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Decay and continuous dependence estimates for harmonic vibrations of micropolar elastic cylinders

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A PRINCIPLE OF SAINT-VENANT-TYPE is established for a right cylinder composed of an anisotropic, linear, homogeneous micropolar elastic solid and subjected to harmonic loading on one of its ends. The amplitude of the harmonic vibrations of this cylinder is also shown to depend continuously on the prescribed data.

1. Introduction

FOR CYLINDRICAL DOMAINS, the Saint-Venant principle in linear micropolar elastostatics was established by BERGLUND [1] and CHIRITA and ARON [2] (see also [3]). By adapting certain ideas originally due to TOUPIN [4], these authors have shown that certain global measures of the displacements which depend upon the distance from the loaded end of the cylinder, decay exponentially with that distance.

More recently, certain aspects of the dynamical version of the Saint-Venant principle have been investigated within the framework of the theory of linear elasticity. In particular, FLAVIN and KNOPS [5], FLAVIN, KNOPS and PAYNE [6] and KNOPS [7] have considered the Saint-Venant principle for the harmonic vibrations of linearly elastic cylinders subjected to harmonic type loadings on one of their ends.

In this paper we deal with the Saint-Venant principle for the harmonic vibrations of right cylinders composed of an anisotropic, linear, homogeneous micropolar elastic solid. Following [5–7] we show that when one end of the cylinder is subjected to prescribed harmonic tractions, and provided that the prescribed frequency of vibrations is strictly less than a certain critical frequency, the energy $E(z)$ stored in a part of the cylinder that lies above a cross-section which is at the distance z from the loaded end, decays faster than a certain exponentially decreasing function of z. The critical frequency depends upon the characteristics of the material and upon the geometry of the body. Additionally, we establish here an estimate for the total energy of the considered cylinder which implies that the amplitude of vibrations depends continuously on the prescribed data and which, when coupled with the Saint-Venant-type estimate mentioned above, provides us with a more explicit description of the way in which $E(z)$ decays as a function of z. Other continuous dependence results in micropolar elasticity have been obtained previously by ARON [8] who considered the case of physically nonlinear micropolar elastostatics.

2. Preliminaries

Consider a body B composed of a homogeneous anisotropic micropolar linear elastic material ([1]). The motion of a particle which belongs to such a body is described by the displacement vector field $\mathbf{u}$ and the microrotation vector field $\boldsymbol{\varphi}$. In what follows we will be employing a six-dimensional vector field $\mathbf{U}$ defined by

$$\mathbf{U} \equiv (\mathbf{u}, j\boldsymbol{\varphi}), \qquad j = \text{const}, \qquad j > 0, \tag{2.1}$$

where j is the square root of the smallest eigenvalue of the microinertia tensor $\mathbf{j}$. The microinertia tensor is assumed to be symmetric and positive definite [9]. As usual, the inner product in the six-dimensional vector space is defined by

$$\mathbf{U} \cdot \mathbf{V} \equiv u_i v_i + j^2 \varphi_i \psi_i\,, \qquad i = 1, 2, 3, \tag{2.2}$$

where u_i, v_i, φ_i, ψ_i are the components of the vectors $\mathbf{u}$, $\mathbf{v}$, $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$, respectively, with respect to a Cartesian system of coordinates $Ox_1x_2x_3$ and where the summation convention over repeated indices has been adopted. In view of (2.2), the magnitude of the vector field $\mathbf{V} \equiv (\mathbf{v}, j\boldsymbol{\psi})$ is given by

$$|\mathbf{V}| \equiv (\mathbf{V} \cdot \mathbf{V})^{1/2} = (v_i v_i + j^2 \psi_i \psi_i)^{1/2}. \tag{2.3}$$

The theory of micropolar elasticity employs two strain tensors, e_{rs} and κ_{rs}, which are defined by

$$e_{rs}(\mathbf{U}) \equiv u_{s,r} + \varepsilon_{srk}\varphi_k\,, \qquad \kappa_{rs}(\mathbf{U}) \equiv \varphi_{s,r}\,, \qquad r, s = 1, 2, 3, \tag{2.4}$$

where ε_{srk} denotes the well-known alternating symbol and a comma followed by r stands for the partial differentiation with respect to x_r. These are related to the stress tensor t_{kl} and couple stress tensor m_{kl} by the equations

$$\begin{aligned} t_{kl}(\mathbf{U}) &= a_{klrs}\, e_{rs}(\mathbf{U}) + b_{klrs}\, \kappa_{rs}(\mathbf{U}), \\ m_{kl}(\mathbf{U}) &= b_{rskl}\, e_{rs}(\mathbf{U}) + c_{klrs}\, \kappa_{rs}(\mathbf{U}), \\ & k, l, r, s = 1, 2, 3, \end{aligned} \tag{2.5}$$

where a_{klrs} etc. are material constants which satisfy

$$a_{klrs} = a_{rskl}\,, \qquad c_{klrs} = c_{rskl}\,. \tag{2.6}$$

([1]) The theory of linear micropolar elasticity was introduced by ERINGEN in [9].

Accordingly, the strain energy density corresponding to $\mathbf{U}$ is given by

$$W(\mathbf{U}) \equiv \frac{1}{2}a_{klrs}e_{kl}(\mathbf{U})e_{rs}(\mathbf{U}) + b_{klrs}e_{kl}(\mathbf{U})\kappa_{rs}(\mathbf{U}) + \frac{1}{2}c_{klrs}\kappa_{kl}(\mathbf{U})\kappa_{rs}(\mathbf{U}) \tag{2.7}$$

and in what follows we shall assume that this quadratic form is positive definite.

Following IESAN [3] we label the nine independent index combinations (rs) or (kl) by capital Greek letters Γ, Δ, etc. so that the constitutive equations can be written as

$$\begin{aligned} t_\Gamma(\mathbf{U}) &= a_{\Gamma\Delta}e_\Delta(\mathbf{U}) + b_{\Gamma\Delta}\kappa_\Delta(\mathbf{U}), \\ m_\Gamma(\mathbf{U}) &= b_{\Delta\Gamma}e_\Delta(\mathbf{U}) + c_{\Gamma\Delta}\kappa_\Delta(\mathbf{U}), \\ &\Gamma, \Delta = 1, 2, ..., 9. \end{aligned} \tag{2.8}$$

Introducing the further notations

$$\begin{aligned} &T_\Gamma \equiv t_\Gamma\,, \qquad T_{9+\Gamma} \equiv j^{-1}m_\Gamma\,, \qquad E_\Gamma \equiv e_\Gamma\,, \\ &E_{9+\Gamma} \equiv j\kappa_\Gamma\,, \qquad A_{\Gamma\Delta} \equiv a_{\Gamma\Delta}\,, \qquad A_{\Gamma(9+\Delta)} \equiv j^{-1}b_{\Gamma\Delta}\,, \\ &A_{(9+\Gamma)\Delta} \equiv j^{-1}b_{\Delta\Gamma}\,, \qquad A_{(9+\Gamma)(9+\Delta)} \equiv j^{-2}c_{\Gamma\Delta} \end{aligned} \tag{2.9}$$

we can rewrite the constitutive equations (2.8) in the form

$$T_K(\mathbf{U}) = A_{KL}\,E_L(\mathbf{U}), \qquad K, L = 1, 2, ..., 18. \tag{2.10}$$

We define now the tensor

$$\mathbf{T}(\mathbf{U}) \equiv [t_{rs}(\mathbf{U}),\ j^{-1}m_{rs}(\mathbf{U})] \tag{2.11}$$

whose magnitude is

$$|\mathbf{T}(\mathbf{U})| \equiv [t_{rs}(\mathbf{U})t_{rs}(\mathbf{U}) + j^{-2}m_{rs}(\mathbf{U})m_{rs}(\mathbf{U})]^{1/2}, \tag{2.12}$$

and, using the notations (2.9), we write the strain-energy density function (2.7) as

$$W(\mathbf{U}) = \frac{1}{2}A_{KL}E_K(\mathbf{U})E_L(\mathbf{U}), \qquad K, L = 1, 2, ..., 18, \tag{2.13}$$

where A_{KL} are the components of a symmetric and positive definite tensor. According to GURTIN [10, p. 197], IESAN [3, p. 97] and MEHRABADI, COWIN and HORGAN [11], if μ_M denotes the largest characteristic value of A_{KL}, we have

$$|\mathbf{T}(\mathbf{U})|^2 = T_K(\mathbf{U})T_K(\mathbf{U}) = A_{KL}A_{KS}E_L(\mathbf{U})E_S(\mathbf{U}) \le 2\mu_M W(\mathbf{U}). \tag{2.14}$$

Denoting by $s_i(\mathbf{U})$ and $q_i(\mathbf{U})$ the components of the stress and couple stress vectors acting on the surface ∂B of B, respectively, we have

$$s_i(\mathbf{U}) = t_{ri}(\mathbf{U})n_r\,, \qquad q_i(\mathbf{U}) = m_{ri}(\mathbf{U})n_r\,, \tag{2.15}$$

where n_r are the components of the outwardly directed unit vector normal to ∂B.

In the absence of body forces and couples, the equations of motion are written as

$$(2.16)\qquad \begin{aligned}[t_{si}(\mathbf{U})]_{,s} &= \varrho\, \ddot{u}_i\,, \\ [m_{si}(\mathbf{U})]_{,s} + \varepsilon_{irs}\, t_{rs}(\mathbf{U}) &= \varrho j_{is}\, \ddot{\varphi}_s\,, \qquad \varrho = \text{const}, \quad \varrho > 0,\end{aligned}$$

where j_{is} are the components of $\mathbf{j}$ and ϱ is the mass-density of the body, and these have to be supplemented by appropriate boundary and initial conditions. As usual, in (2.16), a superposed dot denotes the partial derivative with respect to the time t.

3. An energy decay estimate for the case of harmonic vibrations

In this section B will be assumed to occupy a right cylinder of length L whose cross-section is bounded by one or more piecewise smooth curves. The Cartesian system of coordinates is chosen so that the origin belongs to one of the ends and thus the ends of the cylinder lie in the planes $x_3 = 0$ and $x_3 = L$. An arbitrary cross-section of the cylinder at the distance z from $x_3 = 0$ will be referred to as S_z whereas the part of the cylinder which lies above S_z will be denoted by B_z. Clearly, our notation implies that $B_0 = B$.

We further assume here that

$$(3.1)\qquad \begin{aligned} &u_i(\mathbf{x}, t) = 0, \qquad \varphi_i(\mathbf{x}, t) = 0, \\ &\qquad \mathbf{x} \equiv (x_1, x_2, x_3) \in \partial B \backslash S_0\,, \qquad t \in (0, \infty)\end{aligned}$$

and that

$$(3.2)\qquad \begin{aligned} s_i(\mathbf{U}) &= t_i(\mathbf{x}) \sin(\omega t), \\ q_i(\mathbf{U}) &= p_i(\mathbf{x}) \sin(\omega t), \qquad \mathbf{x} \in S_0\,, \quad t \in (0, \infty),\end{aligned}$$

where t_i and p_i are given functions on S_0 and $\omega > 0$ is a given constant.

Seeking solutions (to the problem given by (2.5), (2.16), (3.1) and (3.2)) of the form

$$(3.3)\qquad u_i(\mathbf{x}, t) = v_i(\mathbf{x}) \sin(\omega t), \qquad \varphi_i(\mathbf{x}, t) = \psi_i(\mathbf{x}) \sin(\omega t)$$

we find that the functions v_i and ψ_i must satisfy

$$(3.4)\qquad \begin{aligned}[t_{si}(\mathbf{V})]_{,s} + \varrho\omega^2 v_i &= 0, \\ [m_{si}(\mathbf{V})]_{,s} + \varepsilon_{irs}\, t_{rs}(\mathbf{V}) + \varrho\omega^2 j_{is}\psi_s &= 0, \qquad \mathbf{V} \equiv (v_i, j\psi_i),\end{aligned}$$

$$(3.5)\qquad v_i(\mathbf{x}) = \psi_i(\mathbf{x}) = 0, \qquad \mathbf{x} \in \partial B \backslash S_0\,,$$

and

$$s_i(\mathbf{V}) = t_i(\mathbf{x}), \qquad q_i(\mathbf{V}) = p_i(\mathbf{x}), \qquad \mathbf{x} \in S_0, \tag{3.6}$$

and, according to the theory developed by FICHERA [12], under our assumptions there exists a unique solution $\mathbf{V}$ to the problem (3.4) – (3.6). For the remainder of this section we will be discussing the decay properties of the amplitude function $\mathbf{V}$ of the harmonic vibrations $\mathbf{U}$.

We begin by introducing an auxiliary function given by

$$I(z) \equiv -\int_{S_z} [t_{3r}(\mathbf{V})v_r + m_{3r}(\mathbf{V})\psi_r]\, dA, \qquad z \in [0, L] \tag{3.7}$$

and note that, in view of the Cauchy - Schwarz inequality and the arithmetic-geometric mean inequality, we have the estimate

$$\begin{aligned} I(z) \leq \frac{1}{2}\alpha \int_{S_z} \Big[t_{3r}(\mathbf{V})\, t_{3r}(\mathbf{V}) + j^{-2} m_{3r}(\mathbf{V})\, m_{3r}(\mathbf{V})\Big]\, dA \\ + \frac{1}{2\alpha}\int_{S_z} (v_r v_r + j^2 \psi_r \psi_r)\, dA, \qquad z \in [0, L], \end{aligned} \tag{3.8}$$

where α is an arbitrary positive constant. On combining (3.8) and (2.14), and on taking into account the meaning of the constant j, we infer that

$$I(z) \leq \mu_M \alpha \int_{S_z} W(\mathbf{V})\, dA + \frac{1}{2\alpha\varrho}\int_{S_z} \varrho(v_r v_r + j_{rs}\psi_r\psi_s)\, dA, \qquad z \in [0, L]. \tag{3.9}$$

Using (3.5) and the Divergence Theorem we also find, from (3.7), that

$$\begin{aligned} I(L) - I(z) &= -\int_{\partial B_z} [t_{sr}(\mathbf{V})n_s v_r + m_{sr}(\mathbf{V})n_s\psi_r]\, d\Sigma \\ &= -\int_{B_z} [(t_{sr}v_r)_{,s} + (m_{sr}\psi_r)_{,s}]\, dV, \qquad z \in [0, L]. \end{aligned} \tag{3.10}$$

Since, by assumption, we have $I(L) = 0$, it follows, from (3.10), (3.4) and (2.4) - (2.6), that

$$I(z) = \int_{B_z} \Big[2W(\mathbf{V}) - \varrho\omega^2(v_r v_r + j_{rs}\psi_r\psi_s)\Big]\, dV, \qquad z \in [0, L]. \tag{3.11}$$

We now consider (analogously with equation (2.17) in FLAVIN and KNOPS [5])

$$\omega_m^2(h, L) \equiv \inf \int_R 2W(\mathbf{U})\, dV \Big/ \int_R \varrho(u_r u_r + j_{rs}\varphi_r\varphi_s)\, dV; \tag{3.12}$$

a) within the class of all right cylinders R which share the end $x_3 = L$ and whose lengths belong to the interval $[h, L]$ (where h is an arbitrary fixed number strictly less than L), and

b) within the class of smooth vector fields $\mathbf{U}$ which are such that $u_r = 0$, $\varphi_r = 0$, $r = 1, 2, 3$, on both $x_3 = L$ and the lateral boundaries of these cylinders.

$\omega_m/2\pi$ therefore, represents the minimum fundamental frequency of vibration of cylinders described in (a) whose lateral surfaces and ends $x_3 = L$ are clamped and whose other ends are free. Equations (3.11) and (3.12) lead to

$$I(z) \geq \left(1 - \frac{\omega^2}{\omega_m^2}\right) \int_{B_z} 2W(\mathbf{V})\, dV, \qquad z \in [0, L-h], \tag{3.13}$$

which, when combined with (3.9), gives

$$\left(1 - \frac{\omega^2}{\omega_m^2}\right) \int_{B_z} 2W(\mathbf{V})\, dV \leq \frac{\mu_M \alpha}{2} \int_{S_z} 2W(\mathbf{V})\, dA + \frac{1}{2\alpha\varrho} \int_{S_z} \varrho(v_r v_r + j_{rs}\psi_r\psi_s)\, dA, \qquad z \in [0, L-h]. \tag{3.14}$$

Following TOUPIN [4], we integrate in (3.14) from z to $z+h$, $z \in [0, L-h]$, and on making use of the notations

$$E(z) \equiv \int_{B_z} 2W(\mathbf{V})\, dV, \qquad Q(z,h) \equiv \frac{1}{h} \int_z^{z+h} E(\zeta)\, d\zeta, \tag{3.15}$$

we obtain

$$\left(1 - \frac{\omega^2}{\omega_m^2}\right) Q(z,h) \leq \frac{\alpha\mu_M}{2h} \int_{B(z,h)} 2W(\mathbf{V})\, dV + \frac{1}{2\alpha\varrho h} \int_{B(z,h)} \varrho(v_r v_r + j_{rs}\psi_r\psi_s)\, dV, \tag{3.16}$$

where $B(z,h)$ stands for the cylindrical slice $B_z \backslash B_{z+h}$.

Denoting by $\omega_0(h)/2\pi$ the lowest frequency of vibrations of the slice $B(z,h)$ (whose lateral surface is clamped and whose ends are free) we have, analogously with (3.12),

$$\int_{B(z,h)} 2W(\mathbf{V})\, dV \geq \varrho\omega_0^2(h) \int_{B(z,h)} (v_r v_r + j_{rs}\psi_r\psi_s)\, dV. \tag{3.17}$$

Now (3.17), together with (3.16), leads to

$$\left(1-\frac{\omega^2}{\omega_m^2}\right)Q(z,h)\le\frac{1}{2h}\left[\mu_M\alpha+\frac{1}{\alpha\varrho\omega_0^2(h)}\right]\int\limits_{B(z,h)}2W(\mathbf{V})\,dV,\qquad z\in[0,L-h]. \tag{3.18}$$

Since

$$\frac{dQ}{dz}(z,h)=\frac{1}{h}\left[E(z+h)-E(z)\right]=-\frac{1}{h}\int\limits_{B(z,h)}2W(\mathbf{V})\,dV \tag{3.19}$$

and

$$\frac{1}{2}\left[\mu_M\alpha+\frac{1}{\alpha\varrho\omega_0^2(h)}\right]\ge\frac{1}{\omega_0(h)}(\mu_M/\varrho)^{1/2}, \tag{3.20}$$

we obtain from (3.18) the following first-order differential inequality

$$\frac{1}{\omega_0(h)}(\mu_M/\varrho)^{1/2}\frac{dQ}{dz}(z,h)+\left(1-\frac{\omega^2}{\omega_m^2}\right)Q(z,h)\le 0,\qquad z\in[0,L-h], \tag{3.21}$$

which, by integration, leads to the estimate

$$Q(z,h)\le Q(0,h)\exp\left[-z\left(1-\frac{\omega^2}{\omega_m^2}\right)\omega_0(h)(\mu_M/\varrho)^{-1/2}\right],\qquad z\in[0,L-h]. \tag{3.22}$$

Since $E(z)$ is a non-increasing function of z we have, for all $\zeta\in[z,z+h]$,

$$E(z)\ge E(\zeta)\ge E(z+h) \tag{3.23}$$

which, on account of $(3.15)_2$, gives

$$E(z)\ge Q(z,h)\ge E(z+h),\qquad z\in[0,L-h]. \tag{3.24}$$

Equation (3.24) implies

$$Q(0,h)\le E(0) \tag{3.25}$$

and thus, on account of $(3.24)_2$ and (3.22), we have

$$E(z+h)\le E(0)\exp\left[-z\left(1-\frac{\omega^2}{\omega_m^2}\right)\omega_0(h)(\mu_M/\varrho)^{-1/2}\right],\quad z\in[0,L-h], \tag{3.26}$$

which can be re-written as

$$E(z) \leq E(0) \exp\left[-(z-h)\left(1-\frac{\omega^2}{\omega_m^2}\right)\omega_0(h)(\mu_M/\varrho)^{-1/2}\right], \quad z \in [h,L]. \tag{3.27}$$

Inequality (3.27) shows that, in the strain-energy measure, the amplitude $\mathbf{V}$ decays exponentially as a function of $z \in [h, L]$, provided that $\omega < \omega_m$. As such, (3.27) is a Saint-Venant type inequality as originally envisaged by TOUPIN [4]. In the following section we will be obtaining an estimate for $E(0)$ which, in addition to providing us with an explicit estimate for $E(z)$ in terms of the data, implies the continuous dependence on data of solutions to the boundary value problem (3.4) – (3.6).

4. The estimate for $E(0)$

Since, by (3.13), we have (see $(3.15)_1$)

$$E(0) \leq \frac{\omega_m^2}{\omega_m^2-\omega^2} I(0), \qquad \omega < \omega_m, \tag{4.1}$$

we will be estimating in what follows the quantity $I(0)$ which, in view of (3.6), can be written as

$$I(0) = -\int_{S_0} (t_r v_r + p_r \psi_r)\, dA. \tag{4.2}$$

To this end, we note that the Cauchy – Schwarz inequality implies

$$I(0) \leq \left[\int_{S_0} \left(t_r t_r + j^{-2} p_r p_r\right) dA\right]^{1/2} \cdot D(\mathbf{V})^{1/2}, \tag{4.3}$$

where we have employed the notation

$$D(\mathbf{V}) \equiv \int_{S_0} \left(v_r v_r + j^2 \psi_r \psi_r\right) dA. \tag{4.4}$$

Next, we use an appropriate re-scaling of the well-known trace theorem [12, p. 353] which reads

$$\int_{\partial B} w_r w_r\, ds \leq A_1 \int_B w_r w_r\, dV + A_2 \int_B w_{r,s} w_{r,s}\, dV, \tag{4.5}$$

where A_1 and A_2 are positive constants which depend upon B and ∂B and where w_r are functions of class $C^1(B \cup \partial B)$. Accordingly, from (4.4) and (4.5), we infer

$$D(\mathbf{V}) \leq A_1 \int\limits_B \left(v_r v_r + j^2 \psi_r \psi_r\right) dV + A_2 \int\limits_B \left(v_{r,s} v_{r,s} + j^2 \psi_{r,s} \psi_{r,s}\right) dV \tag{4.6}$$

which, in view of the significance of the constant j, leads to (see $(2.4)_2$)

$$D(\mathbf{V}) \leq A_1 \int\limits_B (v_r v_r + j_{rs} \psi_r \psi_s)\, dV + A_2 \int\limits_B \left[v_{r,s} v_{r,s} + j^2 \kappa_{rs}(\mathbf{V}) \kappa_{rs}(\mathbf{V})\right] dV. \tag{4.7}$$

Since $\mathbf{v} = \mathbf{0}$ on $\partial B \backslash S_0$ we have the first Korn inequality

$$\begin{aligned} \int\limits_B v_{r,s} v_{r,s}\, dV &\leq \frac{1}{4} A_3 \int\limits_B (v_{r,s} + v_{s,r})(v_{r,s} + v_{s,r})\, dV, \\ A_3 &= A_3(B) = \text{const}, \qquad A_3 > 0, \end{aligned} \tag{4.8}$$

which, when combined with [8]

$$\frac{1}{4} (v_{r,s} + v_{s,r})(v_{r,s} + v_{s,r}) \leq e_{rs}(\mathbf{V})\, e_{rs}(\mathbf{V}) \tag{4.9}$$

and (4.7), gives

$$D(\mathbf{V}) \leq A_1 \int\limits_B (v_r v_r + j_{rs} \psi_r \psi_s)\, dV + A_4 \int\limits_B \left[e_{rs}(\mathbf{V}) e_{rs}(\mathbf{V}) + j^2 \kappa_{rs}(\mathbf{V})\, \kappa_{rs}(\mathbf{V})\right] dV, \tag{4.10}$$

where

$$A_4 \equiv A_2 \max(1, A_3).$$

The assumption that W is positive definite implies that there exists a positive constant μ_m[2] so that

$$2W(\mathbf{U}) \geq \mu_m \left[e_{rs}(\mathbf{U})\, e_{rs}(\mathbf{U}) + j^2 \kappa_{rs}(\mathbf{U})\, \kappa_{rs}(\mathbf{U})\right] \tag{4.11}$$

(2) μ_m is the lowest characteristic value of A_{KL}. See [10, p. 197].

and, by denoting with $\omega_v/2\pi$ the fundamental frequency of the cylinder B whose entire surface, apart from its end plane S_0, is clamped we also have, as before,

$$(4.12) \qquad \int_B 2W(\mathbf{U})\, dV \geq \varrho\omega_v^2 \int_B (u_r u_r + j_{rs}\varphi_r\varphi_s)\, dV,$$

for any smooth vector field $\mathbf{U}$. Thus, from (4.3), (4.10), (4.11) and (4.12) we find (see $(3.15)_1$)

$$(4.13) \qquad I(0) \leq \left(\frac{A_1}{\varrho\omega_v^2} + \frac{A_4}{\mu_m}\right)^{1/2} E(0)^{1/2} \left[\int_{S_0} \left(t_r t_r + j^{-2} p_r p_r\right) dA\right]^{1/2},$$

which, together with (4.1), gives

$$(4.14) \qquad E(0) \leq \frac{\omega_m^4}{(\omega_m^2 - \omega^2)^2} \left(\frac{A_1}{\varrho\omega_v^2} + \frac{A_4}{\mu_m}\right) \int_{S_0} \left(t_r t_r + j^{-2} p_r p_r\right) dA.$$

Inequality (4.14) expresses the fact that in the strain-energy measure, the solution to the boundary value problem (3.4) – (3.6) depends continuously on the data fot $\omega < \omega_m$. Continuous dependence estimates for stresses, strains and displacements can be obtained by combining (4.14) with (2.14), (4.11) and (4.12), respectively. Finally, we note that on combining (4.14) with (3.27) we can obtain an explicit decay estimate for $E(z)$ in terms of the data prescribed on the boundary of the cylinder.

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