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Lagrange's equations for holonomic systems with rigid bodies

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A HOLONOMIC SYSTEM is considered which consists of rigid bodies and material points. Any rigid body is regarded as a continuous system and its position is described by the so-called angular vectors. Starting from the characterization of the constraints and using some identities for the angular vectors, the motion of the holonomic system is shown to be governed by the usual Lagrange's equations. The essential role of the angular vectors is emphasized through comparison with a previous approach.

1. Introduction

A RIGID BODY is a system with a number of degrees of freedom not greater than 6. Nevertheless, treatments of analytical mechanics deal only with material points and hence rigid bodies are modelled as a set of material points though such points are not characterized operatively. The results are then deemed to apply for continuous bodies by merely replacing the summation over the particles by a volume integration, with the point mass becoming a mass density (cf. [1–5]).

Quite naturally, instead, a rigid body might be viewed as a continuous body whose mechanical state in space is characterized by the position of a point and the orientation of a rigidly attached triple of non-coplanar axes. This view is customary in connection with the kinematics of rigid body motion and Euler's equations of motion where angular vectors are used to describe the position of the body (cf. [6–8]).

The standard approach of analytical mechanics can be modified so that both the material points and the rigid bodies are incorporated and, moreover, rigid bodies are considered systematically as continuous bodies with the corresponding number of degrees of freedom. It is the purpose of this note to derive the equations of motion from the characterization of the constraints. The system S under consideration is holonomic and consists of N material points and B rigid bodies. The approach is based on the use of angular vectors [9]. As a result, the motion of the system is shown to be governed by the usual form of Lagrange's equations.

To the author's knowledge, the literature shows one previous approach to Lagrange's equations, where the rigid body was viewed as a continuum [10]. An immediate comparison emphasizes the conceptual difficulty that arises if the angular vectors are not involved.

2. Angular vectors and characterization of the constraints

Let P be any point of a rigid body, G the center of gravity, and $\boldsymbol{\omega}$ the angular velocity. The velocities $\mathbf{v}_P$ and $\mathbf{v}_G$ of P and G are related by

$$\mathbf{v}_P = \mathbf{v}_G + \boldsymbol{\omega}\times(P-G).$$

The time-dependent velocity field $\mathbf{v}_P(t) = \mathbf{v}(P,t)$ is then characterized by the two time-dependent vectors $\mathbf{v}_G(t)$ and $\boldsymbol{\omega}(t)$. Two pairs $\mathbf{v}_G^{(1)}$, $\boldsymbol{\omega}^{(1)}$ and $\mathbf{v}_G^{(2)}$, $\boldsymbol{\omega}^{(2)}$ determine the corresponding fields

$$\mathbf{v}_P^{(1)} = \mathbf{v}_G^{(1)} + \boldsymbol{\omega}^{(1)}\times(P-G), \qquad \mathbf{v}_P^{(2)} = \mathbf{v}_G^{(2)} + \boldsymbol{\omega}^{(2)}\times(P-G).$$

A field of virtual velocity $\boldsymbol{\nu}$ is defined to be the difference of any pair of velocity fields. Analogously, a virtual angular velocity $\boldsymbol{\varpi}$ is defined to be the difference of any pair of angular velocities. Hence, letting $\boldsymbol{\nu} = \mathbf{v}^{(1)} - \mathbf{v}^{(2)}$, $\boldsymbol{\varpi} = \boldsymbol{\omega}^{(1)} - \boldsymbol{\omega}^{(2)}$ we have

$$\boldsymbol{\nu}_P = \boldsymbol{\nu}_G + \boldsymbol{\varpi}\times(P-G). \tag{2.1}$$

The vectors $\boldsymbol{\omega}$ and $\boldsymbol{\varpi}$ are now related to the generalized coordinates.

Let $\{\mathbf{e}_h\}$ be the unit vectors of a Cartesian set of axes fixed in the rigid body, $h = 1,2,3$. For greater generality we let

$$\mathbf{e}_h = \mathbf{e}_h(q,t),$$

where $q = q(t)$ is a set of generalized (or Lagrangian) coordinates for the body. By definition, the angular velocity is given by

$$\boldsymbol{\omega} = \frac{1}{2}\sum_h \mathbf{e}_h \times \dot{\mathbf{e}}_h = \frac{1}{2}\sum_h \mathbf{e}_h \times \frac{\partial \mathbf{e}_h}{\partial q_j}\dot{q}_j + \frac{1}{2}\sum_h \mathbf{e}_h \times \frac{\partial \mathbf{e}_h}{\partial t},$$

where a superposed dot denotes the (total) time derivative d/dt; the sum over repeated indices is understood. Define the *angular vectors* $\boldsymbol{\Omega}_j$, $\boldsymbol{\Omega}_t$ as

$$\boldsymbol{\Omega}_j = \frac{1}{2}\sum_h \mathbf{e}_h \times \frac{\partial \mathbf{e}_h}{\partial q_j}, \qquad \boldsymbol{\Omega}_t = \frac{1}{2}\sum_h \mathbf{e}_h \times \frac{\partial \mathbf{e}_h}{\partial t}.$$

We have

$$\boldsymbol{\omega} = \boldsymbol{\Omega}_j \dot{q}_j + \boldsymbol{\Omega}_t$$

whence

$$\boldsymbol{\Omega}_j = \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_j}. \tag{2.2}$$

Let $\{\dot{q}_j^{(1)}\}$, $\{\dot{q}_j^{(2)}\}$ be the sets of generalized velocities associated with $\mathbf{v}_P^{(1)}$, $\mathbf{v}_P^{(2)}$. Letting $\eta_j = \dot{q}_j^{(1)} - \dot{q}_j^{(2)}$ we have

$$\boldsymbol{\nu}_\alpha = \frac{\partial P_\alpha}{\partial q_j}\eta_j , \qquad \boldsymbol{\varpi} = \boldsymbol{\Omega}_j \eta_j . \tag{2.3}$$

For later use we need the expression of the time derivative $\dot{\boldsymbol{\Omega}}_j$. Letting

$$\Omega_{jp} = \boldsymbol{\Omega}_j \cdot \mathbf{e}_p$$

and

$$\mathbf{e}_{h,j} = \partial \mathbf{e}_h / \partial q_j , \qquad \mathbf{e}_{h,t} = \partial \mathbf{e}_h / \partial t,$$

we obtain

$$\Omega_{jp} = \frac{1}{2}\sum_h (\mathbf{e}_h \times \mathbf{e}_{h,j}) \cdot \mathbf{e}_p = \frac{1}{2}\sum_h (\mathbf{e}_p \times \mathbf{e}_h) \cdot \mathbf{e}_{h,j} = \frac{1}{2}\sum_{h,l} \varepsilon_{phl} \mathbf{e}_{h,j} \cdot \mathbf{e}_l$$

and

$$\mathbf{e}_{h,k}\dot{q}_k + \mathbf{e}_{h,t} = \boldsymbol{\Omega}_k \times \mathbf{e}_h \dot{q}_k + \boldsymbol{\Omega}_t \times \mathbf{e}_h = \boldsymbol{\omega} \times \mathbf{e}_h .$$

Substitution and some rearrangement yield

$$\begin{aligned}
\dot{\Omega}_{jp} &= \frac{1}{2}\varepsilon_{phl}(\mathbf{e}_{h,jk}\dot{q}_k + \mathbf{e}_{h,jt}) \cdot \mathbf{e}_l + \frac{1}{2}\varepsilon_{phl}\mathbf{e}_{h,j} \cdot \boldsymbol{\omega} \times \mathbf{e}_l \\
&= \frac{\partial}{\partial q_j}\frac{1}{2}\varepsilon_{phl}\, \boldsymbol{\omega} \times \mathbf{e}_h \cdot \mathbf{e}_l - \frac{1}{2}\varepsilon_{phl}\, \boldsymbol{\omega} \times \mathbf{e}_h \cdot \mathbf{e}_{l,j} + \frac{1}{2}\varepsilon_{phl}\mathbf{e}_{h,j} \cdot \boldsymbol{\omega} \times \mathbf{e}_l \\
&= \frac{\partial \omega_p}{\partial q_j} - \frac{1}{2}(\mathbf{e}_p \times \mathbf{e}_h) \cdot \mathbf{e}_l (\boldsymbol{\omega} \times \mathbf{e}_h) \cdot (\boldsymbol{\Omega}_j \times \mathbf{e}_l) + \frac{1}{2}(\mathbf{e}_p \times \mathbf{e}_h) \cdot \mathbf{e}_l (\boldsymbol{\Omega}_j \times \mathbf{e}_h) \cdot (\boldsymbol{\omega} \times \mathbf{e}_l) \\
&= \frac{\partial \omega_p}{\partial q_j} - \frac{1}{2}(\boldsymbol{\omega} \times \mathbf{e}_h) \cdot [\boldsymbol{\Omega}_j \times (\mathbf{e}_p \times \mathbf{e}_h)] + \frac{1}{2}(\boldsymbol{\Omega}_j \times \mathbf{e}_h) \cdot [\boldsymbol{\omega} \times (\mathbf{e}_p \times \mathbf{e}_h)] \\
&= \frac{\partial \omega_p}{\partial q_j} - \frac{1}{2}(\boldsymbol{\omega} \times \mathbf{e}_h) \cdot \Omega_{jh}\mathbf{e}_p + \frac{1}{2}(\boldsymbol{\Omega}_j \times \mathbf{e}_h) \cdot \omega_h \mathbf{e}_p .
\end{aligned}$$

Accordingly we have

$$\dot{\Omega}_{jp} = \frac{\partial \omega_p}{\partial q_j} + \boldsymbol{\Omega}_j \times \boldsymbol{\omega} \cdot \mathbf{e}_p .$$

Hence the time differentiation of $\boldsymbol{\Omega}_j = \Omega_{jp}\mathbf{e}_p$ yields

$$\dot{\boldsymbol{\Omega}}_j = \frac{\partial \omega_p}{\partial q_j}\mathbf{e}_p . \tag{2.4}$$

Let $\boldsymbol{\phi}_i$ be the force of constraint at any point i of S, namely, at any material point or at any point of the rigid bodies. Denote by $\boldsymbol{\nu}_i$ the virtual velocity of the point i and let A be the set of labels for the constrained points. Hence we characterize the constraints by assuming that

$$\sum_{i \in A} \boldsymbol{\phi}_i \cdot \boldsymbol{\nu}_i = 0 \tag{2.5}$$

for every set of virtual velocities $\{\boldsymbol{\nu}_i\}$ compatible with the constraints.

For formal convenience we separate the values of i pertaining to the material points from those pertaining to rigid bodies; we label by $\alpha = 1, .., N$ the material points, and by the pair $b\,\beta_b$, $b = 1, ..., B$, $\beta_b = 1, ..., N_b$, we denote the constrained points of the B rigid bodies. Denote by $\mathbf{R}_b^r$ and $\mathbf{M}_{G_b}^r$ the total constraint force and the total constraint torque acting on the body b, i.e.

$$\mathbf{R}_b^r = \sum_{b,\beta_b} \boldsymbol{\phi}_{b\beta_b}, \qquad \mathbf{M}_{G_b}^r = \sum_{b,\beta_b} (P_{b\beta_b} - G_b) \times \boldsymbol{\phi}_{b\beta_b}.$$

The total applied force $\mathbf{R}_b^a$ and the total applied torque $\mathbf{M}_{G_b}^a$ are defined analogously by replacing the constraint forces with the applied forces. By means of (2.1) we have

$$\sum_{b,\beta_b} \boldsymbol{\phi}_{b\beta_b} \cdot \boldsymbol{\nu}_{b\beta_b} = \sum_{b,\beta_b} \boldsymbol{\phi}_{b\beta_b} \cdot \boldsymbol{\nu}_{G_b} + \sum_{b,\beta_b} \boldsymbol{\phi}_{b\beta_b} \cdot \boldsymbol{\varpi}_b \times (P_{b\beta_b} - G_b) = \mathbf{R}_b^r \cdot \boldsymbol{\nu}_{G_b} + \mathbf{M}_{G_b}^r \cdot \boldsymbol{\varpi}_b\,.$$

For any body b, the balance of linear momentum, $\mathbf{P}_b$, and of angular momentum, $\mathbf{L}_b$, is written as

$$\dot{\mathbf{P}}_b = \mathbf{R}_b^a + \mathbf{R}_b^r, \qquad \dot{\mathbf{L}}_b = \mathbf{M}_b^a + \mathbf{M}_b^r\,.$$

The equation of motion for any material point α is given in the form

$$\mu_\alpha \mathbf{a}_\alpha = \mathbf{F}_\alpha + \boldsymbol{\phi}_\alpha\,,$$

where μ_α is the mass, $\mathbf{a}_\alpha$ – the acceleration, $\mathbf{F}_\alpha$ – the applied force. Substitution enables us to write the condition (2.5) in the form

$$\sum_\alpha (\mu_\alpha \mathbf{a}_\alpha - \mathbf{F}_\alpha) \cdot \boldsymbol{\nu}_\alpha + \sum_b (\dot{\mathbf{P}}_b - \mathbf{R}_b^a) \cdot \boldsymbol{\nu}_{G_b} + \sum_b (\dot{\mathbf{L}}_b - \mathbf{M}_{G_b}^a) \cdot \boldsymbol{\varpi}_b = 0. \tag{2.6}$$

3. Lagrange's equations

Let now $q = (q_1, ..., q_n)$ be the set of generalized coordinates for the whole holonomic system. Substitution of (2.3) into (2.6) yields

$$\sum_\alpha (\mu_\alpha \mathbf{a}_\alpha - \mathbf{F}_\alpha) \cdot \frac{\partial P_\alpha}{\partial q_j} \eta_j + \sum_b (\dot{\mathbf{P}}_b - \mathbf{R}_b^a) \cdot \frac{\partial G_b}{\partial q_j} \eta_j + \sum_b (\dot{\mathbf{L}}_b - \mathbf{M}_{G_b}^a) \cdot \boldsymbol{\Omega}_j^b \eta_j = 0.$$

The arbitrariness of the n-tuple $\eta_1, \ldots, \eta_n$ implies that

$$\tau_j - Q_j = 0, \qquad j = 1, \ldots, n, \tag{3.1}$$

where

$$\tau_j = \sum_\alpha \mu_\alpha \mathbf{a}_\alpha \cdot \frac{\partial P_\alpha}{\partial q_j} + \sum_b \dot{\mathbf{P}}_b \cdot \frac{\partial G_b}{\partial q_j} + \sum_b \dot{\mathbf{L}}_b \cdot \mathbf{\Omega}_j^b , \tag{3.2}$$

$$Q_j = \sum_\alpha \mathbf{F}_\alpha \cdot \frac{\partial P_\alpha}{\partial q_j} + \sum_b \mathbf{R}_b \cdot \frac{\partial G_b}{\partial q_j} + \sum_b \mathbf{M}_b \cdot \mathbf{\Omega}_j^b . \tag{3.3}$$

It is natural to view τ_j (Q_j) as the j-th component of the generalized inertia force (generalized force).

To find a convenient form of τ_j we observe that, for any material point P of mass μ, by means of the known identities, we have

$$\mu \mathbf{a} \cdot \frac{\partial P}{\partial q_h} = \mu \frac{d\mathbf{v}}{dt} \cdot \frac{\partial P}{\partial q_h} = \frac{d}{dt}\left(\mu \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial q_h}\right) - \mu \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial q_h} = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_h}\frac{1}{2}\mu \mathbf{v}^2 - \frac{\partial}{\partial q_h}\frac{1}{2}\mu \mathbf{v}^2 .$$

In the same manner, since $\mathbf{P} = m\mathbf{v}_G$, we have

$$\dot{\mathbf{P}} \cdot \frac{\partial G}{\partial q_h} = m\mathbf{a}_G \cdot \frac{\partial G}{\partial q_h} = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_h}\frac{1}{2}m\mathbf{v}_G^2 - \frac{\partial}{\partial q_h}\frac{1}{2}m\mathbf{v}_G^2 .$$

Let $\mathbf{I}$ be the inertia tensor of a body, relative to the corresponding center of gravity. Hence $\mathbf{L} = \mathbf{I}\,\boldsymbol{\omega}$. We now use (2.2) and (2.4) to obtain

$$\begin{aligned} \dot{\mathbf{L}} \cdot \mathbf{\Omega}_h &= \frac{d(\mathbf{I}\,\boldsymbol{\omega})}{dt} \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_h} = \frac{d}{dt}\left[(\mathbf{I}\,\boldsymbol{\omega}) \cdot \frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_h}\right] - (\mathbf{I}\,\boldsymbol{\omega}) \cdot \frac{d}{dt}\frac{\partial \boldsymbol{\omega}}{\partial \dot{q}_h} \\ &= \frac{d}{dt}\left[\frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}\,\boldsymbol{\omega}\right] - (\mathbf{I}\,\boldsymbol{\omega}) \cdot \frac{d\mathbf{\Omega}_h}{dt} = \frac{d}{dt}\left[\frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}\,\boldsymbol{\omega}\right] - \frac{\partial}{\partial q_h}\left[\frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}\,\boldsymbol{\omega}\right] . \end{aligned}$$

The expression of the kinetic energy of the system, viz.

$$T = \sum_\alpha \frac{1}{2}\mu_\alpha \mathbf{v}_\alpha^2 + \sum_b \frac{1}{2}m_b \mathbf{v}_{G_b}^2 + \sum_b \frac{1}{2}\boldsymbol{\omega}_b \cdot \mathbf{I}_b\,\boldsymbol{\omega}_b$$

allows τ_j to be written as

$$\tau_j = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} . \tag{3.4}$$

Accordingly, the conditions (3.1) become

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j , \qquad j = 1, \ldots, n, \tag{3.5}$$

namely Lagrange's equations of the second kind.

4. Comparison with a previous approach

The view that the rigid body is a continuum rather than a set of material points, is expressed in [10]. A comparison is then necessary to assess the conceptual improvement in the present approach.

The approach in [10] starts from the D'Alembert principle for a single body which, in the notation of this note, may be written in the form

$$\mathbf{R}^a \cdot \boldsymbol{\nu}_G + \mathbf{M}_G^a \cdot \boldsymbol{\varpi} - \int_{\mathcal{R}} \varrho \boldsymbol{\nu} \cdot \mathbf{a}\, dv = 0, \tag{4.1}$$

where ϱ is the mass density; the integral over the region $\mathcal{R}$, occupied by the body, is regarded as the power of inertia forces. The assumption (2.5) seems to be more convincing. Yet it follows easily that Eqs. (2.6) and (4.1) are equivalent when a single body is involved, since the observation that

$$\frac{\partial P}{\partial q_j} \cdot \mathbf{a} = \frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} \frac{1}{2} \mathbf{v}^2 - \frac{\partial}{\partial q_j} \frac{1}{2} \mathbf{v}^2$$

and substitution of $\boldsymbol{\nu} = (\partial P / \partial q_j) \eta_j$ yields

$$\int_{\mathcal{R}} \varrho \boldsymbol{\nu} \cdot \mathbf{a}\, dv = \tau_j \eta_j,$$

where τ_j has the form (3.4) in terms of the kinetic energy. Here, the expression (3.2) also leads to (3.4).

The crucial point consists in expressing the power $\mathbf{R}^a \cdot \boldsymbol{\nu}_G + \mathbf{M}_G^a \cdot \boldsymbol{\varpi}$ in terms of the generalized coordinates. First, the "primitive" coordinates λ_s are considered and the power $\mathbf{R}^a \cdot \boldsymbol{\nu}_G + \mathbf{M}_G^a \cdot \boldsymbol{\varpi}$ is written as a linear form in the virtual time derivatives of λ_s; the corresponding coefficients are denoted by Λ_s. Hence, for holonomic systems $\lambda_s = \lambda_s(q, t)$ and it follows that

$$\mathbf{R}^a \cdot \boldsymbol{\nu}_G + \mathbf{M}_G^a \cdot \boldsymbol{\varpi} = \sum_j Q_j \eta_j ,$$

where

$$Q_j = \sum_s \Lambda_s \frac{\partial \lambda_s}{\partial q_j} .$$

Accordingly, the arbitrariness of the set $\{\eta_j\}$ implies that Lagrange's equations (3.5) hold. Unfortunately, without the angular vectors, the quantities Q_j are not defined per se. Indeed, Q_j can be viewed as the coefficient of η_j in the expression of the virtual power. The use of the angular vectors, instead, allows us to write Q_j in the form (3.3). The occurrence of the angular vectors $\boldsymbol{\Omega}_j$ makes it apparent why we are unable to write the expression for Q_j if the angular vectors are not considered.

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