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An integrity basis for plane elasticity tensors

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An isotropic functional basis of 5 polynomials is shown to be also an integrity basis for the space of plane elasticity tensors. A decomposition of each element in this space into a direct sum of "harmonic" tensors is used to compute or estimate the distance between an arbitrary elasticity tensor and the three non-trivial symmetry classes, to allow for the determination of the material symmetry when the elastic coefficients are known only to within a given approximation.

1. Introduction

Let $\mathbb{E}$la be the space of two-dimensional elasticity tensors, which describe the constitutive equations for plane linarly elastic bodies, and let $O(2)$ be the group of orthogonal transformations on the two-dimensional Euclidean space. A function ψ defined on $\mathbb{E}$la is isotropic, or, equivalently, an $O(2)$-invariant, if $\psi(\mathbb{C}) = \psi(\mathbf{Q} * \mathbb{C})$ for all $\mathbb{C} \in \mathbb{E}$la and $\mathbf{Q} \in O(2)$, where, as we shall see more precisely later on, the asterisk denotes an action of $O(2)$ on $\mathbb{E}$la. A finite collection $\mathcal{B}$ of such invariants is a *functional basis* if each other invariant is a function of the elements of $\mathcal{B}$. If these elements are polynomials, and *all* isotropic polynomials are also expressible as *polynomial* functions of them, this collection is an *integrity basis* (or *Hilbert basis*) for the action of $O(2)$. A similar set of definitions covers the case in which the action of the group of proper rotations $SO(2)$ is considered, and the corresponding invariants are said to be *hemitropic*.

It is a classical result that every integrity basis is also a functional basis. The proof, which is far from trivial, is based on a lemma which shows that "polynomials separate the orbits". More explicitly, this statement means that whenever two elements do not lie on the same orbit, there is at least one invariant polynomial which takes different values on them. For a modern proof of this important result we refer to the paper by Wineman and Pipkin [17, Sec. 6]. On the other hand, it is not difficult to provide counterexamples showing that, in general, a functional basis is *not* an integrity basis.

In Sec. 4 we construct a functional basis of 5 polynomials I_n for the isotropic invariants on $\mathbb{E}$la. Similar results were recently obtained by Zheng [18] and by Blinowski, Ostrowska-Maciejewska and Rychlewski [3]. Indeed, the technique used in the present paper is very similar to the discussion contained there, and the basis found is essentially equivalent. However, in addition, here it is shown that the set $\{I_n\}$ is also an *integrity* basis for the action of $O(2)$ on $\mathbb{E}$la, which is the main goal of this paper.

For the sake of clarity and self-completeness we choose to offer a detailed presentation of some mathematical preliminaries, even if this can be seen as an alternative derivation of similar results contained in [3].

The key mathematical step is the decomposition of an elasticity tensor into a quadruplet formed by: two scalars λ and μ, a second-order tensor $\mathbf{H}$ and a fourth-order tensor $\mathbb{K}$, both symmetric and traceless. A description of this technique, when applied for other goals to the three-dimensional case, is contained in some papers by BACKUS [1], BAERHEIM [2], COWIN [6], FORTE and VIANELLO [8] and, moreover, in a classical treatise by SCHOUTEN [15]. However, except for reference [3], we are not aware of any other presentation of a similar decomposition for plane elasticity.

The insight coming from this approach is used to represent the action of $O(2)$ on $\mathbb{E}$la through a pair of orthogonal transformations on the two-dimensional spaces to which $\mathbf{H}$ and $\mathbb{K}$ belong. This point of view allows for a natural construction of a functional basis, thus providing a confirmation, with a slightly different approach, of a similar conclusion reached in [3]. Moreover, the proof that the set $\{I_n\}$ is an integrity basis is strongly dependent on the isomorphism between the action of $O(2)$ on $\mathbb{E}$la and the action of the same group on products of complex planes, which can be easily deduced only in view of the previous considerations.

Constitutive equations for two-dimensional linearly elastic bodies are divided into *four* symmetry classes by a relation stating that two elasticity tensors are equivalent when their symmetry groups are conjugate in $O(2)$. Once a functional basis has been established, it is not difficult, through its geometric interpretation, to obtain a complete characterization of the symmetry classes as *zero-sets* of appropriate collections of invariant polynomials. As noticed in [3], this is a useful result in itself, since it allows for an easy determination of the symmetry class of an elasticity tensor. Moreover, it shows clearly that the collection of tensors with non-minimal symmetry group is a set of *measure zero*.

An interesting problem originates from the experimental errors contained in the numerical data describing elasticity tensors, as it was recently noted also by FRANÇOIS, BERTHAUD and GEYMONAT [5]. In view of the above considerations, the question of symmetry class has, with "probability one", the same answer: The material has no special symmetry. What is really important is a comparison between the precision of our experimental apparatus and the distance between $\mathbb{C}$ and the closest tensor of a given symmetry. If this distance is smaller than a certain value, we may reasonably say that, within the approximation allowed, the material described by $\mathbb{C}$ does belong to that symmetry class. In view of our geometric approach, we propose some formulas, ready for applications, which allow for a quick evaluation of the relevant distances. We believe some of the results to be new.

2. Symmetry groups and symmetry classes

We use small (resp., capital) boldface letters for vectors (resp., second-order tensors) of $\mathcal{V}$, the translation space of a *two*-dimensional Euclidean space $\mathcal{E}$.

Scalars are denoted by Greek letters and fourth-order tensors are written with a blackboard bold font, such as $\mathbb{C}$. A superscript T is used for the transpose and the space of symmetric tensors is called Sym. We use subscripts for the components of vectors or tensors with respect to a fixed orthonormal basis $\mathbf{e}_i$ ($i = 1, 2$). Thus, for instance, $\mathbf{v} = v_i\mathbf{e}_i$ and $\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, where the sum over repeated indexes is understood and the symbol $\otimes$ stands for the tensor product. The subspace of Sym formed by all traceless tensors (such that $A_{ii} = 0$) is Dev, while the space of all fourth-order tensors $\mathbb{H}$ which are symmetric and traceless is $\mathbb{D}$ev. More precisely, $\mathbb{H} \in \mathbb{D}$ev if H_{ijkl} is unchanged by any permutation of the indexes and, moreover, $H_{iikl} = 0$. The group of orthogonal transformations of $\mathcal{V}$ is $O(2)$, where the unit element is denoted by $\mathbf{I}$, and the subgroup of *rotations*, formed by all $\mathbf{Q} \in O(2)$ with determinant equal to one, is $SO(2)$. We write $\mathbf{Q}(\theta)$ for the rotation such that

$$\mathbf{Q}\,\mathbf{e}_1 = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2\,, \qquad \mathbf{Q}\,\mathbf{e}_2 = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2\,, \tag{2.1}$$

and we denote by $\tilde{\mathbf{Q}}$ the reflection with respect to the $\mathbf{e}_1$ direction: $\tilde{\mathbf{Q}}\mathbf{e}_1 = \mathbf{e}_1$, $\tilde{\mathbf{Q}}\mathbf{e}_2 = -\mathbf{e}_2$. Obviously, $O(2)$ is generated by $SO(2)$ and $\tilde{\mathbf{Q}}$.

For an extensive introduction to linear elasticity we refer to classical conventions (see, e.g., GURTIN [10]). Here, we simply recall that an *elasticity tensor* $\mathbb{C}$ is a symmetric linear map of Sym, which gives the stress tensor $\mathbf{T}$ as a function of the infinitesimal strain $\mathbf{E}$: $\mathbf{T} = \mathbb{C}[\mathbf{E}]$. Thus, the components of $\mathbb{C}$ satisfy the following index symmetries:

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}\,.$$

The *symmetry group* $g(\mathbb{C})$ is the collection of all orthogonal transformations $\mathbf{Q}$ such that

$$\mathbb{C}[\mathbf{Q}\,\mathbf{E}\,\mathbf{Q}^T] = \mathbf{Q}\,\mathbb{C}[\mathbf{E}]\mathbf{Q}^T, \qquad \forall\,\mathbf{E} \in \mathrm{Sym}.$$

It is convenient to define an *action* of $O(2)$ on $\mathbb{E}$la, the 6-dimensional space of (plane) elasticity tensors. For each $\mathbf{Q} \in O(2)$ and each $\mathbb{C} \in \mathbb{E}$la, let $\mathbf{Q} * \mathbb{C}$ be defined by

$$(\mathbf{Q} * \mathbb{C})_{pqrs} := Q_{pi}Q_{qj}Q_{rk}Q_{sl}C_{ijkl}\,.$$

Thus, the symmetry group is

$$g(\mathbb{C}) := \{\mathbf{Q} \in O(2) \mid \mathbf{Q} * \mathbb{C} = \mathbb{C}\}.$$

A straightforward consequence of this definition is that $g(\mathbf{Q} * \mathbb{C}) = \mathbf{Q}g(\mathbb{C})\mathbf{Q}^T$. Moreover, by continuity, $g(\mathbb{C})$ is *closed*. Hence, as a consequence of classical results (see, e.g., the book by GOLUBITSKY, STEWART and SCHAEFFER [9, Ch. XIII, Th. 6.1]), we know that $g(\mathbb{C})$ is *conjugate* to exactly one of the elements in the following collection:

$$\Sigma := \{\mathbf{I}, Z_n, D_n, SO(2), O(2)\} \qquad (n \geq 2),$$

where Z_n and D_n denote, respectively, the *cyclic* and *dihedral* groups of order n (for an extensive coverage of this topic see also MILLER'S book [11]).

The space $\mathbb{E}$la is divided into *symmetry classes* by a relation defining $\mathbb{C}_1$ and $\mathbb{C}_2$ as equivalent when $g(\mathbb{C}_1)$ is conjugate to $g(\mathbb{C}_2)$ in $O(2)$. Let $\mathbb{E}\mathrm{la}(G)$ be the collection of all elasticity tensors such that their symmetry groups are conjugate to $G \in \Sigma$. Then, $\mathbb{C}_1$ and $\mathbb{C}_2$ have conjugate symmetry groups if and only if they belong to the same $\mathbb{E}\mathrm{la}(G)$, and the problem of finding the number and type of symmetry classes is equivalent to the problem of determining which $\mathbb{E}\mathrm{la}(G)$ are empty and which are not. The answer is known (see, e.g., RYCHLEWSKI [14, Sec. 8]), even if some contradictory statements can still be found in the literature (cf., e.g., ZHENG [19, Sec. 3.3], where the Author seems to suggest otherwise). However, the discussion of Sec. 3 has the following statement as a direct consequence: *There are exactly four non-empty sets* $\mathbb{E}\mathrm{la}(G)$, *for* $G = Z_2, D_2, D_4, O(2)$.

We use the following terminology to classify the symmetries, depending on which element of Σ the group $g(\mathbb{C})$ is conjugate to: *anisotropic* for Z_2, *orthotropic* for D_2, *tetragonal* for D_4 and *isotropic* for $O(2)$. Notice that only $\mathbb{E}\mathrm{la}(O(2))$ is a linear subspace of $\mathbb{E}$la.

As mentioned before, it is almost impossible that an elasticity tensor obtained from experimental data might have any special symmetry at all. As we recall in Sec. 5, the set of tensors with symmetry D_2, D_4 or $O(2)$ has the structure of an *algebraic manifold* of measure zero, formed by the null-set of a finite number of polynomials. Thus, anisotropic elasticity tensors are *dense* in $\mathbb{E}$la. From this point of view, the question of interest becomes a different one: We would like to know *how close* a given $\mathbb{C}$ is to classes of non-minimal symmetry.

The final section contains a computation of the distance between $\mathbb{C}$ and $\mathbb{E}\mathrm{la}(G)$, for $G = D_2$, D_4 or $O(2)$, which is defined to be the infimum of the distance between $\mathbb{C}$ and $\mathbb{C}^*$, as the latter varies over $\mathbb{E}\mathrm{la}(G)$ (an obvious Euclidean norm and a corresponding distance are defined in the space of elasticity tensors).

3. A decomposition for the space of elasticity tensors

A finite-dimensional vector space is decomposed into a direct sum of subspaces which are irreducible under the action of a compact group (see, e.g., [9] or [11]). In our particular context it is possible to show that the decomposition of $\mathbb{E}$la is described by an $SO(2)$-invariant isomorphism which maps $\mathbb{C}$ into a quadruplet $(\lambda, \mu, \mathbf{H}, \mathbb{K})$, where λ and μ are scalars, while $\mathbf{H}$ and $\mathbb{K}$ belong to Dev and $\mathbb{D}$ev, respectively. More explicitly, for a given $\mathbb{C} \in \mathbb{E}$la:

$$\lambda = (3C_{ppqq} - 2C_{pqpq})/8, \qquad \mu = (2C_{pqpq} - C_{ppqq})/8,$$

$$H_{ik} = [2C_{ipkp} - C_{pqpq}\delta_{ik}]/12,$$

$$K_{ijkl} = C_{ijkl} - [\delta_{ij}C_{kplp} + \delta_{kl}C_{ipjp} + \delta_{ik}C_{lpjp} + \delta_{lj}C_{ipkp} + \delta_{il}C_{jpkp} + \delta_{jk}C_{iplp}]/6 \\ + [C_{pqpq}(5\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{lj} - \delta_{il}\delta_{jk})]/12 - [C_{ppqq}(3\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{lj} - \delta_{il}\delta_{jk})]/8,$$

(δ_{ij} is Kronecker's delta). *Vice versa*, the elasticity tensor $\mathbb{C}$ corresponding to $(\lambda, \mu, \mathbf{H}, \mathbb{K})$ is:

$$C_{ijkl} = K_{ijkl} + \delta_{ij}H_{kl} + H_{ij}\delta_{kl} + \delta_{ik}H_{lj} + H_{ik}\delta_{lj} + \delta_{il}H_{jk} + H_{il}\delta_{jk} \\ + \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk}).$$

The validity of this decomposition can be directly checked through substitutions followed by lengthy computations. Moreover, it is not difficult to see that this is a variation, and an indirect confirmation, of a quite similar result presented by BLINOWSKI *et al.* [3]. However, it is perhaps useful to spend a few words on a short description of the rationale behind our derivation, for which we followed the scheme adopted by BAERHEIM [2] in three dimensions. The first step consists in writing C_{ijkl} as the sum of a completely symmetric part S_{ijkl} and an "asymmetric" part A_{ijkl}:

$$S_{ijkl} := (C_{ijkl} + C_{iklj} + C_{iljk})/3, \qquad A_{ijkl} := (2C_{ijkl} - C_{iklj} - C_{iljk})/3.$$

This corresponds to a decomposition of $\mathbb{E}$la into a direct sum of two orthogonal subspaces. Since the dimension of $\mathbb{E}$la is 6 and the space of completely symmetric fourth-order tensors has dimension 5, it follows that A_{ijkl} is a scalar multiple of a fixed asymmetric tensor, say:

$$A_{ijkl} = \alpha(2\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{lj} - \delta_{il}\delta_{jk}).$$

Next, we use the fact that for each S_{ijkl} there is a unique pair of tensors $\mathbf{A} \in$ Sym and $\mathbb{K} \in \mathbb{D}$ev such that

$$S_{ijkl} = K_{ijkl} + \delta_{(ij}A_{kl)},$$

where the parenthesis denotes full symmetrization with respect to the enclosed set of indexes or, more precisely,

$$\delta_{(ij}A_{kl)} = \delta_{ij}A_{kl} + A_{ij}\delta_{kl} + \delta_{ik}A_{lj} + A_{ik}\delta_{lj} + \delta_{il}A_{jk} + A_{il}\delta_{jk}\,.$$

This property is a reformulation of a well-known result on polynomials, which naturally correspond to symmetric tensors, as discussed in [9, Ch. XIII, Sec. 7, Prop. 7.1].

Finally, we use the decomposition of each element of Sym into the sum of a "spherical" part (i.e., a multiple of $\mathbf{I}$) and an element $\mathbf{H}$ of Dev, so that we may write

$$A_{ij} = H_{ij} + \beta\delta_{ij}/2.$$

Trivial substitutions followed by an appropriate change of names yield the decomposition, which, with obvious meaning, is written as

$$\mathbb{C} = (\lambda, \mu, \mathbf{H}, \mathbb{K}). \tag{3.1}$$

An action of $O(2)$ on Dev is defined by

$$\mathbf{Q} * \mathbf{A} := \mathbf{Q}\mathbf{A}\mathbf{Q}^T, \quad \forall\, \mathbf{Q} \in O(2), \quad \forall\, \mathbf{A} \in \mathrm{Dev}.$$

It is a matter of simple computations to check that

$$\mathbf{Q} * \mathbb{C} = (\lambda, \mu, \mathbf{Q} * \mathbf{H}, \mathbf{Q} * \mathbb{K}), \quad \forall\, \mathbf{Q} \in O(2),$$

and, consequently, $g(\mathbb{C}) = g(\mathbf{H}) \bigcap g(\mathbb{K})$, where $g(\mathbf{H})$ is defined in the natural way. It is now clear why the action of $O(2)$ on Dev and $\mathbb{D}$ev is of great interest, and the importance of the geometric description of this action which is obtained in the final part of this section.

It is convenient to define an appropriate *orthonormal* basis in each of these spaces. For Dev we use:

$$\mathbf{E}_1 := \frac{\sqrt{2}}{2}(\mathbf{e}_1 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2), \qquad \mathbf{E}_2 := \frac{\sqrt{2}}{2}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1).$$

The basis for $\mathbb{D}$ev is more complex:

$$\begin{aligned}\mathbb{E}_1 := \frac{\sqrt{8}}{8}(&\mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2\\ &- \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1),\\ \mathbb{E}_2 := \frac{\sqrt{8}}{8}(&\mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1 \otimes \mathbf{e}_1\\ &- \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 - \mathbf{e}_1 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2 \otimes \mathbf{e}_2).\end{aligned}$$

In view of (2.1), through direct substitution it is not difficult to show that

$$\mathbf{Q}(\theta) * \mathbf{E}_1 = \cos(2\theta)\mathbf{E}_1 + \sin(2\theta)\mathbf{E}_2\,, \qquad \mathbf{Q}(\theta) * \mathbf{E}_2 = -\sin(2\theta)\mathbf{E}_1 + \cos(2\theta)\mathbf{E}_2\,,$$

while more lengthy computations are needed to prove that

$$\mathbf{Q}(\theta) * \mathbb{E}_1 = \cos(4\theta)\mathbb{E}_1 + \sin(4\theta)\mathbb{E}_2\,, \qquad \mathbf{Q}(\theta) * \mathbb{E}_2 = -\sin(4\theta)\mathbb{E}_1 + \cos(4\theta)\mathbb{E}_2\,.$$

Moreover, since $\tilde{\mathbf{Q}}\mathbf{e}_1 = \mathbf{e}_1$ and $\tilde{\mathbf{Q}}\mathbf{e}_2 = -\mathbf{e}_2$,

$$\tilde{\mathbf{Q}} * \mathbf{E}_1 = \mathbf{E}_1\,, \qquad \tilde{\mathbf{Q}} * \mathbf{E}_2 = -\mathbf{E}_2\,, \qquad \tilde{\mathbf{Q}} * \mathbb{E}_1 = \mathbb{E}_1\,, \qquad \tilde{\mathbf{Q}} * \mathbb{E}_2 = -\mathbb{E}_2\,.$$

In conclusion, each $\mathbf{Q}(\theta)$ acts on Dev as a rotation of 2θ and on $\mathbb{D}$ev as a rotation of 4θ, while $\tilde{\mathbf{Q}}$ is simply a reflection with respect to the "horizontal" axes spanned by $\mathbf{E}_1$ and $\mathbb{E}_1$. The geometric insight provided by this point of view makes easy a proof of the fact that there are only symmetry classes corresponding to groups Z_2, D_2, D_4, and $O(2)$.

4. An integrity basis

The Euclidean structure of Dev and $\mathbb{D}$ev is obtained by introducing the inner products $\mathbf{A}\cdot\mathbf{B} = A_{ij}B_{ij}$ and $\mathbb{H}\cdot\mathbb{K} = H_{ijkl}K_{ijkl}$. We use the symbol $|\cdot|$ to denote the norm in both spaces. For a given $\mathbb{C} = (\lambda, \mu, \mathbf{H}, \mathbb{K})$, let α be the angle between $\mathbf{H}$ and $\mathbf{E}_1$, and let β be the angle between $\mathbb{K}$ and $\mathbb{E}_1$. Furthermore, we need the following definitions:

$$\begin{aligned} H_1 &:= |\mathbf{H}|\cos\alpha = \mathbf{H}\cdot\mathbf{E}_1\,, \qquad & H_2 &:= |\mathbf{H}|\sin\alpha = \mathbf{H}\cdot\mathbf{E}_2\,, \\ K_1 &:= |\mathbb{K}|\cos\beta = \mathbb{K}\cdot\mathbb{E}_1\,, \qquad & K_2 &:= |\mathbb{K}|\sin\beta = \mathbb{K}\cdot\mathbb{E}_2\,. \end{aligned} \tag{4.1}$$

The geometric view of the action of $O(2)$ on Dev and $\mathbb{D}$ev makes the choice of *four* independent polynomial invariants quite obvious:

$$I_1 = \lambda, \qquad I_2 := \mu, \qquad I_3 := |\mathbf{H}|^2, \qquad I_4 := |\mathbb{K}|^2.$$

Thus, we only need to find a fifth invariant, and, to this end, we consider the angle $\gamma := 2\alpha - \beta$. Since the action of $\mathbf{Q}(\theta)$ maps α onto $\alpha + 2\theta$ and β onto $\beta + 4\theta$, it follows that γ is left fixed. However, it is also straightforward to see that, under $\widetilde{\mathbf{Q}}$, γ is mapped onto $-\gamma$. Thus, the conclusion is that this angle is an $SO(2)$-invariant, but *not* an $O(2)$-invariant. A natural choice for the fifth isotropic invariant $\mathcal{I}$ is the cosine of γ:

$$\mathcal{I} := \cos\gamma = \cos(2\alpha - \beta).$$

This function is not a polynomial and thus we expand it as

$$\mathcal{I} = (\cos^2\alpha - \sin^2\alpha)\cos\beta + 2\sin\alpha\cos\alpha\sin\beta$$

and use definitions (4.1) to obtain the fifth polynomial isotropic invariant:

$$I_5 := |\mathbf{H}|^2|\mathbb{K}|\,\mathcal{I} = (H_1^2 - H_2^2)K_1 + 2H_1H_2K_2\,.$$

The steps followed for the construction of the collection $\{I_n\}$ show that a necessary and sufficient condition for $\mathbb{C}_1$ and $\mathbb{C}_2$ to be on the same orbit is that $I_n(\mathbb{C}_1) = I_n(\mathbb{C}_2)$ $(1 \le n \le 5)$. It is a well-known result that this condition is necessary and sufficient for $\{I_n\}$ to be a *functional basis* (see, e.g., WEYL [16], WINEMAN and PIPKIN [17, Sec. 4, p.190]).

As an additional remark, we notice that if the $SO(2)$-invariant polynomial

$$I_6 := |\mathbf{H}|^2|\mathbb{K}|\sin\gamma = 2H_1H_2K_1 - (H_1^2 - H_2^2)K_2$$

is added to the previous list, we obtain a functional basis for $SO(2)$-invariant functions on $\mathbb{E}$la. However, in this case, there is a *relation* (or *syzygy*) among the elements of the collection $\{I_m\}$ $(1 \leq m \leq 6)$: $I_5^2 + I_6^2 = I_3^2 I_4$. This is obviously due to the trigonometric identity between $\sin\gamma$ and $\cos\gamma$.

Our aim is now to prove that the collection of invariants $\{I_n\}$ is indeed an integrity basis, and not only a functional basis.

TEOREM 1. *For each $O(2)$-invariant real-valued polynomial p on* $\mathbb{E}$la*, there is a polynomial π in 5 variables such that*

$$p(\mathbb{C}) = \pi(I_1(\mathbb{C}), I_2(\mathbb{C}), I_3(\mathbb{C}), I_4(\mathbb{C}), I_5(\mathbb{C})), \quad \forall\, \mathbb{C} \in \mathbb{E}\text{la}.$$

A convenient technique of proof is based on the idea of looking at the action of $O(2)$ on Dev and $\mathbb{D}$ev as an action on the complex plane $\mathcal{C}$, and then to apply straightforward considerations from the complex number theory. This method was applied by PIERCE [12] to a similar problem.

More precisely, the product between Dev and $\mathbb{D}$ev is seen as $\mathcal{C}^2$. Then, the action of a rotation $\mathbf{Q}(\theta) \in SO(2)$ on this space is defined through the unit complex number $\exp(i\theta)$ as

$$\mathbf{Q} * (z_1, z_2) := (\exp(i2\theta)z_1, \exp(i4\theta)z_2), \quad \forall\, (z_1, z_2) \in \mathcal{C}^2.$$

Moreover, the action of $\widetilde{\mathbf{Q}}$ (reflection with respect to the "horizontal" axes) corresponds to complex conjugation: $\widetilde{\mathbf{Q}} * (z_1, z_2) := (\bar{z}_1, \bar{z}_2)$. According to this point of view, we rewrite three of the invariants as

$$I_3 = |z_1|^2, \qquad I_4 = |z_2|^2, \qquad I_5 = \Re(z_1^2 \bar{z}_2). \tag{4.2}$$

In view of the decomposition of $\mathbb{E}$la described in Sec. 3, we now choose to look at polynomial functions of elasticity tensors as being defined on $\mathcal{R}^2 \times \mathcal{C}^2$. Moreover, we notice that each polynomial in the real variables x and y can be written as a polynomial in the complex variables z and $\bar{z}$, where $z = x + iy$. For this reason, we have

$$p(\mathbb{C}) = \sum c_{lmrstu} \lambda^l \mu^m z_1^r \bar{z}_1^s z_2^t \bar{z}_2^u, \tag{4.3}$$

where the index range depends on the degree of p. However, since we are only interested in real-valued polynomials, the restriction $c_{lmrstu} = \bar{c}_{lmsrut}$ must be satisfied. Moreover, invariance under the action of $\widetilde{\mathbf{Q}}$ is guaranteed by $c_{lmrstu} = c_{lmsrut}$, which combined with the previous condition, implies that all the coefficients are *real*.

The action of $\mathbf{Q}(\theta) \in SO(2)$ yields

$$p(\mathbf{Q} * \mathbb{C}) = \sum c_{lmrstu} \lambda^l \mu^m z_1^r \bar{z}_1^s z_2^t \bar{z}_2^u \exp[i(2r - 2s + 4t - 4u)]$$

and, from $p(\mathbb{C}) = p(\mathbf{Q} * \mathbb{C})$, we deduce that invariance under the action $SO(2)$ is guaranteed when the non-zero coefficients in (4.3) satisfy a relation which simplifies to

$$r - s = 2(u - t).$$

Thus, by inspection, we deduce that there are three types of non-zero terms in the sum defining p: (a) Those for which $r = s$ and $u = t$; (b) Those for which $\tau := u - t$ and $r - s = 2\tau$ are positive integers; (c) Those for which $\tau := t - u$ and $s - r = 2\tau$ are positive integers.

Case (a) is simple, because we rewrite each such addendum as

$$c_{lmrruu}\lambda^l\mu^m(z_1\bar{z}_1)^r(z_2\bar{z}_2)^u = c_{lmrruu}\lambda^l\mu^m|z_1|^{2r}|z_2|^{2u}, \qquad \text{(no sum)},$$

and, in view of (4.2), this is a monomial in the invariants I_3 and I_4. The symmetries of the coefficients c_{lmrstu} imply that the sum of the terms corresponding to cases (b) and (c) can be written as

$$\sum c_{lmrstu}\lambda^l\mu^m[z_1^r\bar{z}_1^s z_2^t\bar{z}_2^u + z_1^s\bar{z}_1^r z_2^u\bar{z}_2^t], \qquad r < s, \quad t < u,$$

which is

$$2\sum c_{lmrstu}\lambda^l\mu^m\Re[z_1^r\bar{z}_1^s z_2^t\bar{z}_2^u], \qquad r < s, \quad t < u.$$

Since $r = s + 2\tau$ and $u = t + \tau$, we conclude that this sum is

$$2\sum c_{lmrstu}\lambda^l\mu^m|z_1|^{2s}|z_2|^{2t}\Re[(z_1^2\bar{z}_2)^\tau], \qquad r < s, \quad t < u.$$

Finally, in view of the binomial formula, the real part of z^τ is always a polynomial in the variables $x := \Re z$ and $y^2 := (\Im z)^2 = |z|^2 - x^2$. Thus, we deduce that $\Re[(z_1^2\bar{z}_2)^\tau]$ is a polynomial in I_3, I_4 and I_5, and this concludes the proof that the collection $\{I_n\}$ is an integrity basis. As a final remark, we wish to draw the reader's attention to the fact that, with a similar technique, it is possible to prove that this collection, plus the sixth invariant I_6, is also an integrity basis for the action of the group $SO(2)$ on $\mathbb{E}\text{la}$.

5. Symmetry classes and invariants

A complete characterization of each one of the three non-trivial symmetry classes mentioned in Theorem 1 as the intersection of the zero-sets of isotropic polynomials is directly deducible from the geometric interpretation of the invariants introduced. This was also shown in [3], but, for the reader's convenience, we repeat here a formulation of this result, which can be easily proved using the concepts previously introduced.

PROPOSITION 1.

$$\mathbb{C} \in \mathbb{E}\text{la}(O(2)) \Leftrightarrow I_3 = I_4 = 0, \qquad \mathbb{C} \in \mathbb{E}\text{la}(D_4) \Leftrightarrow I_3 = 0, \quad I_4 \neq 0,$$

$$\mathbb{C} \in \mathbb{E}\text{la}(D_2) \Leftrightarrow \begin{cases} I_3 \neq 0, & I_4 = 0, \\ I_3 \neq 0, & I_4 \neq 0, \quad I_5^2 - I_3^2 I_4 = 0, \end{cases}$$

$$\mathbb{C} \in \mathbb{E}\text{la}(Z_2) \Leftrightarrow I_3 \neq 0, \quad I_4 \neq 0, \quad I_5^2 - I_3^2 I_4 \neq 0.$$

We are now left with the problem of determining the distance between an elasticity tensor obtained through experimental observations of a given material of unknown symmetry and the symmetry classes $\mathbb{E}\mathrm{la}(O(2))$, $\mathbb{E}\mathrm{la}(D_4)$ and $\mathbb{E}\mathrm{la}(D_2)$. As we shall see, only the distance with the first two classes can be computed explicitly, while for the third one the problem is left in a more general setting.

Before completing this discussion, it is important to make clear a further point. In principle, we are not so much interested in the distance between a given $\mathbb{C}$, which here we shall assume to be *anisotropic*, and the other three symmetry classes, but, rather, in the distance between them and the *orbit* of $\mathbb{C}$. The reason is clear when we think that two different elasticity tensors $\mathbb{C}_1$ and $\mathbb{C}_2$ lying on the same orbit (i.e., such that there is an orthogonal $\mathbf{Q}$ with the property that $\mathbb{C}_1 = \mathbf{Q} * \mathbb{C}_2$) represent the *same* material differently rotated in space. Thus, properly speaking, physical meaning pertains to the orbits, rather than to the elasticity tensors themselves. This observation, which is also discussed by BOEHLER, KIRILLOV and ONAT [4], shows the importance of having at our disposal a functional basis of isotropic invariants, to separate the orbits and decide when two elasticity tensors correspond to the same material body. Incidentally, we note that a functional basis for three-dimensional elasticity is not yet known, even if a partial answer is provided in [4], and a complete solution was recently announced by ZHENG and BETTEN [20, Abstract] and is expected to be published in a forthcoming paper by the same Authors.

However, we now prove that all the elasticity tensors on the same orbit have equal distance from any given symmetry class. Direct substitution shows that the action of $O(2)$ on $\mathbb{E}\mathrm{la}$ is distance-preserving: $d(\mathbb{C}_1, \mathbb{C}_2) = d(\mathbf{Q} * \mathbb{C}_1, \mathbf{Q} * \mathbb{C}_2)$, for all $\mathbf{Q} \in O(2)$. In other words, this action is a homomorphism of $O(2)$ into the group of orthogonal transformations of $\mathbb{E}\mathrm{la}$. For convenience of notation, we let $\mathcal{S}$ be any one of the four symmetry classes of elasticity tensors. Then $\mathbf{Q} * \mathcal{S} = \mathcal{S}$ for all orthogonal $\mathbf{Q}$. Thus,

$$d(\mathbf{Q} * \mathbb{C}, \mathcal{S}) := \inf_{\mathbb{C}^* \in \mathcal{S}} d(\mathbf{Q} * \mathbb{C}, \mathbb{C}^*) = \inf_{\mathbb{C}^* \in \mathcal{S}} d(\mathbf{Q} * \mathbb{C}, \mathbf{Q} * \mathbb{C}^*)$$
$$= \inf_{\mathbb{C}^* \in \mathcal{S}} d(\mathbb{C}, \mathbb{C}^*) =: d(\mathbb{C}, \mathcal{S}).$$

The interested reader will find a more complete discussion of many aspects of the geometry of the orbits of elasticity tensors under the action of the orthogonal group in a paper by RYCHLEWSKI [13].

Our goal is now to compute explicitly the square of the distance between a given tensor $\mathbb{C} = (\lambda, \mu, \mathbf{H}, \mathbb{K})$, which is assumed to be aelotropic, and each one of the three remaining symmetry classes. We write this quantity as follows:

$$\Delta(\mathbb{C}, \mathcal{G}) := |d(\mathbb{C}, \mathbb{E}\mathrm{la}(\mathcal{G}))|^2.$$

Let $\mathcal{G} = O(2)$. Then, for a generic isotropic $\mathbb{C}^*$ we may write the decomposition (3.1) as $\mathbb{C}^* = (\lambda^*, \mu^*, 0, 0)$. Thus,

$$|d(\mathbb{C}, \mathbb{C}^*)|^2 = (\lambda - \lambda^*)^2 + (\mu - \mu^*)^2 + |\mathbf{H}|^2 + |\mathbb{K}|^2.$$

It is now obvious that minimization as $\mathbb{C}^*$ varies over $\mathbb{E}\mathrm{la}(O(2))$ requires $\mathbb{C}^* = (\lambda, \mu, 0, 0)$ and, consequently,

$$\Delta(\mathbb{C}, O(2)) = |\mathbf{H}|^2 + |\mathbb{K}|^2 = I_3 + I_4\,.$$

A geometric interpretation of this result is straightforward: $\mathbb{C}^*$ is simply the orthogonal projection of $\mathbb{C}$ onto the subspace of isotropic tensors, and $\Delta(\mathbb{C}, O(2))$ is the square of the distance between the two. The problem of determining the isotropic elasticity tensor which is the closest to a given $\mathbb{C}$ is classical and, for three-dimensional elasticity, this solution is discussed in many textbooks (see, e.g., FEDOROV [7, Ch. 5, Sec. 26, pp. 174–175].

We now address the issue of determining $\Delta(\mathbb{C}, D_4)$. The decomposition of a generic tetragonal elasticity tensor is: $\mathbb{C}^* = (\lambda^*, \mu^*, 0, \mathbb{K}^*)$. Thus,

$$|d(\mathbb{C}, \mathbb{C}^*)|^2 = (\lambda - \lambda^*)^2 + (\mu - \mu^*)^2 + |\mathbf{H}|^2 + |\mathbb{K} - \mathbb{K}^*|^2,$$

and minimization implies that $\mathbb{C}^* = (\lambda, \mu, 0, \mathbb{K})$. In conclusion,

$$\Delta(\mathbb{C}, D_4) = |\mathbf{H}|^2 = I_3\,.$$

The computation of $\Delta(\mathbb{C}, D_2)$ is more complex. In view of Proposition 1, the symmetry class $\mathbb{E}\mathrm{la}(D_2)$ can be seen as the union of two disjoint subsets $\mathcal{S}_1$ and $\mathcal{S}_2$, formed, respectively, by elasticity tensors such that $I_4 = 0$ and such that $I_4 \neq 0$ with $I_5^2 = I_3^2 I_4$. Minimization of the distance between a given $\mathbb{C}$ and $\mathcal{S}_1$ yields the inequality

$$\Delta(\mathbb{C}, D_2) \leq I_4\,,$$

which, in any case, is a useful estimate of $\Delta(\mathbb{C}, D_2)$. To complete our analysis we need a better description of the set $\mathcal{S}_2$, which is characterized by the condition $\cos\gamma = \pm 1$. Let ψ and ϕ be the angles that the two tensor components in the decomposition (3.1) of a generic element of $\mathcal{S}_2$ form, respectively, with $\mathbf{E}_1$ and $\mathbb{E}_1$. Then, $\psi = \phi/2 + k\pi/2$, for some integer k. The element of $\mathcal{S}_2$ minimizing the distance from $\mathbb{C} = (\lambda, \mu, \mathbf{H}, \mathbb{K})$ is obviously $\mathbb{C}^* = (\lambda, \mu, \mathbf{H}^*, \mathbb{K}^*)$, where $\mathbf{H}^*$ and $\mathbb{K}^*$ are chosen in such a way that the sum $|\mathbf{H} - \mathbf{H}^*|^2 + |\mathbb{K} - \mathbb{K}^*|^2$ is an absolute minimum. We may now use elementary geometry considerations to show that

$$|\mathbf{H} - \mathbf{H}^*|^2 + |\mathbb{K} - \mathbb{K}^*|^2 = (K_1 \sin\phi - K_2 \cos\phi)^2 + (H_1 \sin(\phi/2) - H_2 \cos(\phi/2))^2.$$

Let Δ^* be the minimum of this distance as ϕ varies over $[0, 2\pi)$. In view of the definitions (4.1) we deduce that

$$\Delta^* = \min_{\phi\in[0,2\pi)} \{|\mathbb{K}|^2 \sin^2(\phi - \beta) + |\mathbf{H}|^2 \sin^2(\phi/2 - \alpha)\}.$$

Moreover, since this quantity is invariant under the action of $O(2)$ on $\mathbb{C}$ we may also assume that $\alpha = 0$ and, as a consequence, $\gamma = -\beta$. Thus, in conclusion,

$$\Delta^* = \min_{\phi \in [0,2\pi)} \{I_4 \sin^2(\phi + \gamma) + I_3 \sin^2(\phi/2)\},$$

and

$$\Delta(\mathbb{C}, D_2) = \min\{I_4, \Delta^*\}.$$

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