**P r o o f.** To perform the classification of a singular point of a dynamic system we consider the asymptotic behaviour of its solutions as  $z \rightarrow -\infty$  or, respectively,  $z \rightarrow \infty$ .

To this end we linearize, first, Eq. (4.1) around  $(\bar{w}, 0)$  and obtain a linear, second order differential equation whose characteristic equation reads

$$(4.19) \quad \alpha A(\bar{w})\lambda^2 + s\mu(\bar{w})\lambda + f'(\bar{w}) = 0.$$

The similar procedure yields in the case of Eq. (4.9)

$$(4.20) \quad \alpha A^2 + sM(\bar{U})A + F'_U(\bar{U}) = 0.$$

But from (4.6), (4.11) we have

$$F'_U(U) = \frac{D(w(U))}{w_b} [B(w(U)) - A'_w(w(U))] f(w(U)) \frac{dw}{dU} \\ + \frac{D(w(U))}{A(w(U)) \int_{w_a}^{w_b} D(\zeta) d\zeta} f'_w(w(U)) \frac{dw}{dU}.$$

But, as it follows from (4.7)

$$\frac{D(w(U))}{w_b} \frac{dw}{dU} = \int_{w_a}^{w_b} D(\zeta) d\zeta = 1.$$

Therefore

$$(4.21) \quad F'_U(U) = \frac{B(w(U)) - A'_w(w(U))}{A^2(w(U))} f(w(U)) + \frac{f'_w(w(U))}{A(w(U))}.$$

Consequently

$$(4.22) \quad F'_U(\bar{U}) = \frac{f'_w(\bar{w})}{A(\bar{w})}.$$

Using (4.10) and (4.22) in Eq. (4.20) we see immediately that it coincides with Eq. (4.19). The proof is complete.

As we assume that neither  $A(w)$  nor  $B(w)$  depend on  $\alpha$ , we can apply transformation (4.6), (4.7) to the viscosity shock wave problem (2.29), (2.18), (2.19), and (2.20)<sub>1</sub>. This problem changes to

$$(4.23) \quad sM(U; w_a, w_b, s)U' + F(U; w_a, w_b, s) = 0,$$

and the limit conditions (4.12), (4.13), and (4.14)<sub>1</sub>. Here  $M$  and  $F$  are the same as previously. Also, we assume that (4.15) and (4.16) hold.

Finally, (2.22) and (2.23) take the form

$$(4.24) \quad F'_U(U_a) < 0, \quad F'_U(U_b) > 0,$$

where  $U_a = U(w_a)$ ,  $U_b = U(w_b)$ .

We transfer to here the terminology of the previous section, and the waves described by Eq. (4.9) are called the viscosity-capillarity waves, whereas those described by Eq. (4.23) are called the viscosity waves.

i) *The oscillating viscosity-capillarity waves*

Setting in Eq. (4.20),  $\bar{U} = U_a$  and using (4.24) we see that the point  $(U_a, 0)$  is the saddle in the  $(U, U')$ -plane. The similar analysis for  $\bar{U} = U_b$  shows that the characteristic exponents at this state of equilibrium can have nonzero imaginary parts, so the waves can be oscillatory downstream. Similarly as previously, the viscosity shock waves are never oscillatory, what follows from Eq. (4.23) and assumption (4.24).

ii) *The impending shock splitting*

Let there exist  $w_a = \bar{w}_a$ ,  $w_b = \bar{w}_b$ ,  $s = \bar{s}$  such that

$$F(1; \bar{w}_a, \bar{w}_b, \bar{s}) = 0,$$

and let there be  $\bar{U}$  that  $0 < \bar{U} < 1$ , and

$$F(\bar{U}; \bar{w}_a, \bar{w}_b, \bar{s}) = 0.$$

We take sequences  $w_a^{(n)} \rightarrow \bar{w}_a$ ,  $w_b^{(n)} \rightarrow \bar{w}_b$ ,  $s^{(n)} \rightarrow \bar{s}$  as  $n \rightarrow \infty$ , such that

$$F(1; w_a^{(n)}, w_b^{(n)}, s^{(n)}) = 0, \quad F(U; w_a^{(n)}, w_b^{(n)}, s^{(n)}) \neq 0 \quad \text{for } 0 < U < 1.$$

The shock profiles obtained from Eq. (4.9) for such values of the parameters and sufficiently large  $n$  will exhibit the impending shock splitting (cf. Proposition 1).

#### 4.1. The numerical procedure

In order to solve numerically the limit value problem of the type (2.15) and (2.18)–(2.20) we reduce first Eq. (2.15) to the form (4.9) by means of the transformation (4.17), (4.18). We have to distinguish two cases: the first one concerns large values of  $\alpha$ , whereas the second – the small ones. In the first case, the classical fourth-order Runge–Kutta method works well. In the second case,  $\alpha$  is a small singular parameter since it multiplies the highest derivative of the equation. Owing to that, the problem becomes numerically stiff and an implicit scheme has to be used. In the case of Eq. (4.9) the problem is relatively simple. Namely, with the substitution

$$(4.25) \quad U' = V(U)$$

we reduce our problem to

$$(4.26) \quad \frac{dV}{dU} = -\frac{F(U)}{\alpha V} - \frac{s}{\alpha} M(U),$$

with the boundary conditions

$$(4.27) \quad V(U = 0) = 0, \quad V(U = 1) = 0.$$

We decided upon the so-called trapezoidal rule [10], which consists in the following. Let us consider the generic initial value problem

$$y' = \phi(x, y),$$

subject to the initial condition

$$y(0) = x_0.$$

The numerical scheme is

$$(4.28) \quad y_{n+1} - y_n = \frac{1}{2}h [\phi(x_n, y_n) + \phi(x_{n+1}, y_{n+1})],$$

where  $h$  is the step.

In our case  $x = U$ ,  $y = V$ , and  $\phi$  is the right-hand side of Eq. (4.26). The numerical scheme yields, in our case, an algebraic quadratic equation for  $V_{n+1}$  whose solution provides a recursive relation between this quantity and  $U_n$ ,  $U_{n+1}$ , and  $V_n$ .

We start the iteration process from a point  $(U_0, V_0)$  close to  $(0,0)$  (the latter is a saddle in the  $(U, V)$ -plane) and continue the calculations until we reach a small vicinity of the point  $(1,0)$  (which is the stable node in the same plane). In this way we obtain the function  $V(U)$ , i.e. the right-hand side of Eq. (4.25). Integrating it in the standard way we find  $U = U(z)$ . Finally we use (4.18) and obtain  $w = w(z)$ .

### 5. The case of small $\alpha$

In this section we prove that the viscosity-capillarity and viscosity shock profiles do not differ much, provided that  $\alpha$  is sufficiently small.

We assume in this section that  $w_a$ ,  $w_b$ , and  $s$  are such that Eq. (4.16) is satisfied. First, we take the following

DEFINITION

For  $i = 0, 1, 2, 3$

$$\|y\|_i = \sup_{z \in \mathbb{R}} (|y(z)| + |y'(z)| + \dots + |y^{(i)}(z)|),$$

and

$$\mathcal{B}_i = \left\{ y \in C^i(\mathbb{R}) : \|y\|_i < \infty, \quad y'(z), \dots, y^{(i)}(z) \rightarrow 0 \quad \text{as } |z| \rightarrow \infty \right\}.$$

Also let for  $i = 0, 1$

$$\mathcal{B}_i^0 = \{y \in \mathcal{B}_i : y(z) \rightarrow 0 \text{ as } |z| \rightarrow \infty\}$$

and

$$\mathcal{B}_2^0 = \{y \in \mathcal{B}_2 : y(0) = 0 \text{ and } y(z) \rightarrow 0 \text{ as } |z| \rightarrow \infty\}.$$

THEOREM.

$\alpha$ ) Let the function  $M(U)$  satisfy

i)  $M(U) \in C^2((-\delta, 1 + \delta))$  for some positive  $\delta$ ,

ii)  $M(U) \geq M_0 > 0$  for  $U \in (-\delta, 1 + \delta)$ ,

where  $M_0$  is a constant.

$\beta$ ) Let the function  $F(U)$  satisfy

i)  $F(U) \in C^2((-\delta, 1 + \delta))$  for the same  $\delta$  as previously,

ii)  $F(0) = 0, F(1) = 0$ ,

iii)  $F(U) < 0$  for  $U \in (0, 1)$ ,

iv)  $F'_U(0) < 0, F'_U(1) > 0$ .

Then

1. Equation (4.21) has a unique solution  $U_0(z) \in \mathcal{B}_1$  satisfying the limit conditions (4.12), (4.13) and the initial condition  $U_0(0) = 1/2$ ;

2. Equation (4.9) has a unique solution  $U(z) \in \mathcal{B}_2$  satisfying the same limit conditions and the initial condition  $U(0) = 1/2$ , provided that  $\alpha$  is sufficiently small;

3. There is a constant  $C > 0$ , independent of  $\alpha$ , such that

$$(5.1) \quad \|U(z) - U_0(z)\|_1 < \alpha C,$$

for  $\alpha$  sufficiently small.

Only Part 3 of the assertions of the Theorem needs a proof; Parts 1 and 2 are presented for sake of completeness of the theses and can be easily proved by the phase plane analysis, even without the assumption that  $\alpha$  is small (see [8] for the case of reaction-diffusion systems, and [9] for the case of conservation laws).

The following observation collects some properties of  $U_0(z)$  which will be used in the proof of Part 3 of the Theorem.

OBSERVATION 2. Under the assumptions on  $M(U)$  and  $F(U)$  taken in the Theorem,

$$(5.2) \quad sU'(z) > 0 \quad \text{for } z \in \mathbb{R},$$

$$(5.3) \quad \frac{U_0''(z)}{U_0'(z)} \in \mathcal{B}_1,$$

and

$$(5.4) \quad U'_0(z), U''_0(z) = \begin{cases} O(e^{\kappa-z}) & \text{as } z \rightarrow -\infty, \\ O(e^{\kappa+z}) & \text{as } z \rightarrow \infty, \end{cases}$$

where

$$(5.5) \quad \begin{aligned} \kappa_- &= \begin{cases} -\frac{F'_U(0)}{sM(0)} & \text{for } s > 0, \\ -\frac{F'_U(1)}{sM(1)} & \text{for } s < 0, \end{cases} \\ \kappa_+ &= \begin{cases} -\frac{F'_U(1)}{sM(1)} & \text{for } s > 0, \\ -\frac{F'_U(0)}{sM(0)} & \text{for } s < 0. \end{cases} \end{aligned}$$

The proof is immediate, hence it is omitted.

Now, we make the substitution in Eq. (4.9)

$$(5.6) \quad U = U_0 + h.$$

To prove Parts 2 and 3 of the Theorem it is sufficient to show that the equation for  $h$  resulting from Eq. (4.9) has a solution in  $\mathcal{B}_2^0$ , and that

$$(5.7) \quad \|h\|_1 < \alpha C$$

for some positive constant  $C$  independent of  $\alpha$ , the latter being sufficiently small.

To this end we write the equation for  $h$  in the form

$$(5.8) \quad \alpha h'' + sM(U_0)h' - sM(U_0)\frac{U''_0}{U'_0}h = -\alpha U''_0 - G(h),$$

where  $G : \mathcal{B}_i^0 \rightarrow \mathcal{B}_{i-1}$ ,  $i = 1, 2$ , is a nonlinear differential operator defined by

$$G(h) = s [M(U_0 + h) - M(U_0)] h' + s [M(U_0 + h) - M(U_0) - M'_U(U_0)h] U'_0 + [F(U_0 + h) - F(U_0) - F'_U(U_0)h].$$

The linear differential operator generated by the left-hand side of Eq. (5.8) we denote by  $L_\alpha$ , i.e.

$$(5.9) \quad L_\alpha h = \alpha h'' + sM(U_0)h' - sM(U_0)\frac{U''_0}{U'_0}h.$$

$$L_\alpha h : \mathcal{B}_2^0 \rightarrow \mathcal{B}_0^0.$$

Hence, according to this scheme we have to show first, that  $L_\alpha$  has the inverse  $L_\alpha^{-1} : \mathcal{B}_0^0 \rightarrow \mathcal{B}_2^0$ .

LEMMA. For sufficiently small positive  $\alpha$ , the linear operator  $L_\alpha$  has the inverse  $L_\alpha^{-1} : \mathcal{B}_0^0 \rightarrow \mathcal{B}_2^0$  and, when treated as an operator from  $\mathcal{B}_0^0$  into  $\mathcal{B}_1^0$ , is bounded, i.e. for every  $g \in \mathcal{B}_0^0$  there is a positive constant, say  $C$ , such that

$$(5.10) \quad \|L_\alpha^{-1}g\|_1 \leq C\|g\|_0.$$

P r o o f. We rewrite the equation  $L_\alpha h = g$  in the form

$$(5.11) \quad \alpha h'' + \left[ sM(U_0) - \alpha \frac{U_0''}{U_0'} \right] h' - \left[ sM(U_0) \frac{U_0''}{U_0'} + \alpha \left( \frac{U_0''}{U_0'} \right)' \right] h = g - \left( \frac{U_0''}{U_0'} h \right)' \equiv \tilde{g}$$

or

$$\alpha \left[ h' - \frac{U_0''}{U_0'} h \right]' + sM(U_0) \left[ h' - \frac{U_0''}{U_0'} h \right] = \tilde{g}.$$

Integrating this linear equation we get

$$(5.12) \quad h' - \frac{U_0''}{U_0'} h = V\tilde{g},$$

where  $V$  is a linear integral operator  $V : \mathcal{B}_0^0 \rightarrow \mathcal{B}_1^0$  defined by

$$(5.13) \quad V\tilde{g} = \begin{cases} \frac{1}{\alpha q(z)} \int_{-\infty}^z q(\zeta) \tilde{g}(\zeta) d\zeta, & \text{if } s > 0, \\ -\frac{1}{\alpha q(z)} \int_z^{-\infty} q(\zeta) \tilde{g}(\zeta) d\zeta, & \text{if } s < 0, \end{cases}$$

and

$$(5.14) \quad q(z) = \exp \left[ \frac{s}{\alpha} \int_0^z M(U_0(\zeta)) d\zeta \right].$$

The operator  $V : \mathcal{B}_0^0 \rightarrow \mathcal{B}_1^0$  is bounded. First, let us notice that if  $\lim_{|z| \rightarrow \infty} \tilde{g}(z) = 0$ , then  $\lim_{|z| \rightarrow \infty} (V\tilde{g})(z) = 0$ . To prove this use the de l'Hospital rule.

Secondly, there is a constant  $C > 0$ , independent of  $\alpha$ , such that

$$(5.15) \quad \|V\tilde{g}\|_0 \leq C\|\tilde{g}\|_0.$$

Indeed, let  $s > 0$ . We have obviously

$$|V\tilde{g}| \leq \frac{\|\tilde{g}\|_0}{\alpha} \int_{-\infty}^z \exp \left[ -\frac{s}{\alpha} \int_{\zeta}^z M(U_0(\tau)) d\tau \right] d\zeta.$$

Now, making use of Assumption  $(\alpha)(ii)$  of the Theorem we obtain

$$\int_{-\infty}^z \exp \left[ -\frac{s}{\alpha} \int_{\zeta}^z M(U_0(\tau)) d\tau \right] d\zeta \leq \int_{-\infty}^z \exp \left[ -\frac{sM_0}{\alpha} (z - \zeta) \right] dz = \frac{\alpha}{sM_0}.$$

Hence, (5.15) is proved. The case of  $s < 0$  is treated in a similar way. Now we proceed to Eq. (5.12). Its solution is

$$(5.16) \quad h(z) = U'_0(z) \int_0^z \frac{(V\tilde{g})(\zeta)}{U'_0(\zeta)} d\zeta.$$

Of course,  $h(0) = 0$ . Next, making use of Observation 2 we check easily that  $h(\pm\infty) = 0$ . So,  $h(z)$  as given by (5.16) is an element of  $\mathcal{B}_2^0$ , indeed.

From (5.16) the estimate follows

$$|h(z)| \leq \|V\tilde{g}\|_0 \left| U'_0(z) \int_0^z \frac{d\zeta}{U'_0(\zeta)} \right|.$$

It follows from Observation 2 that there is a constant, say  $C_1 > 0$ , such that

$$\left| U'_0(z) \int_0^z \frac{d\zeta}{U'_0(\zeta)} \right| \leq C_1.$$

Of course,  $C_1$  does not depend on  $\alpha$ , because  $U_0(z)$  is independent of  $\alpha$ .

Hence, the following estimate holds true

$$(5.17) \quad \|h(z)\| \leq C_1 \|V\tilde{g}\|_0.$$

Next, making use of Eq. (5.12), Observation 2, and (5.17) we conclude that there is a constant  $C_2$ , independent of  $\alpha$  such that

$$\|h'(z)\|_0 \leq C_2 \|V\tilde{g}\|_0.$$

Therefore, there is a constant  $C_3 > 0$ , independent of  $\alpha$ , such that

$$(5.18) \quad \|h(z)\|_1 \leq C_3 \|V\tilde{g}\|_0.$$

Equation (5.16) is an integro-differential equation equivalent to (5.12), and thus to  $L_\alpha h = g$ . Explicitly, this equation is of the form

$$(5.19) \quad h(z) = U'_0(z) \int_0^z \frac{(Vg)(\zeta)}{U'_0(\zeta)} d\zeta - \alpha U'_0(z) \int_0^z \frac{\left[ V \left( \frac{U''_0}{U'_0} h \right)'(\zeta) \right]}{U'_0(\zeta)} d\zeta \equiv T(h).$$

To solve this equation we apply the method of successive approximations setting

$$(5.20) \quad \begin{aligned} h_0 &= T(0), \\ h_{k+1} &= T(h_k), \quad k = 0, 1, 2, \dots \end{aligned}$$

By induction we show that, for sufficiently small  $\alpha$ , there is a constant  $C > 0$ , independent of  $\alpha$ , such that

$$(5.21) \quad \|h_k\|_1 \leq C \|g\|_0, \quad k = 0, 1, 2, \dots$$

Next, we check easily that for small  $\alpha$  the following is true:

$$\|T(h_1) - T(h_2)\|_1 \leq \alpha C \|h_1 - h_2\|_1,$$

i.e. for small  $\alpha$ ,  $T(h)$  is a contraction mapping from  $\mathcal{B}_1^0$  into itself. Thus, the sequence defined by (5.20) is convergent in  $\mathcal{B}_1^0$ , its limit, as  $k$  tends to  $\infty$ ,  $h(z)$  satisfies Eq. (5.19), and it is the unique solution. But, any solution of Eq. (5.19) in  $\mathcal{B}_1^0$  is also its solution in  $\mathcal{B}_2^0$ . Hence, the limit function  $h(z)$  is also the unique solution of this equation in  $\mathcal{B}_2^0$ . Consequently, it is the unique solution to the equation  $L_\alpha h = g$ . Moreover, since (5.21) holds for every  $k$ , it holds also for the limit of the sequence. It means that  $h(z)$  satisfies

$$\|h\|_1 \leq C \|g\|_0,$$

and this is (5.10). The proof is complete.

**P r o o f** of the Theorem. Solving Eq. (5.8) in  $\mathcal{B}_2^0$  we get

$$(5.22) \quad h_{k+1} = -\alpha L_\alpha^{-1} U''_0 - L_\alpha^{-1} G(h_k), \quad k = 0, 1, 2, \dots$$

with  $h_0 = 0$ .

The operator  $G(h)$  has the following two properties:

i)  $G(0) = 0$ ,

ii) for every  $\varepsilon > 0$ , there is  $\delta(\varepsilon) > 0$  such that, if  $\|h_1\|_1 \leq \delta(\varepsilon)$ ,  $\|h_2\| \leq \delta(\varepsilon)$ , then  $\|G(h_1) - G(h_2)\|_1 \leq \varepsilon \|h_1 - h_2\|_1$ .

Using them and the Lemma we show easily that there is a constant  $C > 0$ , independent of  $\alpha$ , such that

$$(5.23) \quad \|h_k\|_1 \leq \alpha C, \quad k = 0, 1, 2, \dots,$$

and that the sequence  $\{h_k\}$  is convergent as  $k \rightarrow \infty$ , to the limit  $h(z) \in \mathcal{B}_1^0$ . This limit function is the unique solution of Eq. (5.8) in  $\mathcal{B}_1^0$ , so it is its unique solution in  $\mathcal{B}_2^0$ . Additionally, the limit  $h(z)$  satisfies  $\|h\|_1 \leq \alpha C$ , with the constant  $C$  being the same as in (5.23), provided that  $\alpha$  is sufficiently small. The proof of the Theorem is complete.

## Acknowledgment

The author thanks Professor T. PŁATKOWSKI whose comments were very helpful in avoiding serious mistakes.

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*Received December 16, 1996.*

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