

A nonexistence theorem of small periodic traveling wave solutions to the generalized Boussinesq equation

Y. CHEN (FAYETTEVILLE)

THE GENERALIZED Boussinesq equation, $u_{tt} - u_{xx} + [f(u)]_{xx} + u_{xxxx} = 0$, and its periodic traveling wave solutions are considered. Using the transform $z = x - \omega t$, the equation is converted to a nonlinear ordinary differential equation with periodic boundary conditions. An equivalent relation between the ordinary differential equation and a Hammerstein-type integral equation is then established by using the Green's function method. This integral equation generates compact operators in $(C_{2T}, \|\cdot\|)$, a Banach space of real-valued continuous periodic functions with a given period $2T$. We prove that for small $T > 0$, there exists an $r > 0$ such that there is no nontrivial solution to the integral equation in the ball $B(0, r) \subseteq C_{2T}$. And hence, the generalized Boussinesq equation has no $2T$ -periodic traveling wave solutions having amplitude less than r .

1. Introduction

IN THE 1870's, Boussinesq derived some model equations for the propagation of small amplitude long wave on the surface of water [1]. These equations possess special traveling wave solutions called solitary waves. Boussinesq's theory was the first to give a satisfactory and scientific explanation of the phenomenon of solitary waves discovered by Scott Russell in his experimental observation of shallow water propagation some thirty years earlier [2].

The original equation due to Boussinesq is not the only mathematical model for small amplitude planar long waves on the surface of shallow water. Different choices of the independent variables and the possibility of modifying lower order terms by the use of the leading order relationships can lead to a whole range of equations, all of which have the same formal validity [3]. All of these models possess one obvious characteristic, however, which is that they are perturbations of the linear wave equation that takes account of small effects of nonlinearity and dispersion.

In this paper, we consider a generalized Boussinesq equation of the form

$$(1.1) \quad u_{tt} - u_{xx} + [f(u)]_{xx} + u_{xxxx} = 0,$$

where $u = u(t, x)$ and $f(u)$ is a C^1 function in its argument.

We shall prove a nonexistence theorem of periodic traveling wave solutions to this equation following the idea of LIU and PAO [4], SOEWONO [5], and CHEN and HE [6]. Of course, this does not establish the nonexistence of periodic traveling wave solutions to the generalized Boussinesq equation. We merely prove that

for a given small $T > 0$, under certain conditions, $2T$ -periodic traveling wave solutions with small amplitude to the generalized Boussinesq equation do not exist. There is no doubt about the existence of periodic traveling wave solutions to the generalized Boussinesq equation when $f(u) = u^n$ for some positive integer n . It is well known that the periodic cnoidal wave solutions exist and can be representable as infinite sums of solitons when $f(u) = u^n$ and $n = 1, 2$ [7, 8, 9].

The plan of this paper is as follows. In Sec. 2, the generalized Boussinesq equation is transformed to an ordinary differential equation with periodic boundary conditions. We then apply the Green's function method to derive a nonlinear integral equation equivalent to the ordinary differential equation. The nonexistence of small periodic solutions to the integral equation is proved in Sec. 3. Therefore, the nonexistence of small $2T$ -periodic traveling wave solutions to the generalized Boussinesq equation is established.

2. Formulation of the problem

We start from the generalized Boussinesq equation of the form

$$(2.1) \quad u_{tt} - u_{xx} + [f(u)]_{xx} + u_{xxxx} = 0,$$

where the function f is C^1 in its argument. We are interested in the periodic traveling wave solutions of the form $u(x, t) = U(z) = U(x - \omega t)$, where $\omega > 0$ is the wave speed and $z = x - \omega t$ is the characteristic variable. Substitution of the $U(z)$ into Eq. (2.1) then leads to the fourth order nonlinear ordinary differential equation

$$(2.2) \quad U^{(4)}(z) = CU''(z) - [f(U(z))]'',$$

where $C = (1 - \omega^2)$. To obtain periodic solutions, we impose the following boundary conditions

$$(2.3) \quad U^{(n)}(0) = U^{(n)}(2T), \quad n = 0, 1, 2, 3,$$

where T is a preassigned positive number. To eliminate nontrivial constant solutions to the ordinary differential equation (2.2), another condition is introduced

$$(2.4) \quad \int_0^{2T} U(z) dz = 0.$$

It is obvious that any solution $U(z)$ of the boundary value problem consisting of Eqs. (2.2)–(2.4) can be extended to a $2T$ -periodic traveling wave solution to the original evolution equation (2.1).

Integrating both sides of Eq. (2.2) with respect to z twice and using Eqs. (2.3), (2.4) yield

$$(2.5) \quad U''(z) - CU(z) = E - f(U(z)),$$

$$(2.6) \quad U^{(n)}(0) = U^{(n)}(2T), \quad n = 0, 1,$$

where

$$E = \int_0^{2T} f(U(z)) dz / (2T).$$

Conversely, integrating both sides of Eq. (2.5) from 0 to $2T$ and using Eqs. (2.6) will give us Eq. (2.4), and direct differentiations of Eq.(2.5) will give us Eqs. (2.2), (2.3). Therefore, we have proved the following theorem.

THEOREM 1. *Suppose $C \neq 0$, a function $U(z)$ is a $2T$ -periodic traveling wave solution to Eq. (2.1) satisfying the boundary conditions Eqs. (2.3), (2.4) if and only if it is a solution to the boundary value problem consisting of Eqs. (2.5), (2.6).*

From now on we only consider the two cases: 1. $C > 0$ and 2. $C < 0$ but $-C \neq (k\pi/T)^2$ with k being any integer. Treating the right-hand side of Eq. (2.5) as a forcing term and using the Green's function method [4, 10], the boundary value problem Eqs. (2.5), (2.6) can be converted to a nonlinear integral equation of the form

$$(2.7) \quad U(z) = \int_0^{2T} K_i(z, s) f(U(s)) ds, \quad \forall z \in [0, 2T],$$

where the kernels $K_i, i = 1, 2$, are defined as follows:

1. When $C > 0$, we denote $\lambda_1 = \sqrt{C}$, then

$$(2.8) \quad K_1(z, s) = \frac{\cosh \lambda_1(T - |z - s|)}{2\lambda_1 \sinh \lambda_1 T} - \frac{1}{2\lambda_1^2 T}, \quad \forall z, s \in [0, 2T].$$

2. When $C < 0$ but $-C \neq (k\pi/T)^2$ with k being any integer, let $\lambda_2 = \sqrt{-C}$, then

$$(2.9) \quad K_2(z, s) = \frac{\cos \lambda_2(T - |z - s|)}{2\lambda_2 \sin \lambda_2 T} - \frac{1}{2\lambda_2^2 T}, \quad \forall z, s \in [0, 2T].$$

LEMMA 1. The kernels K_1 and K_2 have the following properties:

$$\begin{aligned} K_i(0, s) &= K_i(2T, s), & \forall s \in [0, 2T], \quad i = 1, 2, \\ K_i(z, 2T - s) &= K_i(2T - z, s), & \forall z, s \in [0, 2T], \quad i = 1, 2, \\ \int_0^{2T} K_i(z, s) ds &= 0, & \forall z \in [0, 2T], \quad i = 1, 2. \end{aligned}$$

P r o o f. Straightforward computations based on the definitions of the $K_1(z, s)$ and $K_2(z, s)$ given in Eqs. (2.8), (2.9). □

THEOREM 2. *A function $U(z)$ is a solution of the boundary value problem consisting of Eqs. (2.5), (2.6) if and only if it is a solution of the integral equation (2.7).*

P r o o f. The “if” part can be proved by direct differentiations of Eq. (2.7) and the “only if” part is based on the Green’s function method by treating the right-hand side of Eq. (2.5) as a nonhomogeneous term. □

3. Nonexistence theorem

It is seen from the Theorem 1 and 2 that $U(z)$ is a solution to the integral equation (2.7) if and only if it is a solution to Eq. (2.1) satisfying the boundary conditions Eqs. (2.3), (2.4). Therefore, to show the nonexistence of small $2T$ -periodic traveling wave solutions to Eq. (2.1) with the boundary conditions Eqs. (2.3), (2.4), it is sufficient to show that small solution to Eq. (2.7) does not exist.

To this end we define C_{2T} as a collection of real-valued continuous functions, $v(z)$, on $[0, 2T]$ such that $v(0) = v(2T)$. Equip C_{2T} with the sup norm $\|\cdot\|$ as $\|v\| = \sup_{0 \leq z \leq 2T} |v(z)|$, for each $v \in C_{2T}$, then $(C_{2T}, \|\cdot\|)$ becomes a Banach space and $\|v\|$ is the amplitude of v .

We now define the operators \mathcal{A}_i , $i = 1, 2$, on C_{2T} as

$$(3.1) \quad \mathcal{A}_i v(z) = \int_0^{2T} K_i(z, s) f(v(s)) ds, \quad \forall v \in C_{2T},$$

where the kernels K_i , $i = 1, 2$, are defined in Eqs. (2.8), (2.9). Notice that the operator \mathcal{A}_i depends on T and λ_i , $i = 1, 2$.

We shall show that for any given small $T > 0$, there exists an $r > 0$ such that $\|\mathcal{A}_i v\| < \|v\|$ for any nontrivial function $v \in B(0, r) \subseteq C_{2T}$, $i = 1, 2$. This implies that the equation $\mathcal{A}_i v = v$ has no nontrivial solution with amplitude smaller than r . And hence, the nonexistence of nontrivial small solution to the boundary value problem Eqs. (2.4), (2.5) is established. This, in turn, leads to the nonexistence of small $2T$ -periodic traveling wave solution $U(z)$ to the generalized Boussinesq equation satisfying the boundary conditions Eqs. (2.3), (2.4).

A consequence of Lemma 1 can be stated now.

LEMMA 2. Let v be an element of C_{2T} . If $v(z) = v(2T - z)$ for $z \in [0, 2T]$, then $\mathcal{A}_i v(z) = \mathcal{A}_i v(2T - z)$, $i = 1, 2$.

Let $r > 0$ and $B(0, r)$ be a bounded ball in C_{2T} , we then have the following theorem.

THEOREM 3. $\mathcal{A}_i : C_{2T} \rightarrow C_{2T}$, $i = 1, 2$, is compact and $\|\mathcal{A}_i v\| < \|v\|$ for all nontrivial $v \in B(0, r)$ when T is small enough, $i = 1, 2$.

P r o o f. First we show $\mathcal{A}_i : C_{2T} \rightarrow C_{2T}$, $i = 1, 2$. Since it is obvious from Lemma 1 that $\mathcal{A}_i v(0) = \mathcal{A}_i v(2T)$ for each $v \in C_{2T}$, $i = 1, 2$, it suffices to show that $\mathcal{A}_i v$, $i = 1, 2$, is continuous on $[0, 2T]$.

Let v be an element in C_{2T} , we have

$$(3.2) \quad \begin{aligned} \frac{d\mathcal{A}_1 v(z)}{dz} &= \frac{-1}{2 \sinh \lambda_1 T} \int_0^z \sinh \lambda_1(T - z + s) f(v(s)) ds \\ &\quad + \frac{1}{2 \sinh \lambda_1 T} \int_z^{2T} \sinh \lambda_1(T + z - s) f(v(s)) ds, \end{aligned}$$

$$(3.3) \quad \begin{aligned} \frac{d\mathcal{A}_2 v(z)}{dz} &= \frac{1}{2 \sin \lambda_2 T} \int_0^z \sin \lambda_2(T - z + s) f(v(s)) ds \\ &\quad + \frac{-1}{2 \sin \lambda_2 T} \int_z^{2T} \sin \lambda_2(T + z - s) f(v(s)) ds. \end{aligned}$$

The existence of $d\mathcal{A}_1 v/dz$ and $d\mathcal{A}_2 v/dz$ implies that both $\mathcal{A}_1 v$ and $\mathcal{A}_2 v$ are continuous on $[0, 2T]$, and hence, we have proved $\mathcal{A}_i : C_{2T} \rightarrow C_{2T}$, $i = 1, 2$.

Let \mathbf{S} be any bounded subset of C_{2T} , i.e., there exists an $L_0 > 0$ such that $\|v\| \leq L_0$ for all $v \in \mathbf{S}$. Then since f is C^1 in its argument, there exists an $M_0 > 0$ such that

$$\|f(v)\| = \sup_{0 \leq z \leq 2T} |f(v(z))| \leq \sup_{-L_0 \leq w \leq L_0} |f(w)| \leq M_0, \quad \forall v \in \mathbf{S}.$$

Since $\sin \lambda_2 T \neq 0$ and

$$\max_{0 \leq z \leq 2T} \int_0^{2T} |K_i(z, s)| ds \leq 2/\lambda_i^2, \quad i = 1, 2,$$

we obtain from Eqs. (3.1)–(3.3)

$$\begin{aligned} \|\mathcal{A}_i v\| &\leq \frac{2M_0}{\lambda_i^2}, \quad \forall v \in \mathbf{S}, \quad i = 1, 2, \\ \|d\mathcal{A}_i v/dz\| &\leq \frac{TM_0}{\tau_i}, \quad \forall v \in \mathbf{S}, \quad i = 1, 2, \end{aligned}$$

where $\tau_1 = 1$ and $\tau_2 = |\sin \lambda_2 T|$. Thus, $\mathcal{A}_i \mathbf{S}$, $i = 1, 2$, is uniformly bounded and equi-continuous, and by the Ascoli–Arzela Theorem both \mathcal{A}_1 and \mathcal{A}_2 are compact operators from C_{2T} into C_{2T} .

From the definition of K_i , Eqs. (2.8), (2.9), we see that for any fixed $z \in (0, 2T)$, the graph of $K_i(z, s)$, $i = 1, 2$, is just a translation of the graph of $K_i(0, s)$, $i = 1, 2$. Therefore, we have the following inequalities [5, 6]

$$(3.4) \quad \int_0^{2T} |K_1(z, s)| ds = \int_0^{2T} |K_1(0, s)| ds \leq \frac{2}{\lambda_1^2} \left(1 - \frac{\lambda_1 T}{\sinh \lambda_1 T} \right),$$

$$(3.5) \quad \int_0^{2T} |K_2(z, s)| ds = \int_0^{2T} |K_2(0, s)| ds \leq \frac{2}{\lambda_2^2} \left(\frac{\lambda_2 T}{\sin \lambda_2 T} - 1 \right).$$

It should be noticed that T is small and the right-hand sides of the above two inequalities (3.4), (3.5) vanish when T goes to zero.

Let v be a nontrivial function in $B(0, r)$. We define

$$I = \{v(s) : v \in B(0, r), 0 \leq s \leq 2T\}.$$

It is obvious that $I \subseteq [-r, r]$ and $0 \in I$, since otherwise the equation $\mathcal{A}_i v = v$ has no solution in $B(0, r)$ because of the condition Eq. (2.4) and we are done. Using the Mean Value Theorem, we then have

$$f(v(s)) = f(0) + f'(c)v(s), \quad \forall s \in [0, 2T],$$

where $c = c(s) \in I$. Hence, since f' is continuous in its argument, there exists an $N > 0$ such that

$$\|f'(c)\| \leq \sup_{0 \leq s \leq 2T} |f'(v(s))| \leq \sup_{-r \leq w \leq r} |f'(w)| \leq N.$$

From the Lemma 1, we know

$$\int_0^{2T} K_i(z, s) f(0) ds = 0.$$

Therefore, we have

$$\begin{aligned} \|\mathcal{A}_1 v\| &= \sup_{0 \leq z \leq 2T} \left| \int_0^{2T} K_1(z, s) f(v(s)) ds \right|, \\ &= \sup_{0 \leq z \leq 2T} \left| \int_0^{2T} K_1(z, s) [f(0) + f'(c)v(s)] ds \right|, \end{aligned}$$

$$\begin{aligned}
 &= \sup_{0 \leq z \leq 2T} \left| \int_0^{2T} K_1(z, s) f'(c) v(s) ds \right|, \\
 &\leq \sup_{0 \leq z \leq 2T} \int_0^{2T} |K_1(z, s)| \|f'(c)\| \|v(s)\| ds, \\
 &\leq N \|v\| \sup_{0 \leq z \leq 2T} \int_0^{2T} |K_1(z, s)| ds, \\
 &\leq \frac{2N}{\lambda_1^2} \|v\| \left(1 - \frac{\lambda_1 T}{\sinh \lambda_1 T} \right), \quad \forall v \in B(0, r).
 \end{aligned}$$

It can be seen that when T is small enough we shall have

$$(3.6) \quad \frac{2N}{\lambda_1^2} \left(1 - \frac{\lambda_1 T}{\sinh \lambda_1 T} \right) < 1.$$

Therefore, we proved that when T is small enough, we shall have $\|\mathcal{A}_1 v\| < \|v\|$ for all nontrivial $v \in B(0, r)$. Similarly, we can also prove that when T is small enough,

$$(3.7) \quad \frac{2N}{\lambda_2^2} \left(\frac{\lambda_2 T}{\sin \lambda_2 T} - 1 \right) < 1,$$

and hence, $\|\mathcal{A}_2 v\| < \|v\|$ for all nontrivial $v \in B(0, r)$. □

By Theorem 3, we see that the equation $\mathcal{A}_i v = v$, $i = 1, 2$, has no nontrivial solution in $B(0, r)$ when the inequalities (3.6), (3.7) hold. This implies that Eq. (2.7) has no nontrivial solution in $B(0, r)$ when $C > 0$ and

$$\frac{2N}{\lambda_1^2} \left(1 - \frac{\lambda_1 T}{\sinh \lambda_1 T} \right) < 1,$$

or when $C < 0$ but $-C/(\alpha a^2) \neq (k\pi/T)^2$ with k being any integer and

$$\frac{2N}{\lambda_2^2} \left(\frac{\lambda_2 T}{\sin \lambda_2 T} - 1 \right) < 1.$$

Therefore, we proved the following nonexistence theorem for small $2T$ -periodic traveling wave solutions to the generalized Boussinesq equation.

THEOREM 4. *For any given small $T > 0$, there exist an $r > 0$ and $N > 0$ such that $\|f'(v)\| \leq N$ when $\|v\| \leq r$. Thus to Eq. (2.1) with the boundary*

conditions Eqs. (2.3), (2.4), there is no $2T$ -periodic traveling wave solution $U(z)$ with amplitude less than r if (1) $C > 0$ and

$$\frac{2N}{\lambda_1^2} \left(1 - \frac{\lambda_1 T}{\sinh \lambda_1 T} \right) < 1,$$

or (2) $C < 0$ but $-C/(\alpha a^2) \neq (k\pi/T)^2$ with k being any integer, and

$$\frac{2N}{\lambda_2^2} \left(\frac{\lambda_2 T}{\sin \lambda_2 T} - 1 \right) < 1.$$

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE,
FAYETTEVILLE STATE UNIVERSITY, FAYETTEVILLE, NORTH CAROLINA, USA.

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