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The influence of deformation path on adaptation process of a solid

J. SACZUK (GDAŃSK)

THE IDEA of a criterion of adaptation process of a body, accounting for the influence of deformation path on the material properties of the body, is proposed. A change of the deformation path, realized either through the change of slip systems and/or by changing external loads, is analyzed within the Finslerian description of the solid behaviour. In this continuum model no yield rule and no intermediate configuration are assumed to exist, and the transition from micro- to macroscales is natural. This approach makes possible the description of yielding, softening, hardening and localization of solids within the unified concept. A shakedown theorem, based on the Finslerian continuum model, is formulated within the theory of differential inequalities. The presented theorem, in which a definite amount of the total strain energy comes into play, has no counterparts in the available literature. It generalizes the classical approaches to the adaptation problems by including arbitrary deformations and material nonlinearities.

1. Introduction

THE COMPLEXITY of inelastic behaviour of a solid is caused primarily by the fact that its internal state is changing during the deformation process as a consequence of glide mechanisms, twinning and other shear transformations. Understanding of the overall deformation resistance of the material and the evolution of its internal structure is also important in the accurate prediction of the long-term average behaviour of structure. On the other hand, in many practical applications both the loading and the initial state of the body are not known with a sufficient accuracy. In these cases the knowledge of the whole evolution has only of limited interest. A desired theory should (i) deliver good estimations of the average behaviour of the structure and thereby correct of theoretical results compatible with experimental ones, and (ii) predict the correct asymptotic behaviour whatever are the initial conditions and the loading programme.

The answer to this question can to some degree, be obtained from the shakedown theory, since the classical theory of limit analysis can sometimes give unsafe estimates of collapse loads in certain cases (KOITER [1]). For that reason the shakedown theory provides a criterion of failure which may be considered as a more realistic basis for design than that of the limit analysis which assumes failure to occur when critical elastic stress is attained. Such an analysis is crucial for the assessment of the structural behaviour under varying loads within the range of time-independent plasticity. The problem is classically solved by analyzing possible residual stress fields in the static approach (MELAN [2], KOITER [1]) or by considering possible mechanisms of plastic deformation in the kinematic method (KOITER [1]), under the assumptions of geometric linearity, elastic-perfectly plastic or linear and unlimited hardening material behaviour, the validity of an associated flow law, etc. The extension of classical shakedown technique to broader classes of problems including the change of temperature, (limited) hardening, the influence of geometric effects are discussed by MANDEL [3], KÖNIG [4], POLIZZOTTO [5], WEICHERT [6, 7], GROSS–WEEGE [8], SACZUK and STUMPF [9], SACZUK [10]. The second direction in the adaptation analysis, being the generalization of the post-yield analysis, is known as the inadaptation analysis (CORRADI and MAIER [11], KÖNIG and SIEMASZKO [12]).

The shakedown criterion characterized by the non-specified definite bound of the plastic work (there exists an instant beyond which no additional plastic deformations occur) has certain shortcomings. A few of them are connected with the impossibility of estimating a (safe) number of load cycles (observed in practice), to estimate lower and upper limits of the plastic work, to take into account a continuous change of material characteristics during its evolution. Different, even of a catastrophic nature, bulk properties of solid deformation, like shear bands, Lüders and Portevin–Le Chatelier bands, hardening and softening, are sensitive to a change of deformation path both at the micro- and macro-levels (cf. KORBEL [13]). On the other hand, the importance of this problem is connected with the fact that the safety problem of structure subjected to variable loads is one of the major problems of structural design. We are still at the initial stage of such analysis.

The aim of this paper is to propose a certain innovation in the assessment of the structural safety, according to the Finslerian modelling of solid behaviour. A measure of adaptation, identified with the boundedness of plastic work, is of course physically justified but is too simple in reality. It is desirable to control the amount of energy necessary to create stable thermodynamic states of the deformation process, and to know how this energy is affected by internal and external parameters. The more correct measure seems to be the definite amount of the total (strain) work. One should stress that the plastic (dissipative) work is not generally easily selected as a part of the total work created during a deformation process. In our case the "plastic" work can be identified with the vertical (internal state) component of deformation process, but not as *a priori* assumption (cf. Sec. 2). We shall try both to propose an improvement of the classical shakedown methodology within the theory of differential inequalities (SZARSKI [14], LAKSHMIKANTHAM and LEELA [15]) and to present its justification within the scope of the generally accepted technique to shakedown problems.

2. Outline of a Finslerian continuum model

The objective in this section is to present the main concepts of the Finslerian modelling of solid behaviour (SACZUK [16]) that need to be known for a thorough understanding of the technique adapted to the shakedown analysis. A reader who whishes to get information beforehand concerning the whole Finsler geometry is asked to consult RUND [17] and MATSUMOTO [18] monographs. The mathematical preliminaries on Finslerian geometry are presented in Appendix A.

2.1. General assumptions

A continuous model of inelastic behaviour of solids modelled by means of the Finsler geometry (SACZUK [16]) is based on the following assumptions:

A.1. A material body (a continuum) $\mathcal{B}$ is assumed to be a 3-dimensional Finsler bundle F^3 whose points will be called line-elements (RUND [17]).

A.2. A motion of the body $\mathcal{B}$ is defined by the mapping:

$$\widehat{\chi} : \mathcal{B} \times R \to E^3 \times E^3 \times R, \qquad (\mathbf{x}, \mathbf{y}, t) \mapsto \mathbf{X} = \widehat{\chi}(\mathbf{x}, \mathbf{y}, t),$$

where E denotes an Euclidean space and R is a real number space.

A.3. A time-space of events is the product $E^3 \times E^3 \times R$.

A.4. The body $\mathcal{B}$ is under external and internal force fields.

A.5. Laws of evolution of the body $\mathcal{B}$ results from a variational principle for the first order functional describing its motion.

2.2. The motion

We sketch an approach (SACZUK [16]) which allows one to describe the irreversible deformation process of the solid taking place from the very beginning of its deformation, in conformity with its real internal nature.

In this approach the body $\mathcal{B}$, identified with the three-dimensional Finsler space F^3, is embedded into the product $E^3 \times E^3$ of Euclidean spaces and its points are called line-elements (oriented particles) (Assumptions (A.1) and (A.2)).

The position vector in an actual configuration is defined to be

$$\mathbf{X} = \widehat{\chi}(\mathbf{x}, \mathbf{y}), \tag{2.1}$$

where a diffeomorphism $\widehat{\chi} : E^6 \supset F^3 \to F^3 \subset E^6$ is a deformation of the body $\mathcal{B}$. The line-element $(\mathbf{x}, \mathbf{y})$ consisting of a position vector and a direction (or an internal variable) vector can be identified with an oriented particle of the body $\mathcal{B}$. The position vector $\mathbf{x}$, identified here with the material point of the configuration space with local coordinates (x^i) at the macro-level, is treated at the micro-level as a separate continuum with coordinates (y^i) at the point $\mathbf{x}$. In special cases we can consider the internal vector $\mathbf{y}$ as the micro-displacement, or the deviation from the mean displacement (KONDO [19]), or the microposition vector (WOŹNIAK [20]).

The deformation gradient is defined as the direct sum of components describing the deformation process of the body (SACZUK [16]) in the form

$$\widehat{\mathbf{F}} = \mathbf{F}^h + \mathbf{F}^v. \tag{2.2}$$

In (2.2) vertical $\mathbf{F}^v$ and horizontal $\mathbf{F}^h$ parts are respectively equal to

$$\mathbf{F}^v = \nabla^v \mathbf{X} = {}_vX^i_k \partial_i \otimes Dl^k, \qquad \mathbf{F}^h = \nabla^h \mathbf{X} = {}_hX^i_k \partial_i \otimes dx^k, \tag{2.3}$$

where $l^k = y^k / L(\mathbf{x}, \mathbf{y})$ are components of the unit tangent vector, $L(\mathbf{x}, \mathbf{y})$ is the fundamental function identified with the energy stored in dislocations and induced by the deformation process, ∂_i is the unit vector in the current configuration of the body and $\otimes$ denotes the tensor product. We shall denote further horizontal and vertical components of any tensor $\mathbf{T}$ by ${}_hT^{i\ldots}_{j\ldots}$ and ${}_vT^{i\ldots}_{j\ldots}$, respectively. The h-derivatives and v-derivatives of the position vector $\mathbf{X} = \mathbf{X}(\mathbf{x}, \mathbf{y})$ are defined as follows (MATSUMOTO [18], RUND [17])

$$(\mathbf{F}^h)^i_k \equiv {}_hX^i_k = \partial_k X^i - \dot{\partial}_l X^i \dot{\partial}_k G^l + \Gamma^{\star i}_{lk} X^l, \tag{2.4}$$

$$(\mathbf{F}^v)^i_k \equiv {}_vX^i_k = L \dot{\partial}_k X^i + A^i_{lk} X^l, \tag{2.5}$$

where $\partial_i \equiv \partial / \partial x^i$, $\dot{\partial}_i \equiv \partial / \partial y^i$ and the remaining unknowns in (2.4), (2.5) are defined by means of components of the metric tensor

$$g_{ij}(\mathbf{x}, \mathbf{y}) = \frac{1}{2} \frac{\partial^2 L^2(\mathbf{x}, \mathbf{y})}{\partial y^i \partial y^j} \tag{2.6}$$

according to

$$\Gamma^\star_{ijk} = \Gamma_{ijk} - C_{jkl} \frac{\partial G^l}{\partial y^i} = \gamma_{ijk} - C_{kjl} \frac{\partial G^l}{\partial y^i} - C_{ijl} \frac{\partial G^l}{\partial y^k} + C_{ikl} \frac{\partial G^l}{\partial y^j}, \tag{2.7}$$

$$\Gamma^\star_{ijk} = g_{jl} \Gamma^{\star l}_{ik}, \qquad \Gamma_{ijk} = g_{jl} \Gamma^l_{ik}, \qquad 2G^l = \gamma^l_{jk} y^j y^k, \tag{2.8}$$

$$\begin{aligned} N^l_k &= \dot{\partial}_k G^l = \frac{\partial G^l}{\partial y^k} = \Gamma^l_{jk} y^j = \Gamma^{\star l}_{jk} y^j, \\ \gamma_{ijk} &= \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ki}}{\partial x^j} \right), \end{aligned} \tag{2.9}$$

$$\begin{aligned} C_{ijk} &= \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}, \qquad C_{ijk} y^k = C_{ijk} y^j = C_{ijk} y^i = 0, \\ C_{ijk} &= g_{jl} C^l_{ik}, \qquad A^i_{jk} = L C^i_{jk}, \end{aligned} \tag{2.10}$$

$$Dl^i = dl^i + N^i_k dx^k. \tag{2.11}$$

2.3. Strain measures

For an orientation-preserving deformation $\widehat{\chi}$ ($\widehat{J} = \det \widehat{\mathbf{F}} > 0$), the Lagrangian strain tensor is defined by

$$\widehat{\mathbf{E}} = \frac{1}{2}(\widehat{\mathbf{C}} - \mathbf{1}), \tag{2.12}$$

where $\widehat{\mathbf{C}} = \widehat{\mathbf{F}}^T \widehat{\mathbf{F}}$ is the right Cauchy–Green deformation tensor. In the representation of the direct sum the relation (2.12), after using (2.2), is equivalent to

$$\widehat{\mathbf{E}} = \begin{pmatrix} \mathbf{E}^h & \mathbf{0} \\ \mathbf{0} & \mathbf{E}^v \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathbf{C}^h - \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}^v - \mathbf{1} \end{pmatrix}.$$

Using (2.4) and (2.5), the horizontal and vertical parts of the Cauchy–Green strain tensor are then respectively equal to

$$\begin{aligned} \mathbf{C}^h = \Big(\partial_h X^i \partial_l X^j &+ \dot{\partial}_m X^j \dot{\partial}_l G^m \dot{\partial}_k X^i \dot{\partial}_h G^k + \Gamma^{\star j}_{nl} \Gamma^{\star i}_{kh} X^n X^k \\ &- \partial_{(l} X^{(j} \dot{\partial}_{|k|} X^{i)} \dot{\partial}_{h)} G^k - \dot{\partial}_m X^{(j} \dot{\partial}_{(l} G^{|m|} \Gamma^{\star i)}_{|k|h)} X^k \\ &+ \partial_{(l} X^{(j} \Gamma^{\star i)}_{|k|h)} X^k \Big) \bar{g}_{ij}\, dx^l \otimes dx^h, \end{aligned} \tag{2.13}$$

$$\begin{aligned} \mathbf{C}^v = L^2 \Big(\dot{\partial}_h X^i \dot{\partial}_l X^j &+ \dot{\partial}_{(l} X^{(j} C^{i)}_{|k|h)} X^k \\ &+ C^i_{kh} C^j_{ml} X^k X^m \Big) \bar{g}_{ij}\, Dl^l \otimes Dl^h, \end{aligned} \tag{2.14}$$

where ( ) means the symmetric part with respect to the enclosed indices, the sign | | enclosing the index is used to exclude it from the symmetrization operation, and $\bar{g}_{ij}$ are components of the metric tensor in the actual configuration. The interrelated pair of measures of any deformation process (2.13) and (2.14) is defined in the invariant way.

In the case when the internal state is neglected, i.e. $\mathbf{y} = \mathbf{0}$, we obtain

$$\mathbf{C}^h = \left(\partial_h X^i \partial_l X^j + \Gamma^{\star j}_{nl} \Gamma^{\star i}_{kh} X^n X^k \right) \bar{g}_{ij}\, dx^l \otimes dx^h, \tag{2.15}$$

$\mathbf{C}^v$ is identically equal zero, and $\mathbf{X}$ and $\mathbf{g}$ are functions of position $\mathbf{x}$ only. The case $\mathbf{y} = \mathbf{y}_r$ with $\mathbf{y}_r$ being a residual or imperfection vector leads to non-singular $\mathbf{C}^h$ and $\mathbf{C}^v$. To specify the connection coefficients C^i_{jk}, G^i_j and $\Gamma^{\star i}_{jk}$ we first have to estimate the local internal (dislocation) structure of the solid under consideration defining its fundamental function L (square root of the internal energy stored in dislocations and induced by the deformation process) or its metric tensor $\mathbf{g}$.

2.4. Equilibrium equations

The equilibrium equations and boundary conditions are obtained from the variation of the action integral (cf. SACZUK [16])

$$I = \int_G \mathcal{L}(\mathbf{x}, \mathbf{y}, \mathbf{X}, \mathbf{F}^h, \mathbf{F}^v)\, dV, \tag{2.16}$$

where G denotes a fixed, closed and simply-connected region in the 6-dimensional space of $(\mathbf{x}, \mathbf{y})$, bounded by a surface ∂G, and

$$dV = \sqrt{\widehat{g}}\, dx^1 dx^2 dx^3 dy^1 dy^2 dy^3 = \sqrt{\widehat{g}}\, d\mathbf{x} d\mathbf{y}$$

with $\widehat{g} = \det(g_{ij} \oplus g_{ij})$ being the volume element. The variational derivative of the action integral I can be written in the form

$$\delta I = \int_G \Big[\mathcal{L}(D_k \delta x^k + \dot{D}_k \delta y^k) + \mathcal{L}_{|i} \delta x^i + L^{-1} \mathcal{L}|_i\, \delta y^i + \frac{\partial \mathcal{L}}{\partial X^k} \delta X^k + \frac{\partial \mathcal{L}}{\partial_h X^k_i} \delta_h X^k_i + \frac{\partial \mathcal{L}}{\partial_v X^k_i} \delta_v X^k_i \Big] dV, \tag{2.17}$$

where

$$D_i(\cdot) = \partial_i(\cdot) + \frac{\partial(\cdot)}{\partial X^n} \partial_i X^n, \qquad \dot{D}_i(\cdot) = \dot{\partial}_i(\cdot) + \frac{\partial(\cdot)}{\partial X^n} \dot{\partial}_i X^n \tag{2.18}$$

are the total partial derivatives with respect to x^i and y^i, and

$$\mathcal{L}_{|i} = \partial_i \mathcal{L} - \dot{\partial}_k \mathcal{L} \dot{\partial}_i G^k - \mathcal{L} \Gamma^{\star k}_{ik}, \qquad \mathcal{L}|_i = L \dot{\partial}_i \mathcal{L} - \mathcal{L} A^k_{ik} \tag{2.19}$$

are h- and v-derivatives of the density function $\mathcal{L}$, respectively.

The components of generalized body forces are defined by

$$_h f_k \equiv (\mathbf{f}^h)_k = \frac{\partial \mathcal{L}}{\partial (\mathbf{X}^h)^k}, \qquad _v f_k \equiv (\mathbf{f}^v)_k = \frac{\partial \mathcal{L}}{\partial (\mathbf{X}^v)^k}, \tag{2.20}$$

where $\mathbf{f}^h$ is identified with the external body force and $\mathbf{f}^v$ can be identified with the internal source of the exchange of momentum between dislocated states (cf. AIFANTIS [21]).

When variations of the independent variables in δI are neglected, i.e. $\delta \mathbf{x} = \mathbf{0}$ and $\delta \mathbf{y} = \mathbf{0}$, then

$$\delta I = \int_G \Big[(\mathbf{f}^h + \mathrm{Div}^h \mathbf{T}) \cdot \delta \mathbf{X}^h + (\mathbf{f}^v + \mathrm{Div}^v \mathbf{T}) \cdot \delta \mathbf{X}^v \Big] dV - \int_{\partial G} \Big[\delta \mathbf{X}^h \cdot (\mathbf{T}^h \mathbf{n} - \mathbf{T}^h \dot{\partial} \mathbf{G}\, \mathbf{m}) + \delta \mathbf{X}^v \cdot L \mathbf{T}^v \mathbf{m} \Big] dS, \tag{2.21}$$

where

$$
\begin{aligned}
(\mathrm{Div}^h\,\mathbf{T})_k &= D_i({}_hT^i_k) - \dot{\partial}_i G^j\,\dot{D}_j({}_hT^i_k) - {}_hT^i_j\Gamma^{\star j}_{ki}\,, \\
(\mathrm{Div}^v\,\mathbf{T})_k &= L\dot{D}_i({}_vT^i_k) - {}_vT^i_jA^j_{ki}
\end{aligned}
\tag{2.22}
$$

are h-divergence and v-divergence of **T**, and n_i, m_i are the components with respect to x^i and y^i of the unit vectors normal to the boundary ∂G, respectively.

One should point out that according to the connections Γ^h and Γ^v, one can distinguish the base space approach and the fibre space approach, respectively (cf. TAKANO [22]). The fundamental lemma of the calculus of variations applied to (2.21) gives the field equations

$$
{}_hf_k + (\mathrm{Div}^h\,\mathbf{T})_k = 0, \qquad {}_vf_k + (\mathrm{Div}^v\,\mathbf{T})_k = 0,
\tag{2.23}
$$

for all variations of $\delta\mathbf{X}^h$ and $\delta\mathbf{X}^v$, or in the component forms

$$
\begin{aligned}
{}_hf_k + \frac{\partial T^i_k}{\partial x^i} - \frac{\partial G^j}{\partial y^i}\frac{\partial T^i_k}{\partial y^j} - T^i_j\Gamma^{\star j}_{ki} &= 0, \\
{}_vf_k + L\frac{\partial T^i_k}{\partial y^i} - T^i_jA^j_{ki} &= 0,
\end{aligned}
$$

which should be satisfied in the interior of the inelastic body. The field equations (2.23), interrelated at the micro-level, form the equilibrium equations for both h- and v-ingredients of the inelastic behaviour of solids.

3. Ideas of a new criterion of adaptation

The classical shakedown criterion which defines the necessary condition of structural safety in the case of variable repeated loads is formulated as follows:

A certain domain of load variations is given and the question arises whether will a given structure will shake down in an arbitrary sequence of the loads contained within this domain.

One of the drawbacks of the classical shakedown theory is that a definite bound of the plastic work is not specified in the shakedown criterion (KÖNIG [23]). A definite amount of this work is at any rate of fundamental value for an adaptation and can be used, among others, to establish a safe number of load cycles for the structure's life. For that reason modifications in the classical criterion of adaptation seem to be necessary.

3.1. Motivation

A more realistic assessment of structural safety demands to change the way of estimation of the energy function used in the shakedown theorem, according to the methodology of Finslerian description of solid behaviour. The source of our idea comes from the proof of shakedown theorem (cf. GROSS–WEEGE [8], SACZUK and STUMPF [9]). In this proof we have to estimate the time-dependent energy function Π in the form

$$\Pi(t) = \frac{1}{2}\int_V \mathbf{T}\cdot\mathbf{F}\, dV,$$

where $\mathbf{T}$ is the first Piola - Kirchhoff stress tensor and $\mathbf{F}$ is the deformation gradient tensor. We analyze the time derivative of Π by decomposing its right-hand side according to the following scheme:

$$\partial_t\Pi \;=\; -\int_V (\check{\mathbf{S}} - \bar{\mathbf{S}})\cdot\partial_t\check{\mathbf{E}}^p dV \quad + \quad \int_V (\check{\mathbf{T}} - \bar{\mathbf{T}})\cdot\partial_t\check{\mathbf{H}}\mathbf{F} dV$$

↓	↓
the convexity conditions	equilibrium conditions
↓	↓
decreases when $\check{\mathbf{E}}^p \neq \mathbf{0}$	$\mathbf{0}$

where $\check{\mathbf{S}}$ is the actual residual (second Piola - Kirchhoff) stress tensor, $\bar{\mathbf{S}}$ is a (fictitious) shakedown stress tensor, $\check{\mathbf{E}}^p$ is the residual plastic strain tensor obtained from the multiplicative decomposition of the deformation gradient tensor, and $\check{\mathbf{H}}$ is the residual displacement gradient.

The above estimation of $\partial_t\Pi$ we replace by

$$\partial_t\Pi(t, \widehat{\mathbf{C}}) \leq \beta \parallel \partial_{\widehat{\mathbf{C}}}\Pi(t, \widehat{\mathbf{C}}) \parallel + \psi(t, \Pi(t, \widehat{\mathbf{C}})), \tag{3.1}$$

where $\widehat{\mathbf{C}}$ is the Cauchy - Green strain tensor, β is a constant connected with a safe domain of admissible strains and ψ is a comparison function being a maximal solution of a comparison differential problem used to estimate Π. The basic problem here is how to define the comparison function. Before that we will introduce the notion of the maximal solution of a differential problem and the comparison differential theorem within the theory of differential inequalities (LAKSHMIKANTHAM and LEELA [15], SZARSKI [14]). Let us note that differential inequalities are extremely important and constitute a very helpful technique in the differential problems to formulate the uniqueness conditions for their solutions and to make their certain estimations.

Assume that $I = [t_0, T) \subset R$, $0 < t_0 < T$ is a time-interval, $G \subset R^2$ is an open set in R^2.

DEFINITION 1. *Let r be a solution of a differential problem*

$$u' = \psi(t,u), \qquad u(t_0) = u_0, \qquad t \in I \tag{3.2}$$

and $\psi \in C(G,R)$. Then r is said to be a maximal solution of (3.2) *if*

$$u(t) \le r(t), \qquad t \in I \tag{3.3}$$

for every solution u of (3.2) *in I.*

COMPARISON THEOREM (LAKSHMIKANTHAM and LEELA [15], Vol. I)
Suppose:
1. $\psi \in C(G,R)$ *and r is the maximal solution of* (3.2).
2. $m \in C(I,R)$, $(t, m(t)) \in G$ *for* $t \in I$, $m(t_0) \le u_0$, *and*

$$Dm(t) \le \psi(t, m(t)), \qquad t \in I \backslash S \tag{3.4}$$

with D being a fixed Dini derivative and S at most a countable subset of I. Then

$$m(t) \le r(t) \qquad \text{in} \quad I. \tag{3.5}$$

3.2. A comparison problem

A comparison function ψ in (3.1) is identically equal zero in the classical case. This takes place when the microstructure-independent equilibrium conditions for $\check{\mathbf{T}}$ and $\bar{\mathbf{T}}$ are satisfied *a priori*. Therefore, the equilibrium conditions, or more strictly the equations of motion, will be used to define the comparison function ψ.

Our comparison problem will be defined by differential equations, deduced from the equations of motion in the continuum with microstructure (cf. Eqs. (2.23))

$$\rho_0\, {}_h\dot{v}_k = {}_hf_k + \frac{\partial T^i_k}{\partial x^i} - \frac{\partial G^j}{\partial y^i}\frac{\partial T^i_k}{\partial y^j} - T^i_j \Gamma^{\star j}_{ki}\,, \tag{3.6}$$

$$\rho_0\, {}_v\dot{v}_k = {}_vf_k + L\frac{\partial T^i_k}{\partial y^i} - T^i_j A^j_{ki}\,, \tag{3.7}$$

where ${}_hv_k$ and ${}_vv_k$ are components of macro- and micro-velocity. The remaining unknowns were defined in Sec. 2. The first equation describes the macro-motion of the body (in the configuration space), while the second one its micro-motion (in the internal state space), or briefly h-motion and v-motion.

For clarity, the above partial differential equations are reduced to the scalar differential equations of the form

$$\dot{v}^h = \psi^h(t, v^h), \tag{3.8}$$

$$\dot{v}^v = \psi^v(t, v^v) \tag{3.9}$$

using an Euclidean norm $\| \mathbf{a} \| = \sqrt{(\mathbf{a} \cdot \mathbf{a})}$ under the following assumptions. If body forces $\mathbf{f}^h$ and $\mathbf{f}^v$ are neglected and classical equilibrium conditions $\partial_i T^i_k = 0$ and $\overset{\bullet}{\partial}_i T^i_k = 0$ are satisfied inside the body, then the right-hand sides in (3.6) and (3.7) lead to the following identifications:

$$\Gamma^* \propto \frac{1}{x}\frac{y^1}{y}, \qquad \mathbf{A} \propto \frac{1}{y}, \tag{3.10}$$

where $y = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}$ is the Euclidean distance in the internal state space. The identification used in (3.10) is due to the following form of L:

$$L^2(\mathbf{x}, \mathbf{y}) = g_{ij}(\mathbf{x}, \mathbf{y}) y^i y^j = \frac{2x^1 y^1}{y}(y^1)^2 + 2x^2 (y^2)^2 + 2x^3 (y^3)^2.$$

This assumed relation, identified here with the internal energy stored in dislocations, is proportional to the square of the microposition vector.

According to equations (3.8) and (3.9) one can write the admissible functions in the form

$$\psi^h = \frac{v^h}{x}\frac{y^1}{y}, \qquad \psi^v = \frac{v^v}{y}$$

and, then, their solutions under homogeneous initial conditions are respectively equal

$$v^h = \exp\left(\frac{y^1}{xy}t\right), \qquad v^v = \exp\left(\frac{t}{y}\right). \tag{3.11}$$

3.3. Method of solution

The language of the theory of differential inequalities (LAKSHMIKANTHAM and LEELA [15], SZARSKI [14]) is used to provide a general definition of "adaptation" of structures. Within this methodology it is sufficient to assume:

1. There exist (experimental or other) time-dependent comparison functions ψ^h and ψ^v defined in a certain domain Ω of the strain-time space.

2. The estimation (3.1) takes place at any instant of time t.

Then, according to the comparison theorem (cf. SZARSKI [14], Chap. 7, LAKSHMIKANTHAM and LEELA [15], Chap. 9) one can make the following estimations

$$\Pi^h(t, \mathbf{C}^h) \leq r^h(t), \tag{3.12}$$

$$\Pi^v(t, \mathbf{C}^v) \leq r^v(t), \tag{3.13}$$

where r^h and r^v are the maximal solutions (see Definition 1) of the differential problems induced by ψ^h and ψ^v, respectively, i.e. at the given time interval the energy function is estimated by known *a priori* time-dependent functions. In this case we have to find the time-dependent comparison functions ψ^h and ψ^v, instead of a time-independent residual stress field postulated by the classical shakedown criterion (MELAN [2], KOITER [1]). The condition (3.1) is the asymptotic estimation of the rate of energy function. Moreover, the energy function $\Pi = \Pi^h + \Pi^v$ is the position-direction dependent function. It should be emphasized that the variable t can here mean either the time or a monotonically-increasing loading parameter. The presented explanation allows us to propose

DEFINITION 2. *It is said that the structure will shake down over any programme of loading if the total energy created during its deformation satisfies* (3.12) *and* (3.13) *at any time interval of that programme.*

Note that the classical shakedown demands only boundedness of the total plastic work in the limit as the time approaches infinity. To define the plastic energy for defining a criterion of adaptation, one has to decompose the total strain tensor into elastic and plastic parts. In the classical approach it is generally realized within the multiplicative decomposition of the deformation gradient using the unstressed intermediate configuration concept, while in the Finslerian one the additive decomposition is given by definition. The shortcoming of such the additive decomposition lies in that a common sense of elastic and plastic part of the deformation is slightly changed. It is due to the fact that any deformation process cannot be strictly decomposed into elastic and plastic parts [16]. For simplicity, one can consider the state of strain in the structure that does not vary with position in it. Then the deformation gradient $\widehat{\mathbf{F}}$ is a function of the microposition vector $\mathbf{y}$. Summarizing the quoted explanations we assume:

1. The total (strain) energy is defined as a time-dependent energy function

$$\Pi(t, \widehat{\mathbf{C}}) = \int\limits_V (\mathbf{T}^h \cdot \mathbf{C}^h + \mathbf{T}^v \cdot \mathbf{C}^v)\, dV,$$

where the Cauchy–Green strain tensors $\mathbf{C}^h$ and $\mathbf{C}^v$ are defined by (2.13) and (2.14), and appropriate stress tensors from (2.23), for given $\mathbf{f}^h$ and $\mathbf{f}^v$, respectively.

2. We also establish by calculation or using experimental results the safe domain Ω, required to any individual shakedown problem, and the comparison functions ψ^h and ψ^v.

3.4. Algorithm

The above explanation can be arranged in the following algorithm.

For $i = 1, n$

$(i, 1)$ Calculate the strain state of the body for a given loading programme and an assumed internal state, or basing on experimental data connected with internal state of its material;

$(i, 2)$ Define the strain energy function;

$(i, 3)$ Define the function of a comparison problem;

$(i, 4)$ Find the maximal solution of the comparison problem;

$(i, 5)$ Verify the conditions of adaptation (see Shakedown Theorem).

The steps from (i,2) to (i,5) are changed according to a demand. The index i can symbolize a number of states which are relevant for a prediction of the safe domain in the space of admissible strains. In general, an optimization technique is necessary to define the safe domain of adaptation for the given structure.

4. A proposition of the shakedown theorem

Under the preparation of Sec. 3 we come to the following theorem

SHAKEDOWN THEOREM:

(i) *Suppose that* $r^h(t, t_0, v_0^h)$ *and* $r^v(t, t_0, v_0^v)$ *are the maximal solutions of the scalar differential problems*

$$(4.1) \qquad \overset{\bullet}{v}^h = \psi^h(t, v^h) \equiv \frac{1}{\rho_0} \parallel \mathrm{Div}^h\, \mathbf{T} + \mathbf{f}^h \parallel, \qquad v^h(t_0) = v_0^h \equiv \parallel \mathbf{C}_0^h \parallel,$$

$$(4.2) \qquad \overset{\bullet}{v}^h = \psi^v(t, v^v) \equiv \frac{1}{\rho_0} \parallel \mathrm{Div}^v\, \mathbf{T} + \mathbf{f}^v \parallel, \qquad v^v(t_0) = v_0^v \equiv \parallel \mathbf{C}_0^v \parallel .$$

(ii) *Suppose that the energy function* $\Pi(t, \widehat{\mathbf{C}}) = \Pi^h + \Pi^v$ *possesses continuous partial derivatives* $\partial_t \Pi$ *and* $\partial_C \Pi$ *on*

$$(4.3) \qquad \Omega = \Big\{(t, C):\; t_0 \le t \le t_0 + a, \quad C = \parallel \widehat{\mathbf{C}} \parallel \le \alpha - \beta(t - t_0)\Big\}.$$

(iii) *The following inequalities are satisfied on* Ω:

$$(4.4) \qquad \partial_t \Pi^h \le \beta \parallel \partial_{\mathbf{C}^h} \Pi^h(t, \mathbf{C}^h) \parallel + \psi^h(t, \Pi^h(t, \mathbf{C}^h)),$$

$$(4.5) \qquad \partial_t \Pi^v \le \beta \parallel \partial_{\mathbf{C}^v} \Pi^v(t, \mathbf{C}^v) \parallel + \psi^v(t, \Pi^v(t, \mathbf{C}^v)).$$

Then,

$$(4.6) \qquad \Pi^h(t_0, \mathbf{C}^h) \le v_0^h ,$$

$$(4.7) \qquad \Pi^v(t_0, \mathbf{C}^v) \le v_0^v$$

for $C \le \beta$ *implies*

$$(4.8) \qquad \Pi^h(t, \mathbf{C}^h) \le r^h(t, t_0, v_0^h),$$

$$(4.9) \qquad \Pi^v(t, \mathbf{C}^v) \le r^v(t, t_0, v_0^v),$$

i.e. the body shakes down under the given loading programme.

REMARK 1. The inequalities (4.6) and (4.7) define initial conditions for the strain state. In fact, they can be neglected as in the classical shakedown theorem, if we shall remember that strains in reality are not arbitrary quantities, but have always definite values. These conditions mean simply that an analyzed initial strain state is inside the domain Ω.

Proof of the Shakedown Theorem is analogous to the proof of Theorem 9.2.1 in LAKSHMIKANTHAM and LEELA [15]. In this proof, in the spirit of the Comparison Theorem, we have to estimate the function

$$m(t) = \max_{\|\widehat{\mathbf{C}}\| \le \alpha - \beta(t-t_0)} \Pi(t, \widehat{\mathbf{C}}), \tag{4.10}$$

which satisfies the differential inequality (3.4), using the extremal solutions $r^h(\cdot, t_0, \| \mathbf{C}_0^h \|)$ and $r^v(\cdot, t_0, \| \mathbf{C}_0^v \|)$ of the corresponding differential equations (4.1) and (4.2) under conditions (4.6) and (4.7), respectively.

There are two observations which we would like to make with respect to the presented technique. In the first place, the presented idea of new criterion of adaptation can be treated as an example, and its extension to more complicated cases is also possible (cf. LAKSHMIKANTHAM and LEELA [15]). For a more detailed and rigorous discussion of the generalized cases the reader is referred to the cited literature. The second observation is that the fundamental conditions used in Shakedown Theorem depend on a particular internal strain distribution. This information is of fundamental importance since the mechanical response of the solid like softening, hardening, localization is only changed by the history of deformation and the applied load system (BASINSKI and JACKSON [24]).

5. Conclusions

The importance of shakedown theorems depends on proving that if the structure shakes down under some particular programme of loading, it will shake down under any loading programme. The proposed shakedown theorem can be used to predict the behaviour of structures based on the properties of the energy function and its internal energy distribution. As final conclusions one may cite:

1. It generalizes the classical approaches (MELAN [2], KOITER [1], CORRADI and MAIER [11], KÖNIG and SIEMASZKO [12]) by including arbitrary deformations and material nonlinearities.

2. It is based on the consistent continuum theory of solid behaviour which allows one to describe, among others, the specific internal structure of the material, the influence of initial deformations or imperfections of the deformation process within the unified concept.

3. It can be used to estimate a safe number of load cycles for the real or predicted structure's life.

4. It can be used to estimate lower and upper limits of the total (or plastic under certain conditions) work in the given time interval.

A. Appendix. Basic notions in Finslerian geometry

In this appendix we give the mathematical preliminaries on Finslerian geometry (RUND [17], MATSUMOTO [18]), especially on the definitions of connections, absolute differential and covariant derivatives.

A Finslerian (generalized metric) geometry is a natural generalization of a Riemannian one and of which a metric tensor depends both on the position and on the direction. Following BUSEMANN [25] the Finsler space is a metric, finitely compact (i.e. every bounded, infinite sequence of points in a metric space contains a converging subsequence) and locally Minkowskian space. The anisotropic character (the Minkowskian metric is not symmetric in general) of this geometry is expressed completely by the physically useful concept of indicatrix. Therefore, common inelastic solid behaviours like anisotropy and hysteresis loop are modelled easily within this geometry. The major obstacle encountered in the practical application of this geometry is caused by its complexity and a difficulty of using its concepts to define mechanical counterparts.

The subject is described in the monographs of RUND [17], ASANOV [26], MATSUMOTO [18], ABATE and PATRIZIO [27], ANTONELLI and MIRON [28] where additional bibliography can be found. In this appendix we shall use the notation employed mainly in [18] and [17] without further comments.

A.1. Basic concepts

We consider an n-dimensional differentiable manifold M (cf. CHOQUET–BRUHAT *et al.* [29, Ch. III], WESTENHOLTZ [30, Parts II and V]) as the space for modelling of a solid behaviour. Let T_xM be the tangent space of M at a point $\mathbf{x}$, and $T(M)$ be the set of all tangent vectors parameterized by M. The mapping (projection) $\pi_T : T(M) \to M$ is defined by $\pi_T(\mathbf{y}) = \mathbf{x}$ for $\mathbf{y} \in T_xM$. Let $L(M)$ be the linear frame bundle of the manifold M. The projection $\pi_L : L(M) \to M$ is given by $(\mathbf{x}, \mathbf{z}) \to \mathbf{z}$, where a frame $\mathbf{z}$ at a point $\mathbf{x}$ of M is by definition a base $\mathbf{z} = \{z_i\}_{1,\ldots,n}$ of the tangent vector space T_xM.

The Finsler bundle $F(M)$ of M is, by definition [18], the principal bundle $\pi_T^{-1}L(M)$ over $T(M)$ induced from the frame bundle $L(M)$ by the projection π_T of the tangent bundle $T(M)$. This construction is represented by the following commutative diagram

$$\begin{array}{ccccc} (\mathbf{y},\mathbf{z}) \in F(M) & \xrightarrow{\pi_2} & L(M) \ni (\mathbf{x},\mathbf{z}) \\ \pi_1 \downarrow & & \downarrow \pi_L \\ (\mathbf{x},\mathbf{y}) \in T(M) & \xrightarrow{\pi_T} & M \quad \ni (\mathbf{x}) \end{array}$$

where π_T, π_L and π_1, π_2 are projections.

To introduce a Finsler connection $F\Gamma$ it is worth reminding that a (nonlinear) connection N on M is a distribution $\mathbf{y} \in T(M) \mapsto N_y$ in the $T(M)$ satisfying $T_y = N_y \oplus T_y^v$, namely, the tangent space T_y at every point $\mathbf{y}$ of $T(M)$ is the direct sum of N_y and the vertical subspace T_y^v.

A vertical connection Γ^v is a distribution $u \in F(M) \mapsto \Gamma_u^v$ in the $F(M)$ such that the restriction $\Gamma^v|_{F(x)}$ of Γ^v of the subbundle $F(x)$ of $F(M)$ over the fibre $\pi_T^{-1}(x)$ over every point $\mathbf{x} \in M$ is an ordinary connection in $F(x)$. In turn, the horizontal connection Γ^h is a distribution $u \in F(M) \mapsto \Gamma_u^h$ in $F(M)$ satisfying

$$T_u F(M) = \Gamma_u^h \oplus \Gamma_u^v \oplus T_u F(M)^v, \tag{A.1}$$

$$\tau_g' \Gamma_u^h = \Gamma_{ug}^h, \tag{A.2}$$

where $T_u F(M)^v$ is the vertical subspace of the tangent space $T_u F(M)$ and τ_g is a right translation of $F(M)$ by $g \in GL(n, R)$. Other equivalent ways leading to the Finsler connection have been discussed by MATSUMOTO [18].

The h-part Γ^h of the Finsler connection $F\Gamma = (\Gamma^h, \Gamma^v)$ is spanned by the h-basic vector field $\mathbf{B}^h(\mathbf{v})$, $\mathbf{v} \in V$, V – a vector space, of the form (MATSUMOTO [31])

$$\mathbf{B}^h(\mathbf{v}) = z_a^i v^a \left(\frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j} - z_b^k F_{ki}^j \frac{\partial}{\partial z_b^j} \right), \tag{A.3}$$

at a point $u = (x^i, y^i, z_a^i)$ and for $\mathbf{v} = (v^a)$, where (x^i, y^i, z_a^i) are local coordinates on $F(M)$ induced from local coordinates (x^i) on M. The connection coefficients F_{ki}^j are defined by $F_{ki}^j = \Gamma_{ki}^j - C_{kr}^j N_i^r$; following RUND [17] they will be denoted by Γ_{ki}^{*j}.

The v-part Γ^v of the Finsler connection $F\Gamma = (\Gamma^h, \Gamma^v)$ is spanned by the v-basic vector field $\mathbf{B}^v(\mathbf{v})$, $\mathbf{v} \in V$, of the form (MATSUMOTO [31])

$$\mathbf{B}^v(\mathbf{v}) = z_a^i v^a \left(\frac{\partial}{\partial y^i} - z_b^i C_{ki}^j \frac{\partial}{\partial z_b^j} \right), \tag{A.4}$$

where functions $C_{ki}^j(\mathbf{x}, \mathbf{y})$ are connection parameters of the vertical connection Γ^v.

The h- and v-covariant derivatives of an arbitrary Finsler tensor field K [18] are defined, within the bundle theory, by

$$\nabla^h K(\mathbf{v}) = \mathbf{B}^h(\mathbf{v}) \cdot K, \tag{A.5}$$

$$\nabla^v K(\mathbf{v}) = \mathbf{B}^v(\mathbf{v}) \cdot K, \tag{A.6}$$

for any $\mathbf{v} \in V$. The components of $\nabla^h(\ )$ and $\nabla^v(\ )$ are usually distinguished by "$_|$" and " | ", respectively (cf. (A.17) and (A.18)).

The torsion tensor fields $\mathbf{T}^1$, $\mathbf{R}^1$, $\mathbf{C}^1$, $\mathbf{P}^1$, $\mathbf{S}^1$ ($\mathbf{T}^1$ and $\mathbf{C}^1$ are often denoted by $\mathbf{T}$ and $\mathbf{C}$, respectively, cf. RUND [17]) and the curvature tensor fields $\mathbf{R}^2$, $\mathbf{P}^2$, $\mathbf{S}^2$ of a Finsler connection $F\Gamma$ are introduced by the structure equations

$$\begin{aligned}
&[\mathbf{B}^h(1), \mathbf{B}^h(2)] = \mathbf{B}^h(\mathbf{T}^1(1,2)) + \mathbf{B}^v(\mathbf{R}^1(1,2)) + \mathbf{Z}(\mathbf{R}^2(1,2)),\\
&[\mathbf{B}^h(1), \mathbf{B}^v(2)] = \mathbf{B}^h(\mathbf{C}^1(1,2)) + \mathbf{B}^v(\mathbf{P}^1(1,2)) + \mathbf{Z}(\mathbf{P}^2(1,2)),\\
&[\mathbf{B}^v(1), \mathbf{B}^v(2)] = \mathbf{B}^v(\mathbf{S}^1(1,2)) + \mathbf{Z}(\mathbf{S}^2(1,2)),
\end{aligned} \tag{A.7}$$

where $\mathbf{v}_1, \mathbf{v}_2 \in V$ are present by their indices 1, 2 only. Here $\mathbf{Z}(A)$ is the fundamental vector field on $F(M)$ corresponding to the element A of the Lie algebra of $GL(n, R)$.

The torsion and curvature tensors are called as follows:

$\mathbf{T}^1$ is $(h)h$-torsion, $\mathbf{R}^1$ is $(v)h$-torsion, $\mathbf{R}^2$ is h-curvature,
$\mathbf{C}^1$ is $(h)hv$-torsion, $\mathbf{P}^1$ is $(v)hv$-torsion, $\mathbf{P}^2$ is hv-curvature,
$\mathbf{S}^1$ is $(v)v$-torsion, $\mathbf{S}^2$ is v-curvature.

DEFINITION A.1. *A Finsler connection $F\Gamma$ of a Finsler space $F(M)$ with a fundamental function L is the Cartan connection if*:

(i) $\nabla^h \mathbf{g} = \mathbf{0}$ *and* $\mathbf{T}^1 \equiv \mathbf{0}$;

(ii) $\nabla^v \mathbf{g} = \mathbf{0}$ *and* $\mathbf{S}^1 \equiv \mathbf{0}$;

(iii) *The deflection tensor field* $\mathbf{D} = \nabla^h \mathbf{y}$ *is given.*

In practice $\mathbf{D} \equiv \mathbf{0}$. This condition means that nonlinear connection coefficients N^i_j are defined by horizontal connection coefficients $\Gamma^{\star i}_{rj}$ as $N^i_j = \Gamma^{\star i}_{rj} y^r$.

The triplet $F\Gamma = (F^i_{jk}, N^i_k, C^i_{jk})$ is known as the Finsler connection. Before determining the Finsler connection $F\Gamma$ one has to introduce the Finsler metric. In this geometry the differentiable manifold M is equipped with a line element $ds = L(\mathbf{x}, d\mathbf{x})$, where the function L, homogeneous of degree one in $d\mathbf{x}$, is called the fundamental function of the Finsler space. Its geometric significance results from the fact that in each tangent space $T_x M$ of M the function $L(\mathbf{x}, \mathbf{y})$ defines the $(n-1)$-dimensional hypersurface,

$$L(\mathbf{x}, \mathbf{y}) = 1, \tag{A.8}$$

called the indicatrix, where $\mathbf{x}$ is assumed to be fixed. The concept of the indicatrix (YASUDA [32], MATSUMOTO [33], WATANABE [34]), as developed by modern geometry, provides a precise explanation of the main geometric properties of a given manifold. The definition (A.8) represents a sphere in the Riemannian case.

Physically, one can construct the fundamental function L using a relation

$$L(\mathbf{x}, \mathbf{y}) = \sqrt{W(\mathbf{x}, \mathbf{y})}, \tag{A.9}$$

where the function W, homogeneous of the second degree with respect to $\mathbf{y}$, can be interpreted as a stored energy in dislocations and induced by the deformation process. One should stress that the geometric structure of a yield surface in elasto-plasticity falls under the general concept of indicatrix of the Finsler space. The most evident analogy between the indicatrix and the classical yield surface is that they both are closed convex hypersurfaces in the 6-dimensional spaces. The first one is in the 6-dimensional (x, y)-space of the Finsler bundle, the second one is in the 6-dimensional x-space of symmetric stress or strain tensors (the stress or strain space). The further analogies are not so evident due to their different geometro-physical meanings. For example, the indicatrix can represent an abstract model of internal structure of the given solid, while the yield surface has a sense of the surface separating an elastic region from a plastic one. In other words, the indicatrix is the fundamental geometric object of underlying (physical) space for any solid, while the yield surface is a certain auxiliary notion of criterion type used to distinguish between loading and unloading criteria. In Finslerian approach such a distinction can be superfluous.

The function $L = L(\mathbf{x}, \mathbf{y})$ satisfies, by assumptions, the following two conditions:

(i) The function L is at least of class C^3 with respect to $\mathbf{x}$;

(ii) The function L is positive homogeneous of degree one with respect to $\mathbf{y}$.

The homogeneity condition (ii) plays an important role in the Finsler geometry. The application of the Euler theorem on homogeneous functions to L^2 gives

$$L^2(\mathbf{x}, \mathbf{y}) = g_{ij}(\mathbf{x}, \mathbf{y}) y^i y^j, \tag{A.10}$$

where

$$g_{ij}(\mathbf{x}, \mathbf{y}) = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2(\mathbf{x}, \mathbf{y}) \tag{A.11}$$

is the Finslerian metric tensor.

For example, the Riemannian space as a special case of the Finsler space demands

$$ds^2 = g_{ij}(x) dx^i dx^j \equiv L^2(\mathbf{x}, d\mathbf{x}),$$

where the metric tensor is defined by

$$g_{ij}(\mathbf{x}) = \frac{1}{2} \frac{\partial^2 L^2}{\partial dx^i \partial dx^j}. \tag{A.12}$$

Using the relation $y_i = g_{ij}(\mathbf{x}, \mathbf{y}) y^j$, from (A.10) we obtain $y_i = L \dot{\partial}_i L$. Then the unit tangent vector

$$l_i = \frac{y_i}{L(\mathbf{x}, \mathbf{y})} = \dot{\partial}_i L(\mathbf{x}, \mathbf{y}) = g_{ij}(\mathbf{x}, \mathbf{y}) l^j. \tag{A.13}$$

The definition (A.11) (cf. Definition A.1) allows us to define, among others, connection coefficients (A.20)–(A.23) as functions of L only. For example, the so-called Cartan torsion tensor is then defined from

$$C_{ijk}(\mathbf{x},\mathbf{y}) = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k L^2(\mathbf{x},\mathbf{y}). \tag{A.14}$$

The remaining torsion tensors, curvature tensors, Cartan structure equations are discussed by RUND [17], MATSUMOTO [18].

A.2. Covariant derivatives

The definitions of covariant derivatives (A.5) and (A.6) restricted to the Cartan connection (Definition A.1) can be introduced as follows. Consider an $(\mathbf{x},\mathbf{y})$-dependent tensor $\mathbf{T} = \mathbf{T}(\mathbf{x},\mathbf{y})$, then the absolute differential and covariant derivatives are defined by

$$DT^j_k = T^j_{k|i}dx^i + T^j_k|_i\, Dl^i, \tag{A.15}$$

or, in the absolute tensor notation, by

$$D\mathbf{T} = \nabla^h_i \mathbf{T}\otimes dx^i + \nabla^v_i \mathbf{T}\otimes Dl^i, \tag{A.16}$$

where

$$T^j_{k|i} = \partial_i T^j_k - \dot{\partial}_i G^l \dot{\partial}_l T^j_k + \Gamma^{\star j}_{lk} T^l_i - \Gamma^{\star l}_{ik} T^j_l\,, \tag{A.17}$$

$$T^j_k|_i = L\dot{\partial}_i T^j_k + A^j_{il} T^l_k - A^l_{ik} T^j_l \tag{A.18}$$

are h-derivative and v-derivative, respectively, and

$$Dl^i = dl^i + \Gamma^i_{jk} y^k dy^j \tag{A.19}$$

is the absolute differential of the unit tangent vector (A.13). To define the remaining quantities in (A.17) and (A.18) one has to use Definition A.1 (RUND [17], MATSUMOTO [18]). In this case they are given as follows:

$$\Gamma^\star_{ijk} = \Gamma_{ijk} - C_{jkl}\frac{\partial G^l}{\partial y^i} = \gamma_{ijk} - C_{kjl}\frac{\partial G^l}{\partial y^i} - C_{ijl}\frac{\partial G^l}{\partial y^k} + C_{ikl}\frac{\partial G^l}{\partial y^j}\,, \tag{A.20}$$

$$\Gamma^\star_{ijk} = g_{jl}\Gamma^{\star l}_{ik}\,, \qquad \Gamma_{ijk} = g_{jl}\Gamma^l_{ik}y^j, \qquad 2G^l = \gamma^l_{jk}y^j y^k, \tag{A.21}$$

$$\dot{\partial}_k G^l = \frac{\partial G^l}{\partial y^k} = \Gamma^l_{jk}y^j = \Gamma^{\star l}_{jk}y^j, \qquad \gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ki}}{\partial x^j}\right), \tag{A.22}$$

$$\begin{aligned} C_{ijk} &= \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}\,, \qquad C_{ijk}y^k = C_{ijk}y^j = C_{ijk}y^i = 0, \\ C_{ijk} &= g_{jl}C^l_{ik}\,, \qquad A^i_{jk} = LC^i_{jk}\,. \end{aligned} \tag{A.23}$$

In particular,

$$X^k_{|j} = \partial_j X^k - \dot{\partial}_j G^n \dot{\partial}_n X^k + \Gamma^{\star k}_{nj} X^n, \tag{A.24}$$

$$X^k|_j = L\dot{\partial}_j X^k + A^k_{ij} X^i \tag{A.25}$$

for a contravariant vector field $\mathbf{X} = \mathbf{X}(\mathbf{x}, \mathbf{y})$, and

$$f_{|j} = \partial_j f - \dot{\partial}_j G^k \dot{\partial}_k f, \tag{A.26}$$

$$f|_j = L\dot{\partial}_j f \tag{A.27}$$

for a scalar field $f = f(\mathbf{x}, \mathbf{y})$.

Instead of using (A.5) and (A.6) to define the absolute differential and covariant derivatives, one can apply the linear mapping

$$\nabla_{\mathbf{X}} : TF(M) \to TF(M), \qquad \mathbf{Y} \mapsto \nabla_{\mathbf{X}}\mathbf{Y} \tag{A.28}$$

for any $\mathbf{X}, \mathbf{Y} \in TF(M)$. To obtain a desirable result we can proceed as follows.

In Finsler spaces all quantities are depend both on the position vector ($\mathbf{x}$) and the direction vector ($\mathbf{y}$). If we define coordinate transformations by

$$\bar{x}^a = f^a(x), \qquad \bar{y}^a = \frac{\partial \bar{x}^a}{\partial x^j} y^j, \qquad \text{rank}\left[\frac{\partial \bar{x}^a}{\partial x^j}\right] = n \tag{A.29}$$

and, if there exist the quantities $N^a_j(\mathbf{x}, \mathbf{y})$ which transform under (A.29) according to the rule

$$\frac{\partial^2 \bar{x}^a}{\partial x^j \partial x^k} y^k = \frac{\partial \bar{x}^a}{\partial x^k} N^k_j - \frac{\partial \bar{x}^b}{\partial x^j} \bar{N}^a_b,$$

then one can define the covariant differential operator by

$$\frac{\delta}{\delta x^k} = \frac{\partial}{\partial x^k} - N^i_k \frac{\partial}{\partial y^i} \qquad \text{or} \qquad \delta_k = \partial_k - N^i_k \dot{\partial}_i\,. \tag{A.30}$$

The quantities N^i_k in the light of Definition A.1 are equal to $\dot{\partial}_k G^i$. It is important to note that the basis $(\partial_i, \dot{\partial}_i)$, with respect to the coordinate transformation (A.29), does not transform as a vector, while the basis $(\delta_i, \dot{\partial}_i)$ has the desired property (cf. ČOMIĆ [35]). The dual basis to $(\delta_i, \dot{\partial}_i)$ is denoted by (dx^i, Dy^i), where

$$Dy^i = dy^i + N^i_k dx^k.$$

Any vector field $\mathbf{X}$ in $TF(M)$ can be represented in the basis $(\delta_i, \dot{\partial}_i)$ in the following form

$$\mathbf{X} = \mathbf{X}^h + \mathbf{X}^v = X^i \delta_i + X^a \dot{\partial}_a\,, \tag{A.31}$$

where we call $\mathbf{X}^h = X^i \delta_i$ the horizontal vector field, and $\mathbf{X}^v = X^a \dot{\partial}_a$ the vertical vector field. Generalization to more complex geometric object is straightforward.

Under this preparation, if we take $X^i = dx^i$ and $X^a = Dy^a$ in $D\mathbf{T} = \nabla_{\mathbf{X}}\mathbf{T}$, the relation (A.16) can be alternatively written as

$$D\mathbf{T} = \nabla_{\delta_i}\mathbf{T} \otimes dx^i + \nabla_{\dot{\partial}_i}\mathbf{T} \otimes Dy^i .$$

Analogously to the Riemannian geometry, the Cartan covariant derivatives are defined to be metric, i.e.

$$g_{ik|j} = 0, \qquad g_{ik}|_j = 0. \tag{A.32}$$

In addition

$$l^k_{|j} = y^k_{|j} = L_{|j} = 0, \qquad L|_j = y_j , \qquad y^i|_j = L\delta^i_j , \qquad y_i|_j = Lg_{ij} . \tag{A.33}$$

The identities (A.32) and (A.33) show that the metric tensor g_{ik}, the metric function L, and the tangent vectors y^i and l^i can be treated as constants for the h-derivative. In the case of v-derivative this is true only for the metric tensor g_{ik}.

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INSTITUTE OF FLUID-FLOW MACHINERY
POLISH ACADEMY OF SCIENCES
e-mail: jsa@imppan.imp.pg.gda.pl

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