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Application of the Fourier cosine series to the approximation of solutions to initial non-Dirichlet boundary-value problems

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THE PAPER deals with an application of the Fourier cosine series to the determination of an approximate solution to some one-dimensional initial boundary-value problems. With the new approach one can approximate solutions of many equations of engineering and physics, without solving the eigenvalue problems. It has been found out that the new method can successfully be used for linear partial differential equations with non-Dirichlet boundary conditions. The heat equation and the wave equation with constant coefficients have been solved using the method described. The solutions have been compared to those obtained by means of the method of seperation of variables. The numerical results show that the new solutions approximate well the classical solutions. For the heat equation, even the boundary conditions at the initial instant of time are satisfied. This does not occur, however, in the case of the wave equation, since the initial displacement of the rod does not satisfy prescribed boundary conditions.

1. Introduction

THERE ARE some useful methods of solving linear initial boundary-value problems of partial differential equations. One of them is the method of seperation of variables, called the Fourier method [1]. It consists first in finding solutions of the corresponding eigenvalue problem for functions of spatial variables and next, in solving the set of decoupled ordinary differential equations for functions of time variable only. Finally the solution to the boundary-value problem is represented by an infinite series of these functions.

In [5] presenting the solution of the heat conduction equation it has been shown, that the solution to the problem can be represented, with an arbitrary accuracy, by the Fourier cosine series whose spatial components do not satisfy the boundary conditions given. In [6] the approach was applied to many other differential equations, both ordinary and partial. Many initial and boundary-value problems of linear and nonlinear ordinary differential equations were solved. Many cases with variable parameters were treated with this method as well.

In the present paper we prove that the Fourier cosine series is the "weak" solution to the heat conduction problem and to the wave equation, which is the solution to the so-called Integro-Differential-Boundary Equations (IDBE) [5] derived for the corresponding equation. The Fourier coefficients are calculated from the corresponding Infinite Set of Ordinary Differential Equations (ISODE) using the Runge–Kutta method. The way how to get the IDBE and ISODE is shown in Secs. 3 and 4 of the paper, as well as in [5, 6].

In this paper, using the Fourier cosine series we solve two initial boundary-value problems with non-Dirichlet boundary conditions without solving the eigenvalue problems. The new approach has been applied to the equation describing the heat conduction subject to non-Dirichlet boundary conditions, and for the wave equation describing the vibrations of a rod also subject to non-Dirichlet boundary conditions. Solving the corresponding ISODE truncated at $N_a = 10$ for the heat equation and at $N_a = 15$ for the wave equation, a satisfactory approximation of the solutions obtained by means of the method of seperation of variables (called classical solutions), truncated at $N_c = 5$ for the heat equation and at $N_c = 10$ for the wave equation, have been achieved. Analysis of the boundary conditions has shown that for the heat conduction equation with prescribed initial condition, the boundary conditions at $t = 0$ are satisfied with an error decreasing as the number of components of the Fourier cosine series N_a increases. Analysing the boundary conditions of the wave equation for a given initial displacement of the rod, we have derived formulas for the boundary conditions at $t = 0$. They are expressed as convergent series of the Fourier cosine coefficients $c_k(0)$ but they do not tend to zero, which means that the new method of solution does not satisfy the prescribed boundary conditions at $t = 0$. The classical solution to the wave equation (derived by the method of separation of variables) is a generalized solution [2] and does not satisfy the prescribed boundary conditions either, since the initial condition u_0 for the problem does not satisfy the boundary conditions given [2].

2. Description of the method

Let us consider two second-order linear partial differential equations of the form:

$$\begin{aligned} &\frac{\partial U}{\partial t} - P\frac{\partial^2 U}{\partial x^2} - R\frac{\partial U}{\partial x} - QU = 0 \quad \text{for} \quad (x,t) \in (0,L) \times (0,t_e), \\ &\frac{\partial^2 U}{\partial t^2} - P\frac{\partial^2 U}{\partial x^2} - R\frac{\partial U}{\partial x} - QU = 0 \quad \text{for} \quad (x,t) \in (0,L) \times (0,t_e), \end{aligned} \tag{2.1}$$

with the boundary conditions

$$\begin{aligned} \alpha U + \beta\frac{\partial U}{\partial x} &= 0 \qquad \text{for} \quad x = 0, \\ \gamma U + \delta\frac{\partial U}{\partial x} &= 0 \qquad \text{for} \quad x = L \end{aligned} \tag{2.2}$$

for $t \in [0, t_e)$, and the initial conditions

$$U(x,0) = u_0(x), \qquad \frac{\partial U}{\partial t}(x,0) = v_0(x), \qquad \text{for} \quad x \in [0,L]. \tag{2.3}$$

P, Q, R in (2.1) are constants or functions of t only, α, β, γ and δ in (2.2) are constants and u_0 and v_0 in (2.3) are given functions of $x \in [0, L]$.

We assume that

$$\beta\delta \neq 0. \tag{2.4}$$

Let ϕ_n with $n = 0, 1, 2, \ldots$, denote functions of one space variable which form an orthogonal set on $\mathcal{L}^2[0, L]$ and let $\phi_n'' = -\mu_n^2\phi_n$ for each n, where the double prime denotes the second derivative. Upon multiplying (2.1) by ϕ_n and integrating over the interval $(0, L)$, we see that

$$\frac{d^i}{dt^i}\int_0^L U(x,t)\phi_n(x)\,dx - P\int_0^L \frac{\partial^2 U}{\partial x^2}(x,t)\phi_n(x)\,dx - R\int_0^L \frac{\partial U}{\partial x}(x,t)\phi_n(x)\,dx - Q\int_0^L U(x,t)\phi_n(x)\,dx = 0, \tag{2.5}$$

where $i = 1$ corresponds to Eq. $(2.1)_1$ and $i = 2$ corresponds to Eq. $(2.1)_2$. Putting the following

$$\int_0^L \frac{\partial U}{\partial x}(x,t)\phi_n(x)\,dx = \phi_n(x)U(x,t)\Big|_0^L - \int_0^L U(x,t)\phi_n'(x)\,dx,$$

$$\int_0^L \frac{\partial^2 U}{\partial x^2}(x,t)\phi_n(x)\,dx = \phi_n(x)\frac{\partial U}{\partial x}(x,t)\Big|_0^L - \phi_n'(x)U(x,t)\Big|_0^L - \mu_n^2\int_0^L U(x,t)\phi_n(x)\,dx$$

into (2.5), we obtain the Integro-Differential-Boundary Equations [5] for the problems

$$\begin{aligned} \frac{d^i}{dt^i}\int_0^L U(x,t)\phi_n(x)\,dx + (P\mu_n^2 - Q)\int_0^L U(x,t)\phi_n(x)\,dx \\ + R\int_0^L U(x,t)\phi_n'(x)\,dx = F_n\,, \\ F_n := U(0,t)\left(P\phi_n'(0) + \left(P\frac{\alpha}{\beta} - R\right)\phi_n(0)\right) \\ - U(L,t)\left(P\phi_n'(L) + \left(P\frac{\gamma}{\delta} - R\right)\phi_n(L)\right), \qquad n = 0, 1, 2, \ldots\,. \end{aligned} \tag{2.6}$$

On the right-hand side of F_n boundary conditions (2.2) have been taken into account. The functions F_n do not describe the case with Dirichlet boundary conditions. They are valid only for $\beta\delta \neq 0$ (non-Dirichlet boundary conditions).

3. "Weak" solutions to some boundary value problems

Let us consider the IDBE (2.6) with $R = 0$

$$\frac{d^i}{dt^i}\int_0^L U(x,t)\phi_n(x)dx + (P\mu_n^2 - Q)\int_0^L U(x,t)\phi_n(x)\,dx = F_n\,,$$

$$F_n = U(0,t)\left(\phi_n'(0) + \frac{\alpha}{\beta}\phi_n(0)\right)P - U(L,t)\left(\phi_n'(L) + \frac{\gamma}{\delta}\phi_n(L)\right)P, \qquad n = 0,1,2,\ldots, \tag{3.1}$$

and introduce

DEFINITION. *A function*

$$u(\cdot,\cdot) : [0,L]\times[0,t_e) \to \mathcal{R}$$

is a "weak" solution to the boundary-value problem $(2.1)_1$, (2.2) *or* $(2.1)_2$, (2.2) *(for* $R = 0$*) with initial conditions* $(2.3)_1$ *or* (2.3), *respectively, if it satisfies the IDBE* (3.1), *that is the function* u *is a solution to*

$$\frac{d^i}{dt^i}\int_0^L u(x,t)\phi_n(x)\,dx + (P\,\mu_n^2 - Q)\int_0^L u(x,t)\phi_n(x)\,dx = F_n\,,$$

$$F_n = u(0,t)\left(\phi_n'(0) + \frac{\alpha}{\beta}\phi_n(0)\right)P - u(L,t)\left(\phi_n'(L) + \frac{\gamma}{\delta}\phi_n(L)\right)P, \qquad n = 0,1,2,\ldots, \tag{3.2}$$

with $i = 1$ *for the heat equation and* $i = 2$ *for the wave equation.*

This definition differs from the definition known from the literature [3]; that is why we put it in quotes and name it "weak".

Now we shall prove the following

PROPOSITION. The "weak" solution to the boundary-value problem $(2.1)_1$, (2.2) or $(2.1)_2$, (2.2) (for $R = 0$) with initial conditions $(2.3)_1$ or (2.3), respectively, can be represented by the Fourier cosine series

$$u(x,t) \sim \frac{c_0(t)}{2} + \sum_{n=1}^{\infty} c_n(t)\cos\left(\frac{n\pi}{L}x\right), \tag{3.3}$$

whose coefficients satisfy an Infinite Set of Ordinary Differential Equations (ISODE):

$$
(3.4)\qquad
\begin{aligned}
&\frac{d^i}{dt^i}c_n + (P\mu_n^2 - Q)c_n = \frac{2}{L}F_n\,,\\
F_n &= \left(u(0,t)\frac{\alpha}{\beta} - u(L,t)\frac{\gamma}{\delta}(-1)^n\right)P,\\
c_n(0) &= \frac{2}{L}\int_0^L u_0(x)\cos\left(\frac{n\pi}{L}x\right)dx \qquad \text{for} \quad i = 1,2,\\
\dot{c}_n(0) &= \frac{2}{L}\int_0^L v_0(x)\cos\left(\frac{n\pi}{L}x\right)dx \qquad \text{for} \quad i = 2 \ \text{only},\\
&n = 0,1,2,\dots\,,
\end{aligned}
$$

where $\mu_n = n\pi/L$.

Proof. Let $u(x,t)$ represented by (3.3) be the solution to (3.2). For this representation, $\phi_n \equiv \cos(n\pi x/L)$ for $n = 0,1,2,\ldots$, constitute the orthogonal bases in $\mathcal{L}^2[0,L]$, with $\mu_n = n\pi/L$ and the Fourier cosine series coefficients of the solution (3.3) can be calculated from

$$
(3.5)\qquad c_n(t) = \frac{2}{L}\int_0^L u(x,t)\cos\left(\frac{n\pi}{L}x\right)dx, \qquad n = 0,1,2,\ldots\,.
$$

If we now multiply $(3.2)_1$ by $2/L$ then we simply come to ISODEs $(3.4)_{1,2}$ for coefficients c_n. The initial conditions $(3.4)_{3,4}$ for the ISODEs follow from (3.5).

4. Main results

We shall consider two initial boundary value problems with non-Dirichlet boundary conditions:

- the heat conduction problem

$$
(4.1)\qquad \frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0 \qquad \text{for} \quad (x,t) \in (0,L)\times(0,t_e)
$$

with boundary conditions

$$
(4.2)\qquad
\begin{aligned}
\frac{\partial U}{\partial x} - \mathrm{Bi}\,U &= 0 \qquad \text{for} \quad x = 0,\\
\frac{\partial U}{\partial x} + \mathrm{Bi}\,U &= 0 \qquad \text{for} \quad x = L
\end{aligned}
$$

for $t \in [0,t_e)$, where Bi is the Biot number, and the initial condition

$$U(x,0) = u_0(x), \qquad \text{for} \quad x \in [0,L], \tag{4.3}$$

and

- the problem of vibration of a rod

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = 0 \qquad \text{for} \quad (x,t) \in (0,L) \times (0,t_e) \tag{4.4}$$

with the boundary conditions

$$\begin{aligned} \frac{\partial U}{\partial x} &= 0 \qquad \text{for} \quad x = 0, \\ \frac{\partial U}{\partial x} + gU &= 0 \qquad \text{for} \quad x = L \end{aligned} \tag{4.5}$$

for $t \in [0, t_e)$, where g is constant,
and the initial conditions

$$U(x,0) = u_0(x), \qquad \frac{\partial U}{\partial t}(x,0) = v_0(x), \qquad \text{for} \quad x \in [0,L]. \tag{4.6}$$

4.1. The heat conduction problem

The corresponding ISODE for the problem is the following one:

$$\begin{aligned} &\dot{c}_k + \mu_k^2 c_k + \frac{2\mathrm{Bi}}{L}\left(\frac{c_0(t)}{2}[1+(-1)^k] + \sum_{n=1}^{\infty} c_n(t)[1+(-1)^k(-1)^n]\right) = 0, \\ &c_k(0) = \frac{2}{L}\int_0^L u_0(x)\cos\left(\frac{k\pi}{L}x\right)dx, \qquad k = 0,1,2,\dots\ . \end{aligned} \tag{4.7}$$

The calculations were carried out for $u_0(x) = 1 + \sin[2\pi(x - L/4)/L]$, $L = 1$ with the Biot number $\mathrm{Bi} = 0.185$. The solution

$$U_a(x,t) \approx c_0(t)/2 + \sum_{k=1}^{N_a} c_k(t)\cos(k\pi x/L) \tag{4.8}$$

for $N_a = 10$ and its spatial derivatives for $N_a = 30$ evaluated for some time instants for every section of the layer, are presented in Figs. 1 and 2, respectively. The new results have been compared to the corresponding results of the classical solution

$$U_c(x,t) = \sum_{k=1}^{\infty} a_k \exp(-\omega_k^2 t)\psi_k(x), \tag{4.9}$$

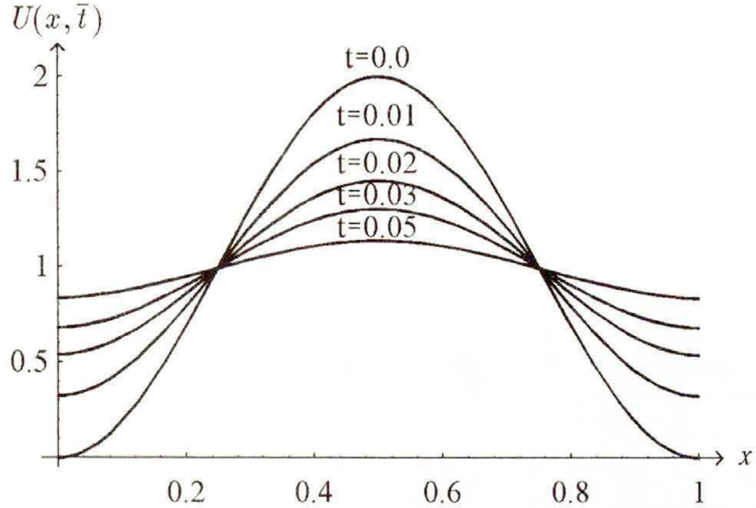

FIG. 1. Solution of the heat equation for some values of $\bar{t}$, for new solution (4.8) and for classical solution (4.9) (they cannot be distinguished).

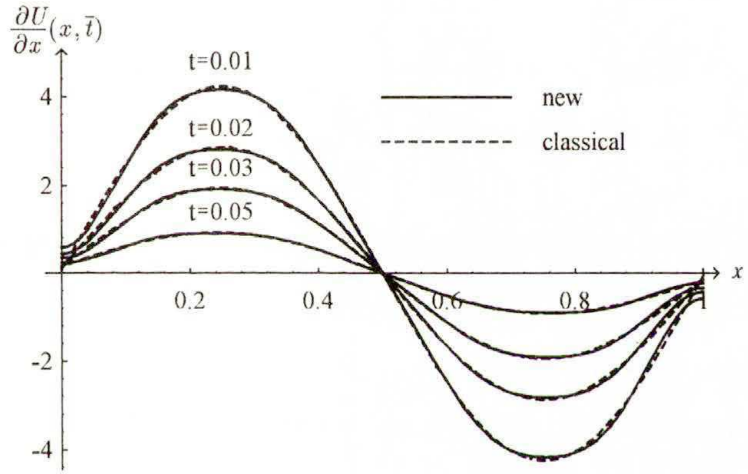

FIG. 2. Spatial derivative of the solution of the heat equation for some values of $\bar{t}$.

where $N_c = 5$ components of the series (4.9) were taken, and

$$\psi_k(x) = \omega_k \cos(\omega_k x) + \mathrm{Bi}\,\sin(\omega_k x)$$

are the eigenfunctions of the problem (4.1)–(4.3), with eigenvalues calculated from the equation

$$\mathrm{ctg}(\omega L) = \frac{\omega^2 - \mathrm{Bi}^2}{2\omega\,\mathrm{Bi}},$$

and

$$a_k = \int_0^L u_0(x)\psi_k(x)\,dx / \|\psi_k(x)\|^2.$$

From the figures presented one can see that the new solution and the classical solution cannot be distinguished even at the boundaries. This shows how the new solution converges well to the classical one. The spatial derivative of the new solution calculated for $N_a = 30$ does not approximate so well the spatial derivative of the classical solution as it happens in the case of the solutions themselves. This is true especially at the boundaries. The error is the largest for $t = 0$. One can show, however, that the error at $t = 0$ tends to 0 as the number N_a increases (see Fig. 3).

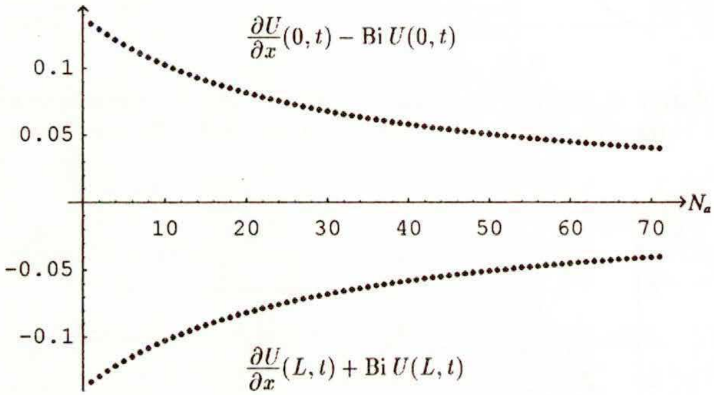

FIG. 3. Boundary conditions for the heat conduction at $t = 0$ according to the new approach.

4.2. Vibrations of a rod

For the vibrations of a rod we solved the following ISODE

$$\begin{aligned} &\ddot{c}_k + \mu_k^2 c_k + (-1)^k \frac{2g}{L}\left(\frac{c_0(t)}{2} + \sum_{n=1}^{\infty} c_n(t)(-1)^n\right) = 0, \\ &c_k(0) = \frac{2}{L}\int_0^L u_0(x)\cos\left(\frac{k\pi}{L}x\right)dx, \qquad (4.10) \\ &\dot{c}_k(0) = \frac{2}{L}\int_0^L v_0(x)\cos\left(\frac{k\pi}{L}x\right)dx, \qquad k = 0, 1, 2, \ldots . \end{aligned}$$

The calculations were carried out for $u_0(x) \equiv a(x - L)$, $v_0(x) \equiv 0$, $g = 2$,

$a = -0.01$. The solution

$$U_a(x,t) \approx c_0(t)/2 + \sum_{k=1}^{N_a=15} c_k(t)\cos(k\pi x/L) \tag{4.11}$$

for some time instants for every cross-section of the rod is presented in Fig. 4, but the solution for chosen cross-sections of the rod in the given time period are shown in Fig. 5. The new results have been compared with the classical solutions

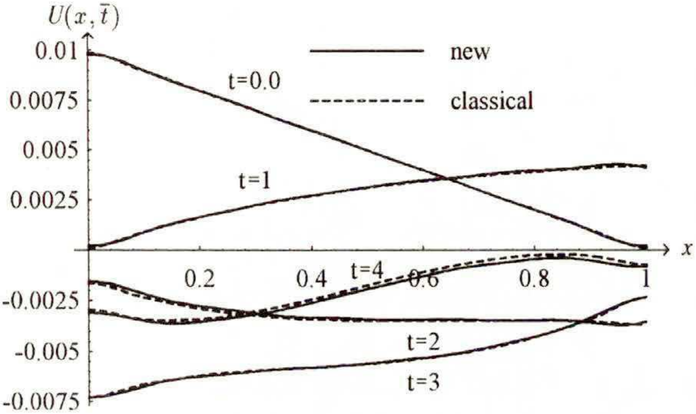

FIG. 4. Solution of the wave equation for some values of $\bar{t}$.

of the problem (4.4) – (4.6),

$$U_c(x,t) = \sum_{k=1}^{\infty}[a_k\cos(\omega_k t) + b_k\sin(\omega_k t)]\cos(\omega_k x), \tag{4.12}$$

with $N_c = 10$ components of the series (4.12) taken, and with eigenvalues calculated from the equation

$$\omega\tan(\omega L) = g$$

and

$$a_k = \int_0^L u_0(x)\cos(\omega_k x)\,dx/\|\cos(\omega_k x)\|^2,$$

$$b_k = \frac{1}{\omega_k}\int_0^L v_0(x)\cos(\omega_k x)\,dx/\|\cos(\omega_k x)\|^2 = 0.$$

From Figs. 4 and 5 one can see that the new solution approximates well the classical solution, although the curves are quite complicated.

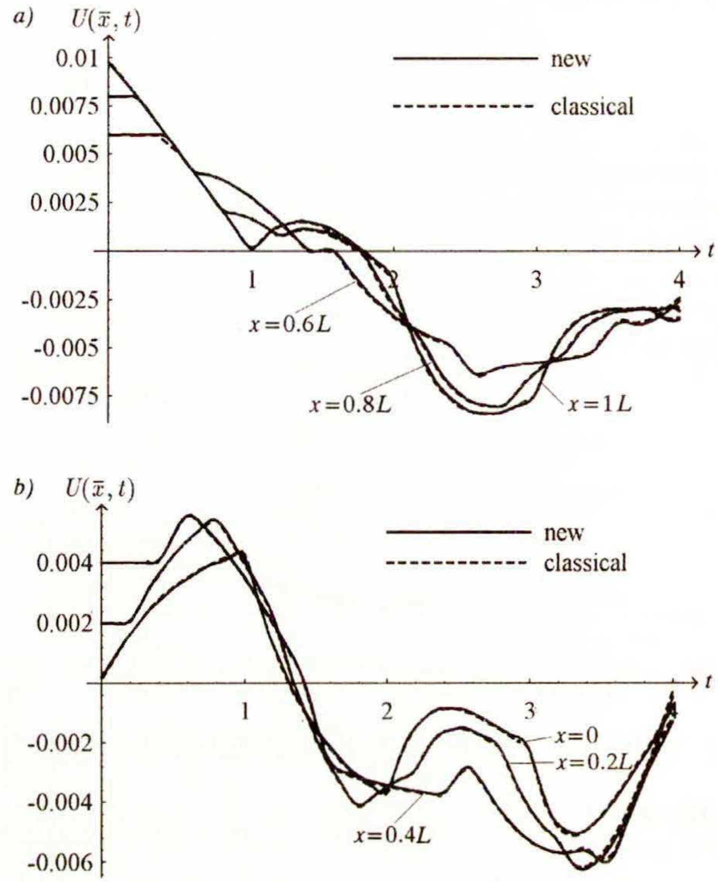

FIG. 5. Solution of the wave equation for some values of $\overline{x}$.

From the boundary conditions of the problem considered, using Theorem 6 from [6], one can derive the following formulas for the boundary conditions at $t = 0$:

$$
(4.13)\qquad
\begin{aligned}
a + \sum_{k=1}^{\infty} \overline{c}_k(0) &= 0 \qquad \text{for} \quad x = 0, \\
a - g\frac{a\,L}{2} + \sum_{k=1}^{\infty} \left[\overline{c}_k(0) + g\,c_k(0)\right](-1)^k &= 0 \qquad \text{for} \quad x = L,
\end{aligned}
$$

where

$$
\overline{c}_k(0) := \frac{16a}{\pi^2} \sum_{j=1}^{\infty} \frac{1}{(2k)^2 - (2j-1)^2}, \qquad k = 1, 2,
$$

and

$$c_k(0) = -\frac{4aL}{(2k-1)^2\pi^2}, \qquad k = 1, 2, \dots .$$

The left-hand sides of (4.13) are covergent series but they do not equal 0. Their value is a for both $x = 0$ and $x = L$. Therefore the boundary conditions for this initial condition are not satisfied in the new approach. The conditions are not satisfied in the classical approach as well, as the classical solution (4.12), for this initial condition u_0 is in a generalized form [2].

5. Remarks

The results obtained in the paper have revealed that the new method can succesfully be used for the solution to other boundary value problems with non-Dirichlet boundary conditions. The experience gained also shows that the new approach can be used for other boundary conditions (e.g. Dirichlet conditions) [6] and for other BVPs that cannot be solved by the method of seperation of variables (e.g. boundary value problems with mixed derivatives).

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