Arch. Mech., 49, 3, pp. 589–609, Warszawa 1997

A gradient theory of finite viscoelasticity

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IN THIS PAPER we present a *gradient* theory of finite viscoelasticity. The theory is founded on the concept of internal fields, in conjunction with a variational principle and the dissipation inequality. The internal variables, which in local theories obey *local* evolution equations, have been replaced by internal fields *and their gradients*, which arise from physical processes that involve non-affine deformation. At variance with the local theory, these fields obey "internal" field equations and appropriate boundary and initial conditions. As a result, *uniform* boundary tractions give rise to inhomogeneous strain fields. This phenomenon is illustrated in one dimension, where it is shown that the creep function, normally a function of time only, is a function of space as well as time, even though the material domain is phenomenologically homogeneous.

1. Introduction

WE BEGIN with the experimental observation that macroscopically *uniform* material domains, under *uniform* surface tractions, develop localized, i.e., non-uniform deformation fields – contrary to predictions of "local" theories. There are other issues such as "regularization" whereby ill-posed boundary and/or initial value problems are rectified by the introduction of gradients in the constitutive parameters. Such issues are less clear and are often due to the inadequacy of the constitutive theories, rather than the material behaviour itself.

Phenomenological theories with higher gradients in mass density began with the work of Van der Waal. More recently we have witnessed the development of elasticity (hyperelasticity) theories with higher deformation gradients. We mention the papers of TOUPIN [1, 2], MINDLIN [3, 4] and GREEN and RIVLIN [5] as characteristic of that era. It is not our purpose to discuss these theories in detail except to say that a more precise formulation of the constitutive response of an elastic material was sought, in the light of the perceived long-range interaction effects, particularly when strong spatial variations in the boundary tractions and/or displacements were present.

In the present paper, the phenomenon of localization of macroscopic deformation was a motivating force for the development of continuum theories with (higher) gradients in the constitutive variables. Their recent advocacy, AIFANTIS [6–8], is due mainly to finer and more convincing experiments, pointing to the need for including gradients of these variables in a constitutive equation.

Other contributions in this area were forthcoming. We note the work of VARDOULAKIS and AIFANTIS [9] and VARDOULAKIS and FRANTZISKONIS [10] in the area of plasticity, where higher order plastic deformation gradients were introduced. The field is still in its infancy with a large scope for development.

Our object, in this paper, is to develop a *gradient theory of* viscoelasticity using the notion of *internal fields under isothermal conditions*. The condition of uniform temperature allows for the developmemt of the theory in a strictly *mechanical setting* without a need for entropy arguments, even though such arguments have been used before, successfully, in the context of local theories. See VALANIS [11–13] and COLEMAN and GURTIN [14].

2. Physical foundations

The basic premise of continuum mechanics is that the deformation of a material region is given mathematically by a one-to-one and on-to mapping:

$$x \to y \tag{2.1}$$

in the usual notation. Both frames x and y are Euclidean or reducible to Euclidean by a coordinate transformation. More importantly, the deformation of a neighbourhood in x, this being a "sphere" of radius $\|dx\| \le \delta$, where parallel bars denote the Euclidean norm and δ is a suitably small number, is given by Eq. (2.1):

$$dy_i = F_{i\alpha}\, dx^{\alpha}, \tag{2.2}$$

where the deformation gradient $F_{i\alpha}$ $(\partial y_i/\partial x^{\alpha})$ is non-singular and constant within the neighbourhood. More precisely, no matter how heterogeneous the deformation is, a sufficiently small δ can be found such that $\|F_{i\alpha}dx^{\alpha}\|$ is of order δ. A fundamental topological consequence of the above assertions, in terms of the motion of discrete particles within a neighbourhood, is that the order of disposition of the particles is invariant under deformation. Thus, a material line contains always the same particles and in an order that remains unchanged with deformation.

Furthermore, neighbourhoods that are disjoint sets of particles before deformation, remain disjoint after deformation, i.e., no particle "diffusion" is allowed and particle membership of the initial material neighbourhood is conserved in the course of deformation. In short, the deformation of a neighbourhood given by Eq. (2.2) is affine. However, the basic physical characteristic of inelastic deformation is the non-affine motion of the particles either through the mechanism of slip, dislocation motion or particle migration.

2.1. Non-affine deformation

We consider a neighbourhood N undergoing non-affine deformation. In this case, at least one particle which, before deformation, occupied a position P in N, now occupies a position P' *not* in N. Note that this position P' cannot be described by Eq. (2.2). If a sufficiently large number of particles leave their original neighbourhoods, one may regard these as constituting a "migratory phase", ideally a continuous material sub-domain of particles whose material coordinates are no longer x^α but $p^K(x^\alpha, t)$. The functions p^K are posited to be continuous and differentiable in x^α and t; furthermore $\mathrm{Det}(\partial p^K/\partial x^\alpha) \neq 0$. We call $p^K(x^\alpha)$, or $p(x)$ for short, the "migration map".

To avoid repetition, lower Latin suffixes will denote vector (tensor) components in the y-frame, upper Latin suffixes in the p-frame and upper Greek – those in the x-frame. Furthermore any such suffixes following a comma, will denote partial differentiation with respect to the corresponding coordinate.

Consider now the deformation of a domain whose particles *initially* at x^α, occupy points y_i in a Euclidean spatial frame. Further, let a subset of these particles, previously referred to as the migratory phase, occupy positions $p^K(x^\alpha, t)$ in the material reference frame. The deformation gradient of the affine phase, i.e. the phase of particles that have *not* migrated, is $\partial y_i/\partial x^\alpha$, symbolically y_x, the suffix x denoting differentiation, while the deformation gradient of the migratory phase is $\partial y_i/\partial p^K$ or y_p. In a more general setting, n migratory phases could exist, each with a different migration map $p^r(x)$.

REMARK. The particles that constitute the affine as well as the other phases, are *indistinguishable* in the deformed configuration y. Their only signature lies in the description of their motion relative to the x-frame. The motion of the ones that deform affinely is given by the map $x \to y$, while the motion of those in a migratory phase is given by the map $p^r \to y$. Thus the position y in the deformed configurations pertains to *all* particles, irrespective of phase, and similarly the traction on the surface of the deformed domain bears on all the particles of a neighbourhood of the deformed surface. The same argument applies in the case of body forces (inertia forces included).

2.2. The free energy density

The physics that underlies the migration process is very complex. Here we shall consider two simple, yet realistic models of this process with a view to obtaining equations that are reasonably tractable. Because viscoelasticity applies most naturally to polymeric materials, we shall consider models that pertain to such materials.

MODEL (i). This is an assembly of polymer networks that are not *elastically* interactive with each other. However they impede each other's motion in a resistive sense, so that they are *viscously* interactive. Initially, particles of the networks are identified by the material coordinate x^α. In the course of deformation however, the networks drift relative to each other, thus constituting migratory motion in the sense discussed above. Thus each network is a phase and the *particles within the phase ineract between themselves elastically*. We may thus posit a cross-linked reference network that deforms affinely, relative to which the other phases (networks) suffer migratory motion. It is clear that in this model the (Helmholtz) free energy density ψ of the whole is the sum of the free energy densities ψ^r of its parts ($r = 0, 1, \ldots, n$), where ψ^0 pertains to the affine phase. The following equations, therefore, are applied

$$
\begin{aligned}
\psi^0 &= \psi^0(y_{i,\alpha}), \\
\psi^r &= \psi^r(y_{i,K}{}^{(r)}) = \psi^r\{y_{i,\alpha}x^{\alpha}{}_{,K^{(r)}}\}, \\
\psi &= \sum_0^r \psi^r,
\end{aligned}
\tag{2.3}
$$

where, in Eq. $(2.3)_2$ the chain rule of differentiation was used. Implicit in Eqs. (2.3) and (2.4) is the stipulation that the interactive forces among particles are of short range. A general statement of Eq. $(2.3)_3$ is Eq. (2.4):

$$
\psi = \psi(y_x, y_p{}^r), \qquad r = 1, 2, \ldots, n. \tag{2.4}
$$

Of interest is the case where the phase drifts relative to the initial configuration but maintains an elastic, albeit weak, connection with that configuration. If this connection is modelled by means of an elastic spring, then there will be an additional contribution to ψ by virtue of the term $(p^K - \delta^k{}_\alpha x^\alpha)$, i.e., the difference in position of the phase at time t and at time zero. Thus now:

$$
\psi = \psi(y_{i,\alpha}; x^\alpha{}_{,K}; p^K - \delta^K{}_\alpha x^\alpha) \tag{2.4$'$}
$$

or

$$
\psi = \psi(y_x; x_p; p - x). \tag{2.4$''$}
$$

MODEL (*ii*). This model is more complex and it represents a different physical situation. Initially, all the networks are cross-linked and elastically interactive, so that the material consists of one single cross-linked network udergoing affine deformation. Thus, initially,

$$
\psi = \psi(y_{i,\alpha}). \tag{2.5}
$$

Since, however, the bonds have strength of statistical variability, one may conceive a situation where at some critical free energy level $\psi = \psi_{(1)}$, one phase, say $r = 1$, will become elastically detached, so that subsequently,

$$
\psi = \psi^\star(y_{i,\alpha}) + \psi^1(y_{i,K}{}^{(1)}) \tag{2.5$'$}
$$

and the domain consists of one affine and one migratory phase. We note parenthetically that $\psi^\star$, at the transition point, need not be equal to $\psi_{(1)}$ because of the loss of elastic energy associated with the fracture of cross-links connecting the migratory to the affine phase.

In a similar fashion, when an energy level $\psi_{(2)}$ is reached such that $\psi^{\star} = \psi_{(2)}$, another phase becomes elastically detached so that two migratory phases are operative. Now:

$$\psi = \psi^{\star\star} + \psi^{1}(y_{i,K}{}^{(1)}) + \psi^{2}(y_{i,K}{}^{(2)}). \tag{2.5''}$$

Thus, the difference between the first and second models is that, in the latter, the migratory phases are not present *ab initio* and the onset of a migratory phase is delayed until the free energy density of the affine phase has reached a "threshold" value, in a manner reminiscent of a yield surface in plasticity. Other models are, of course, also possible.

3. A variational principle

We begin with an integral form of a principle which is of purely mechanical character, in that it avoids questions of entropy and temperature under conditions of irreversibility (even though the question of existence of entropy was dealt with by VALANIS [11, 12], in an earlier work). Furthermore, it is simple and leads to direct results. The principle is in the form of the global statement that applies to a dissipative continuous medium, in this case one with n migratory phases. If $\dot{\Psi}$ is the (virtual) rate of change of the stored energy Ψ (Helmholtz free enrgy in thermodynamics) of such a medium in its reference configuration x, with domain V and surface S, then

$$\dot{\Psi} = \int\limits_{S} T_i v_i \, dS + \int\limits_{V} f_i v_i \, dV - \int\limits_{V} D \, dV, \tag{3.1}$$

where v_i is a virtual velocity field, T_i are the surface tractions and f_i are the body forces (including inertial forces), and D is the internal dissipation density, which is always non-negative, i.e.,

$$D \leq 0. \tag{3.2}$$

The internal dissipation density D is due to the rate of work of the internal forces Q_L acting on the migratory velocity fields v^L, where:

$$v^L = \dot{p}^L = \partial p^L / \partial t|_x \,. \tag{3.3}$$

Thus

$$D = \sum Q_L v^L \geq 0, \tag{3.4}$$

where

$$\sum Q_L v^L = \sum_r Q_L^{(r)} v_{(r)}^L, \qquad r = 1, 2, \ldots, n.$$

Hence, to summarize,

$$\dot{\Psi} = \int_S T_i v_i \, dS + \int_V f_i v_i \, dV - \int_V Q_L v^L \, dV. \tag{3.5}$$

The physical foundations of this principle are given in Appendix II.

Equation (3.5) is a statement of the fact that, for all *admissible* virtual velocity fields v_i and v^L, at constant T_i, f_i and Q_L, the virtual rate of change of the free energy of a region is equal to the virtual rate of work done by the external body as well as surface forces, minus the virtual dissipation due to the virtual rate of work done by the internal forces Q_L.

With regard to the admissibility of the velocity fields v_i and v^L, we point out that while v_i are completely arbitrary, v^L must satisfy the dissipation inequality:

$$Q_L v^L > 0, \qquad \text{if} \quad \|v^L\| > 0, \quad \|Q_L\| > 0 \tag{3.6}$$

for all r, double bars denoting norms, i.e., $\|v_L\|^2 = v_L v^L$, so that equality (3.5) may be written in terms of the Ineq. (3.7)

$$\dot{\Psi} \leq \int_S T_i v_i \, dS + \int_V f_i v_i \, dV \tag{3.7}$$

for all arbitrary virtual velocities v_i, and v^L, subject to the constraint that in V, $Q_L v^L \geq 0$, with the proviso that the equality sign applies only in the case when $\|Q_L\| = 0$ and/or $\|v^L\| = 0$.

We complete the variational statement by stipulating that for all $v_i^\star$, these being velocity vectors associated with virtual rigid body motion,

$$\dot{\Psi} = 0, \qquad \|v^L\| = 0. \tag{3.8}$$

This is a constitutive statement. The fact, as we shall show, that this is also a statement of (dynamic) equilibrium, raises philosophical questions as to whether equilibrium is an independent law, or a form of constitutive law, (common to all materials whose constitution is determined by the dependence of the free energy density on the displacement and internal field gradients), that rests on the stipulation that, under condition of (virtual) rigid body motion, the free energy is invariant and the dissipation is zero, since in fact $\|v^L\| = 0$.

For the purposes of the analysis we introduce, in the variational principle, the Helmholtz free energy density ψ, per unit undeformed volume, such that:

$$\Psi = \int_V \psi \, dV \tag{3.9}$$

assuming short-range interaction among particles. We thus have a variational principle in terms of the following inequality:

$$\int_V \dot{\psi}\, dV \leq \int_S T_i v_i\, dS + \int_V f_i v_i\, dV, \tag{3.10}$$

where $\dot{\psi} \equiv (\partial\psi/\partial t)_x$.

4. Field equations in the presence of internal fields

We begin with the generic Eq. $(2.4)_1$, i.e.,

$$\psi = \psi(y_{i,\alpha}; p^K{}_{,\alpha}; q^K), \tag{4.1}$$

where $p^K{}_{,\alpha}$ is the inverse of $x^\alpha{}_{,K}$, i.e.,

$$x^\alpha{}_{,K} p^K{}_{,\beta} = \delta^\alpha{}_\beta \tag{4.2}$$

and $q^K = p^K - \delta^K_\alpha$.

Thus

$$\dot{\psi} = (\partial\psi/\partial y_{i,\alpha})v_{i,\alpha} + (\partial\psi/\partial p^K{}_{,\alpha})(v^K{}_{,\alpha} + \partial\psi/\partial q^K\, v^K, \tag{4.3}$$

where

$$v^K = \partial q^K/\partial t = \partial p^K/\partial t|_x\,, \tag{4.4}$$

$$v^K{}_{,\alpha} = (\partial p^K{}_{,\alpha}/\partial t)_x\,. \tag{4.5}$$

Hence

$$\dot{\psi} = \psi^\alpha{}_i v_{i,\alpha} + \psi^\alpha{}_K v^K{}_{,\alpha} + \psi_L v^L, \tag{4.6}$$

where

$$\psi^\alpha{}_i = \partial\psi/\partial y_{i,\alpha}\,, \qquad \psi^\alpha{}_K = \partial\psi/\partial p^K{}_{,\alpha}\,, \qquad \psi_L = \partial\psi/\partial q^L. \tag{4.7}$$

We now use Eq. (4.5) in the variational inequality (3.10) to find:

$$\int_V (\psi^\alpha{}_i v_{i,\alpha} + \psi^\beta{}_L v^L{}_{,\beta})dV \leq \int_S T_i\, v_i\, dS + \int_V f_i\, v_i\, dV. \tag{4.8}$$

The left-hand side of Eq. (4.8) is now recast in surface and volume integrals with the aid of the Green–Gauss theorem, and Eq. (4.9) is thereby obtained:

$$\int_S (\psi^\alpha{}_i n_\alpha - T_i)v_i\, dS - \int_V \left[(\psi^\alpha{}_{,i})_{,\alpha} + f_i\right] v_i\, dV + \int_S \psi^\beta{}_L n_\beta v^L\, dS - \int_V \left[(\psi^\beta{}_L)_{,\beta} - \psi_L\right] v^L\, dV \leq 0. \tag{4.9}$$

DISCUSSION. Before we proceed with the consequences of Eq. (2.4) we note that, generally, on a part of the surface S, namely S_T, tractions are applied, while on its complement S_Y displacements or velocities V_i are applied instead. Therefore, on S_T the virtual velocities v_i are arbitrary while on S_Y these are zero. With regard to the boundary conditions of the migrating phases (and following the discussion at the end of Appendix II), the surface S is the sum of the sub-surface S_0 on which the velocities of the phases are unknown and thus the virtual velocities are arbitrary, and the sub-surface S_P, which is impenetrable to phase migration, and on which the migratory velocities $V^L = 0$. No other physical situation is possible (see discussion at the end of Appendix II). Thus on S_0 the virtual velocities v^L are arbitrary while on S_P:

$$p^K = \delta^K{}_\alpha x^\alpha \tag{4.10}$$

and the virtual velocities v^L are zero. In the interior both v_i and v^L are arbitrary except that v^L are admissible only if they satisfy the dissipation inequality.

With the above discussion in mind, let a set of admissible v^L in V be prescribed, in the sense of Ineq. (2.1). The virtual velocity fields v_i on S_T, v_i in V and v^L on S_0, can be independently and arbitrarily prescribed. Thus setting these equal to zero, and noting that v_i are zero on S_Y and v^L are zero on S_P, one finds in view of Ineq. (4.9), that

$$\int_V \left[(\psi^\beta{}_L)_{,\beta} - \partial\psi/\partial q^L\right] v^L \, dV \geq 0. \tag{4.11}$$

Now keeping v_i in V and V^l on S null, one may prescribe v_i on S_T in a manner that violates Ineq. (4.9), if the bracket under the surface integral does not vanish. Thus

$$\psi^\alpha{}_i n_\alpha = T_i \qquad \text{on } S_T\,, \tag{4.12}$$

$$v_i = V_i \qquad \text{on } S_Y\,, \tag{4.13}$$

where V_i are known functions of time and the surface coordinates. Repeating the same argument for the other integrals one finds that

$$(\psi^\alpha{}_i)_{,\alpha} + f_i = 0 \qquad \text{in } V \tag{4.14}$$

and

$$\psi^\beta_L n_\beta = 0 \quad \text{on } S_0\,, \qquad v^L = 0 \quad \text{on } S_P\,. \tag{4.15}$$

DISCUSSION. Equation (4.14) replicates the equation of motion in continuum mechanics when internal fields are absent. Here we show that the equation applies in the presence of internal fields. Therefore if f_i contain inertia forces, as in the dynamic case, then

$$(4.16) \qquad f_i = g_i - \varrho_0 \partial^2 y_i(x^\alpha, t)/\partial t^2,$$

where g_i are body forces other than inertia forces and ϱ_0 is the reference density of the domain. We point out that to obtain Eq. (4.14) in the presence of inertia forces, we choose a virtual velocity field which is accelerationless, i.e. V_i is a function of x^α only and independent of time.

REMARK. As noted above, the physics of the problem is such that tractions are prescribed on S_T ($\leq S$) with full kinematic freedom of the particles on the surface, while the deformation of the surface S_Y ($S_Y \leq S$) is prescribed by means of a relation:

$$(4.17) \qquad {y_i}^S = {y_i}^S({x^\alpha}_S, t),$$

where ${y_i}^S$ and ${x^\alpha}_S$ denote the coordinates of the particles on the deformed and undeformed surface, respectively. In the former case Eq. (4.12) applies. In the latter case v_i are prescribed on the surface since

$$(4.18) \qquad v_i = V_i = (\partial {y_i}^S/\partial t)_x$$

and Eq. (4.13) applies. With regard to the migratory boundary conditions, the surface velocities V^L are arbitrary on S_0 while V^L are zero on S_P.

5. Internal equations of motion

We begin by noting that Eqs. (4.11), (4.12) and (4.13) in conjunction with Eqs. (3.5) and (3.9) lead to the following relation for the dissipative forces Q_L

$$(5.1) \qquad \int\limits_V \left\{({\psi^\beta}_L)_{,\beta} - \partial\psi/\partial q^L - Q_L\right\} v^L \, dV = 0.$$

Since this equation must be true for all arbitrary (including infinitesimal) domains, the local form of Eq. (5.1) results:

$$(5.2) \qquad \left\{({\psi^\beta}_L)_{,\beta} - \partial\psi/\partial q^L - Q_L\right\} v^L = 0.$$

Equation (5.2), however, cannot be satisfied for all admissible fields v^L (see Appendix I), unless:

$$(5.3) \qquad ({\psi^\beta}_L)_{,\beta} - \partial\psi/\partial q^L - Q_L = 0.$$

Equation (5.3) is the equation of internal equilibrium that relates the dissipative force Q_L to the divergence of the bi-vector ψ^β_L.

At this point we recall Ineq. (3.6), i.e., $Q_L v^L \geq 0$, which is a constraint or a requirement of positive dissipation in the presence of non-affine deformation. The constraint demands that Q_L and V^L should be related, otherwise they could be prescribed independently and in a manner that would violate the inequality. The most obvious relation is a linear one of the form:

(5.4)
$$Q_K = b_{KL} v^L,$$

where b_{KL} is a covariant "viscosity tensor". Equation (5.4) is a statement to the effect that the dissipative (resistive) force is a linear and homogeneous function of the migratory velocity of a phase.

Equations (5.3) and (5.4) combine to give Eq. (5.5),

(5.5)
$$\psi^\beta{}_{K,\beta} - \partial\psi/\partial q^K = b_{KL} v^L$$

which is the equation for the motion of the particles in a migratory phase.

The initial conditions

The initial conditions are obtained from the presumption that the material is in a quiescent state at $t = 0$. Thus

(5.6)
$$y_i(x^\alpha, 0) = \delta_{i\alpha} x^\alpha, \qquad q^K(x^\alpha, 0) = 0,$$

(5.7)
$$v_i(x^\alpha, 0) = 0.$$

At this point we summarize the equations pertinent to the motion of the domain, reference being made to the individual phases $r = 1, 2, ..., n$.

Summary of equations

In V

(5.8)
$$\psi = \psi(\partial y_i/\partial x^\alpha; \partial p^K/\partial x_\alpha; g^K),$$

(5.9)
$$(\psi^\alpha{}_i)_{,\alpha} + g_i = \varrho_0 \partial^2 y_i(x^\alpha, t)/\partial t^2,$$

(5.10)
$$\psi^\beta{}_{K,\beta} - \psi_K = b_{KL} v^L.$$

On S_T

$(5.11)_1$
$$\psi^\alpha{}_i n_\alpha = T_i\,.$$

On S_Y

$(5.11)_2$
$$y_i{}^S = y_i{}^S(x^\alpha{}_S, t) \quad \text{or} \quad v_i = V_i\,.$$

On S_0

$(5.11)_3$
$$\psi^\beta{}_L n_\beta = 0.$$

On S_P

$(5.11)_4$
$$v^L = 0.$$

The initial conditions are such as in Eqs. (5.6) and (5.7).

5.1. Invariance under rigid body motion

We restate the two conditions stated previously, to be satisfied under conditions of (virtual) rigid body motion:

$$\dot{\psi} = 0, \qquad \|v^L\| = 0. \tag{5.12}$$

These conditions are fundamental in putting further restrictions on the form of ψ and in identifying physically certain constitutive constraints.

To show this we employ Eq. (3.5), i.e.,

$$\dot{\Psi} = \int_S T_i v_i \, dS + \int_S f_i \, dV - \int_V Q_L v^L \, dV \tag{5.13}$$

which in the presence of rigid body motion then becomes:

$$\int_S T_i v_i \, dS + \int_V f_i v_i \, dV = 0. \tag{5.14}$$

(i) *Rigid body translation*

In this case v_i are constant in V. Thus, in view of Eq. (5.14),

$$v_i \left\{ \int_S T_i \, dS + \int_V f_i \, dV \right\} = 0. \tag{5.15}$$

A set of three linearly independent vectors v_i can be found for which Eq. (5.15) must hold. This is possible iff

$$\int_S T_i \, dS + \int_V f_i \, dV = 0. \tag{5.16}$$

Using the classical argument of applying Eq. (5.16) to a tetrahedron of vanishing dimensions in the undeformed domain, one finds that

$$T^{\alpha}{}_i n_{\alpha} = T_i \,, \tag{5.17}$$

where the tractions T_i are calculated in the x-frame and pertain to the undeformed area. We recognize $T^{\alpha}{}_i$ as the First Piola–Kirchhoff stress tensor. Furthermore, in the light of Eqs. (4.13) and (5.17),

$$T^{\alpha}{}_i = \psi^{\alpha}{}_i \,. \tag{5.18}$$

If we transform the domain of integration in Eq. (5.14) to that of the deformed configuration and apply the same procedures, we find that

$$T_{ij} n_j = T'_i \,, \tag{5.19}$$

where T_{ij} is the Cauchy stress and the traction T_i', where $T_i = \mathrm{Det}(y_{i,\alpha})T_i'$, are calculated in the y-frame and pertain to the deformed area. The following tangent transformations apply:

$$(5.20) \qquad n_\alpha = y_{i,\alpha}n_i\,, \qquad T^\alpha{}_i = J\,T_{ij}x^\alpha{}_{,j}\,, \qquad J\,T_{ij} = T^\alpha{}_i y_{j,\alpha}\,.$$

(ii) *Rigid body rotation*

Here, the virtual velocity v_i is caused by a *virtual* angular velocity Ω_i brought about by rotation of the spatial frame of reference y_i. Therefore there are no induced centrifugal forces as there would be, had Ω_i been actual, i.e., an angular velocity of the domain itself. Thus Ψ does indeed remain invariant in the presence of a virtual angular velocity field Ω_i.

We now begin Eq. (5.21):

$$(5.21) \qquad v_i = e_{ijk}\Omega_j\,y_k\,,$$

where e_{ijk} is the permutation tensor and Ω_i is an arbitrary angular velocity vector, brought about by rotation of the frame of reference y_i. Again, transforming the domain of integration in Eq. (5.14) to that of the deformed configuration and applying classical arguments we find that:

$$(5.22) \qquad T_{ij} = T_{ji}\,.$$

Thus, in view of Eqs. (5.18), $(5.20)_3$ and (5.22):

$$(5.23) \qquad \psi^\alpha{}_i y_{j,\alpha} = \psi^\alpha{}_j y_{i,\alpha}$$

Eq. (5.23) is a restriction on the functional form of ψ.

Further Invariance Considerations. We recall Eq. (5.8):

$$(5.24) \qquad \psi = \psi(y_{i,\alpha}; p^K{}_{,\alpha}).$$

Virtual rigid body rotation leaves ψ as well as $x^\alpha{}_{,K}{}^{(r)}$ invariant. However, $y_{i,\alpha}$ is a bi-vector and represents in fact the three vectors: $y_{i,1}$; $y_{i,2}$; $y_{i,3}$. A clasical theorem in continuum mechanics (see for instance ERINGEN [17]), is that a scalar function ψ of three vectors $\mathbf{a}_r$ remains invariant under rotation of the frame of reference iff it is a function of the inner products $\mathbf{a}_r \cdot \mathbf{a}_s$ and the determinant $|a_{ir}|$. Furthermore, if ψ is centro-symmetric, i.e., invariant under reflection of the frame of reference, as it must be since the choice of the spatial reference frame is arbitrary, then ψ must be an even function of $|a_{ir}|$, since $|a_{ir}|$ changes sign upon reflection.

It follows, therefore, that a necessary and sufficient condition that ψ be invariant under rigid body rotation and reflection of the spatial frame of reference y_i is that

$$(5.25) \qquad \psi = \psi(C_{\alpha\beta}; p^K{}_{,\alpha}; q^K),$$

where $C_{\alpha\beta}$ is the Right Cauchy–Green tensor $y_x{}^T y_x$. One can verify that condition (5.23) is now trivially satisfied.

5.2. Conditions of material isotropy

In strictly affine deformations, the mathematical definition of material isotropy is invariance of a constitutive equation, or property, under rotation and inversion of the material frame of reference x^α. In the present case, however, the situation is more complex and lends the theory a wider scope for material characterization. For instance, a migratory phase may be isotropic initially but may evolve into an anisotropic state as migration proceeds. We thus distinguish between two distinct possibilities:

(i) *Isotropy in the initial state whereby ψ is invariant under rotation and inversion of the material frame x^α*

(ii) *Isotropy of phase "r" in the migrated state, in which event ψ remains invariant under rotation and inversion of the frame ${p^K}_{(r)}$.*

We note that in (ii) we have introduced a formal, rigorous definition of "strain-induced anisotropy" in phase r, a lack of invariance if ψ under rotation and inversion of the frame ${p^K}_{(r)}$.

Restrictions on ψ

(i) *Isotropy in the initial state*

This means invariance of ψ under rotation (and inversion) of the material frame x^α. We begin with Eq. (5.25) which we write in the form:

$$\psi = \psi(C_{\alpha\beta}; {x^\alpha}_K; q^K). \tag{5.26}$$

In the case ψ is an isotropic function of the tensor $C_{\alpha\beta}$, and the three vectors ${x^\alpha}_{,K} : \alpha = 1, 2, 3$. Thus

$$\psi = \psi(I_\alpha : C_{KL}; G_{KL}; q^K), \tag{5.27}$$

where $C_{KL} = C_{\alpha\beta}{x^\alpha}_{,K}{x^\beta}_{,L}$; $G_{KL} = \delta_{\alpha\beta}{x^\alpha}_{,K}{x^\beta}_{,L}$ and I_α are the three principal invariants of $C_{\alpha\beta}$.

(ii) *Isotropy in the initial state and migratory phase 1*

In this case ψ is an isotropic function of $G^{(1)}_{KL}$, ${C_{KL}}^{(1)}$ and q^k but a general function of the tensors ${G_{KL}}^{(r)}$, $C^{(r)}_{KL}$ and $q^{(r)}$, $r = 2, 3, \ldots, n$. In other words, if a phase remains isotropic during its migratory motion then ψ will be an isotropic function of G_{KL}, C_{KL} and q^K of that particular phase.

Thus, more generally, specific material symmetries in the initial configuration x involve invariance under appropriate rotations of frame x, while evolving symmetries in a specific phase r involve invariance under appropriate rotations of frame ${p^K}_{(r)}$.

5.3. Linearization of the field equations

We complete this section by giving the linearized form of the field equations and boundary conditions (5.8) – (5.11)$_4$. The basis of the linearization scheme is the premise of small deformation in the sense that:

$$y_i = \delta_{i\alpha} x^\alpha + \eta u_i \,, \tag{5.28}$$

$$p^K = \delta^K{}_\alpha x^\alpha + \eta q^K \,, \tag{5.29}$$

where η is a small real number. Equations (5.28) and (5.29) are then substituted in Eqs. (5.8)–(5.11)$_4$, terms in η are retained, while terms in η^2 and higher order are neglected. Subsequently η is set equal to unity. Furthrmore since, ultimately, all equations are referred to the reference frame x, following the analysis all indices are replaced by small Latin letters. Note, parenthetically, that since the frame p now collapses onto the frame x, there cannot be any *evolution* of anisotropy of phase, if the phase is initially isotropic.

Thus, beginning with the relations:

$$p^K{}_\alpha = \delta^K{}_\alpha + q^K{}_{,\alpha} \,, \tag{5.30}$$

$$C_{\alpha\beta} = \delta_{\alpha\beta} + 2\eta\varepsilon_{\alpha\beta} \,, \tag{5.31}$$

$$C_{KL} = \delta_{KL} + 2\eta\varepsilon_{KL} - 2\eta q_{KL} \,, \tag{5.32}$$

$$G_{KL} = \delta_{KL} - 2\eta q_{KL} \,, \tag{5.33}$$

where $\varepsilon_{\alpha\beta}$ is the strain tensor while $2q_{KL} = q_{K,L} + q_{L,K}$, the following equations result for all r:

In V

$$\psi = \psi(\varepsilon_{ij}, q_{i,j}{}^{(r)}, q_i{}^{(r)}) \tag{5.34}$$

or

$$\psi = \psi(\varepsilon_{ij}, q_{ij}{}^{(r)}, q_i{}^{(r)}) \tag{5.35}$$

if the domain is initially isotropic. Also

$$(\partial\psi/\partial u_{i,j})_{,j} + g_i = \partial^2 u_i/\partial t^2, \tag{5.36}$$

$$(\partial\psi/\partial q_i{}^{(r)}{}_{,j})_{,j} - \partial\psi/\partial q_i{}^{(r)} = b_{ij}{}^{(r)}\partial q_i{}^{(r)}/\partial t \quad (r \text{ not summed}). \tag{5.37}$$

On S_T

$$(\partial\psi/\partial u_{i,j}) n_j = T_i \,; \tag{5.38$_1$}$$

on S_0

$$(\partial\psi/\partial q_{i,j}^{(r)}) n_j = 0; \tag{5.38$_2$}$$

on S_u

$$u_i = U_i\,; \tag{5.39$_1$}$$

on S_P

$$q_i = 0. \tag{5.39$_2$}$$

Initial conditions

$$u_i(x,0) = (\dot{u})(x,0) = 0; \qquad {q_i}^{(r)}(x,0) = 0. \tag{5.40}$$

6. A worked example

To illustrate the ramifications of the non-local theory, we present in this section a worked example of simple quasi-static shearing in one dimension. Let a half-space be infinite in directions x and z and semi-infinite in direction y. The material domain is in a quiescent state when, at time $t = 0$, a shearing traction $T_0(t)$ is applied in direction x on the surface $y = 0$. Let

$$\psi = (1/2)Au_y^2 + Bu_yq_y + (1/2)Cq_y^2\,, \tag{6.1}$$

where $u_y \equiv \partial u/\partial_y$, $q_y \equiv \partial q/\partial_y$, i.e., a subscript denotes differentiation. The pertinent boundary conditions then are: At $y = 0$, $T = T_0(t)$; $\partial\psi/\partial q_y = 0$. At $y = \infty$, all variables are bounded. At $t = 0$, $u = q = 0$. The equilibrium condition, Eq. (5.27), gives:

$$(\partial\psi/\partial u_y)_y = 0 \tag{6.2}$$

while the equation of motion for the internal variable q is given by Eq. (6.3):

$$(\partial\psi/\partial q_y)_y = b\,q_t\,. \tag{6.3}$$

In view of Eq. (6.2), the shear stress $T(= \partial\psi/\partial u_y)$ is uniform in the domain as in the local theory. However this is not true of the strain. Eqs. (6.2) and (6.3) combine to give the following (diffusion) equation for q:

$$C_1 q_{yy} = b\,q_t\,, \tag{6.4}$$

where $C_1 = C - B^2/A$. We have solved Eq. (6.4) by the usual Laplace Transform technique and obtained the following expression for the shear strain γ

$$\gamma = \int_0^t J(x; t-\tau)(\partial T_0/\partial\tau)\,d\tau. \tag{6.5}$$

Note that memory function $J(y,t)$ plays the role of a creep function except that now, at variance with local theory, it is a function of x as well as t. Equation (6.6) gives the analytical form of J, found from the solution

$$J(y,t) = A^{-1}\left\{H(t) + (B^2/AC_1)\mathrm{erfc}\left[y/(2at^{1/2})\right]\right\}, \tag{6.6}$$

where $a^2 = b/C_1$. In Fig. 1 we show the dependence of γ on time at various y-stations when T_0 has the form of a Heaviside step function, in which event $\gamma(y,t) = J(y,t)$. Evidently the strain "diffuses" into the half-space as time increases.

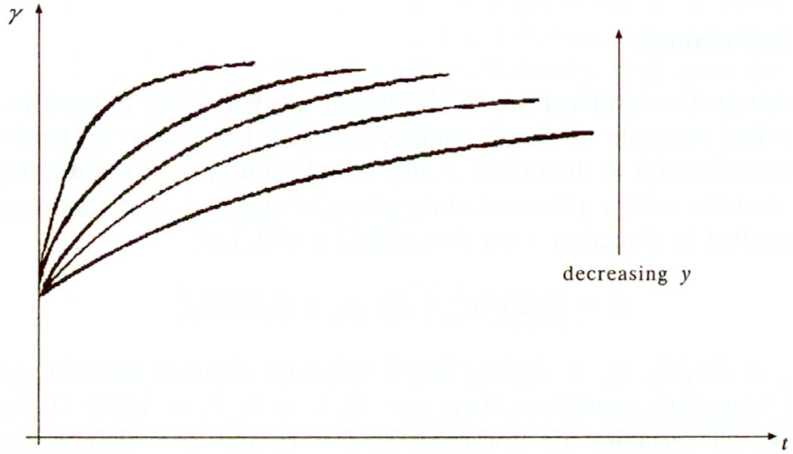

FIG. 1. Shear strain versus time for different values of y.

7. Postscript on plasticity

Previously, VALANIS [15, 16], we have developed the constitutive equations of plasticity and viscoplasticity, in the context of the local theory of thermodynamics of internal variables, by introducing the concept of "intrinsic time" z, and substituting z for t in the equations of evolution of these variables. The theory developed here may be extended to materials that are strain-rate indifferent, or partially indifferent, by the use of a similar procedure, whereby $\dot{z}$ premultiplies the left-hand side of the equations of motion of the internal fields. Eq. (5.10) will now read

$$\dot{z}\left\{\psi^{\beta}{}_{,K^{(r)}} - \psi_{K^{(r)}}\right\} = b^{(r)}_{KL}\, v^{L}_{(r)} \quad (r \text{ not summed}). \tag{7.1}$$

The precise nature of $\dot{z}$ will be discussed in future studies.

Appendix I

The task is to prove that if

$$(Q_L + \psi_L)v^L = 0 \tag{I.1}$$

for all admissible v^L, where $\psi_L \equiv (\psi^\beta{}_{,L})_{,\beta}$, then

$$(Q_L + \psi_L) = 0. \tag{I.2}$$

As discussed in Sec. 2, v^L is admissible if

$$Q_L v^L > 0, \qquad \psi_L v^L < 0 \tag{I.3}$$

for all $||Q_L|| \neq 0$, $||\psi_L|| \neq 0$, $||v^L|| \neq 0$. Otherwise v^L are arbitrary.

P r o o f. If Q_L and ψ_L are collinear and either of the same sign or *unequal*, the proof is trivial. Let $Q_L = \alpha\psi_L$, where $\alpha \neq -1$. Then in view of Eq. (I.2) $\psi_i \delta q_i = 0$. However, the constraints $\psi_L v^L = 0$ and $\psi_L v^L < 0$, cannot be satisfied simultaneously if $||\psi_L|| \neq 0$. Thus, in this case, $\alpha = -1$ and $Q_L + \psi_L = 0$.

If Q_L and ψ_L are not collinear then there exists a vector B_L normal to the plane of the vectors Q_L and ψ_L ($Q_L B^L = 0$, $\psi_L B^L = 0$), such that the scalar product $p = \varepsilon^{LMN} Q_L \psi_M B_N > 0$. The three vectors Q_L, ψ_L and B_L are linearly linearly independent. We now introduce two vectors: R^L, normal to the plane of ψ_L and B_L, and P^L normal to the plane of Q_L and B_L, i.e.,

$$R^L = \varepsilon^{LMN} \psi_{MNk}\,, \qquad P^L = \varepsilon^{LMN} Q_M B_N\,. \tag{I.4}$$

It may now be shown that all vectors v^L of the form:

$$v^L = \alpha R^L + \beta P^L + \gamma B^L, \tag{I.5}$$

where α and β are positive scalars and γ in non-negative, are admissible. In fact

$$Q_L v^L = \alpha p, \qquad \psi_L v^L = -\beta p \tag{I.6}$$

thus satisfying inequalities (I.3).

Three vectors $v^L{}_r$ are then constructed as in Eq. (I.7),

$$\delta v^L{}_r = \alpha_r R_i + \beta_r P_i + \gamma_r B_i \tag{I.7}$$

where $\alpha_r > 0$, $\beta_r > 0$, $\gamma_r \geq 0$. These vectors are admissible and, furthermore *linearly independent* if $\gamma_1 = 1$, $\gamma_2 = 0$, $\gamma_3 = 0$, and the determinant condition (I.8) is satisfied

$$\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} \neq 0. \tag{I.8}$$

Since one can always find positive scalars such that $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$, condition (I.1) now demands that the vector $(Q_L + \psi_L)$ be orthogonal to three linearly independent vectors. This is not possible if the said vector is different from zero. Thus $Q_L = -\psi_L$.

Appendix II

Variational principle

Physical foundations

To derive Eq. (3.1) we begin with the observation that, actually, the rate of (virtual) work $\dot{W}_S$ done by the surface tractions is

$$\dot{W}_S = \int_S \sum T_i^{(r)} v_i^{(r)}\, dS, \tag{II.1}$$

where $\mathbf{T}^{(r)}$ are the surface tractions on the phases of the domain and $\mathbf{v}^{(r)}$ are the corresponding (virtual) velocities of the phases, i.e.,

$$v_i^{(r)} = \partial y_i(p^K{}_{(r)}, t)/\partial t|_x. \tag{II.2}$$

More precisely, and since p^K are migration maps given by the relation

$$P^K = p^K(x^\alpha, t) \tag{II.3}$$

and omitting the index r on the right-hand side of (II.4), for simplicity of notation, the phase velocity $v_i^{(r)}$ is given by Eq. (II.4):

$$v_i^{(r)} = \partial y_i(p^K(x^\alpha, t), t)/\partial t|_x = (\partial y_i/\partial p^K)\partial p^K/\partial t|_x + \partial y_i/\partial t|_p. \tag{II.4}$$

Quite clearly $\partial p^K{}_{(r)}/\partial t|_x (= v^K{}_{(r)}$ is the migratory velocity of phase r while $\partial y_i(p^K{}_{(r)}, t)/\partial t|_p$ $(= v_i{}^{(r)}{}_{(p)})$ is the velocity of the phase relative to the present reference configuration p. Thus:

$$v^{(r)} = \partial y_i/\partial p^K{}_{(r)} v^K{}_{(r)} + v_i{}^{(r)}{}_{(p)}. \tag{II.5}$$

Equation (II.5) is merely the rule of addition of velocities.

In a similar manner the rate of (virtual) work $\dot{W}_V$ done by the body forces is

$$(\dot{W})_V = \int_V \sum_r f_i^{(r)} v_i^{(r)}\, dV. \tag{II.6}$$

Thus, the statement that the rate of change of the free energy of a domain is equal to the rate of work done by the applied surface and body forces minus the rate of dissipation (all rates being virtual), has the analytical form of Eq. (II.7)

$$\dot{\Psi} = \int_S \sum_r T_i^{(r)} v_i^{(r)}\, dS + \int_V \sum f_i^{(r)} v_i^{(r)}\, dV - \int_V D\, dV. \tag{II.7}$$

It is a posited premise in continuum mechanics that the forces that constitute a surface traction are shared equally by all particles of the neighbourhood. With the above in mind, let $n_0, n_1, \ldots, n_n$, be the particle densities of the phases (at the surface) such that $\sum n_r = 1$. It then follows that

$$\mathbf{T}^{(r)} = n_r \mathbf{T}, \qquad \mathbf{T} = \sum \mathbf{T}^{(r)}. \tag{II.8}$$

In a similar manner,

$$\mathbf{f}^{(r)} = n_r \mathbf{f}, \qquad \mathbf{f} = \sum \mathbf{f}^{(r)}, \tag{II.9}$$

where n_r are now the particle densities in V (if different from S). Substituting equations $(\text{II.8})_1$ and $(\text{II.9})_1$ in Eq. (II.7) we recover Eq. (II.6) of the text, i.e.

$$\dot{\Psi} = \int_S T_i v_i \, dS + \int_V f_i v_i \, dV - \int_V D \, dV, \tag{II.10}$$

where

$$v_i = \sum n_r v_i^{(r)} \tag{II.11}$$

i.e. v_i is the *mean*, number-averaged (virtual) velocity and is equal to the one that would be calculated from the first principles, in the case if the (virtual) velocities of the phase were not equal.

The superscript r of the function $y_i^{(r)}$ on the right-hand side of Eq. (II.2) signifies the fact that the deformation of the phases is not compatible, in the sense that, after deformation, each phase r occupies a point $y_i^{(r)}$ in the spatial system, not necessarily the same as $y_i^{(r+1)}$, say, or any other $y_i^{(m)}$, for that matter.

Thus, to be precise, the free energy density ψ^r in model (i) in the Sec. 2. Physical Foundations, should be given by Eq. (II.12)

$$\psi^{(r)} = \psi^r (\partial y_i^{(r)} / \partial p^K{}_{(r)}), \qquad r \text{ not summed.} \tag{II.12}$$

But then the theory would be too complex, and mathematically and physically intractable. In this simpler physical approach the deformation gradient $\partial y_i^{(r)} / \partial p^K{}_{(r)}$ has been replaced in Eq. (II.12) by the *mean* deformation gradient $\partial y_i / \partial p^K{}_{(r)}$ where

$$y_i = \sum_r n_r \, y_i^{(r)}.$$

Thus

$$v_i = \partial y_i / \partial_t |_x = \sum_r n_r \, v_i^{(r)} \tag{II.13}$$

as in Eq. (II.11), assuming n_r to be constant.

Discussion of boundary conditions

When the boundary conditions were discussed in the text, the question was posed whether a diffusive velocity $v^L{}_{(r)}$, of a phase r, could be prescribed at the boundary. This is experimentally not feasible – since at the boundary, separate motions of the phases cannot be distinguished experimentally – unless, of course, $v^L{}_{(r)} \equiv 0$ for all r. This is achievable physically by making the pertaining part of the boundary impenetrable to particle migration. In this case,

$$p^K{}_{(r)} = \delta_\alpha{}^K x^\alpha \tag{II.14}$$

for all r. Thus

$$v^K{}_{(r)} = \partial p^K{}_{(r)}/\delta t|_x = 0, \tag{II.15}$$

and in view of Eq. (II.5),

$$v_i^{(r)} = v_i \, . \tag{II.16}$$

Thus, either the diffusive velocities $v^K{}_{(r)}$ of the migrating phases are not prescribable on the boundary, or if they are, then they all identically vanish.

A footnote on dissipation

In reference to Eq. (II.5), quite clearly the dissipative velocity is the non-affine, migratory velocity v^L. Thus when resistance to such motion exists, through a resistive force Q_L, then the rate of dissipation is the rate of work done by the dissipative forces, i.e., $Q_L v^L$.

We make the statement of "when resistance exists" so as to open the door to the possibility of *elastic* non-affine deformation. This, in principle, could be achieved through breaking of bonds but without resistance to subsequent motion. In this event D would be zero. This case is merely a sub-case of the theory already presented. The relevant equations are obtained by setting the right-hand side of Eq. (5.10) equal to zero.

One thus obtains a theory of non-local elasticity without the need for higher gradients of deformation, by introducing the concept of internal fields.

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Received January 7, 1997.