

Stability of micro-periodic materials under finite deformations

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A NEW APPROACH to the stability analysis for highly-elastic micro-periodic composite materials subjected to finite deformations is proposed. The analysis is based on the refined macrodynamics of periodic structures which describes the effect of the microstructure size on the dynamic body behaviour. It is shown that the loss of stability can take place both on the macro- and micro-level and that the internal dynamic instability depends on the microstructure size. The obtained results are illustrated by a simple example.

1. Introduction

STABILITY OF HOMOGENEOUS elastic materials under finite deformations was investigated in the series of papers [1–9, 11–15, 18]; the main results can be found in the monograph [10]. The aim of this contribution is to outline a new approach to the problem of stability for composite bodies made of perfectly bonded elastic constituents subjected to large strains. It is assumed that in the natural configuration the material structure of the body is micro-periodic. The analysis is based on the refined macro-dynamics of composite materials, introduced in the framework of linear elasticity in [19] and extended to finite elastic deformations in [16, 17]. The effect of the unit cell length dimensions on the dynamic stability of a micro-periodic body and the existence of new kinds of material stability, related to the microstructure of a composite, are most important features of the proposed approach.

Notations

Indices α, β, \dots and i, j, \dots run over 1, 2, 3 and are related to the material and spatial coordinate systems, respectively. Capital Latin indices A, B, \dots run over 1, \dots, N ; $N \geq 1$. Summation convention holds for all aforementioned indices if not otherwise stated. By V_R we denote the region $(-l_1/2, l_1/2) \times (-l_2/2, l_2/2) \times (-l_3/2, l_3/2)$ in a three-space of points $\mathbf{X} = (X^\alpha)$. An averaged value of any integrable V_R -periodic function $f(\cdot)$ of \mathbf{X} will be denoted by

$$\langle f(\mathbf{X}) \rangle := \frac{1}{l_1 l_2 l_3} \int_{V_R} f(\mathbf{X}) dX^1 dX^2 dX^3.$$

Here and in the sequel the subscript R is related to the known reference configuration of the body under investigation.

2. Foundations

Let the highly-elastic composite body in the natural (reference) configuration occupy a region Ω_R in a physical three-space and have in this configuration the V_R -periodic structure. The microstructure length parameter defined by $l := \sqrt{(l_1)^2 + (l_2)^2 + (l_3)^2}$ is assumed to be sufficiently small compared to the smallest characteristic length dimension of Ω_R . The position of an arbitrary point \mathbf{X} , $\mathbf{X} \in \Omega_R$, of the body at an instant t , $t \geq 0$, will be denoted by $\mathbf{x} = \mathbf{p}(\mathbf{X}, t)$, $\mathbf{X} = (X^\alpha) \in \Omega_R$. Hence $\mathbf{u}(\mathbf{X}, t) := \mathbf{p}(\mathbf{X}, t) - \mathbf{X}$ is a displacement vector from the natural configuration. The properties of the composite under consideration are determined by a mass density $\varrho_R(\cdot)$ and a strain energy density function $\varepsilon_R(\cdot, \nabla \mathbf{p})$, which are V_R -periodic functions defined almost everywhere on Ω_R and related, as densities, to the reference configuration.

The idea of the refined macrodynamics, explained in [19] and applied in a series of related papers, is based on the heuristic constraint assumption that the displacements $u_i(\mathbf{X}, t)$ in a periodic composite can be represented by certain averaged displacements $U_i(\cdot, t)$ on which highly-oscillating disturbances are superimposed, caused by the micro-inhomogeneity of a medium. To describe this situation, the concept of a regular macro-function was introduced in [19]; roughly speaking, a function $F: \Omega_R \rightarrow \mathbf{R}$ is called a macro-function (for the known microstructure length parameter l and a certain accuracy ε_F assigned to numerical calculations of the values of F) if for every $\mathbf{X}, \mathbf{Z} \in \Omega_R$ such that $\|\mathbf{X} - \mathbf{Z}\| < l$ condition $|F(\mathbf{X}) - F(\mathbf{Z})| < \varepsilon_F$ holds. If similar conditions also hold for all derivatives of F then F is said to be a regular macro-function. The aforementioned constraint assumption specifies a class of motions given by

$$(2.1) \quad u_i(\mathbf{X}, t) = U_i(\mathbf{X}, t) + h^A(\mathbf{X})Q_i^A(\mathbf{X}, t), \quad \mathbf{X} \in \Omega_R, \quad t \geq 0,$$

where $U_i(\cdot, t)$, $Q_i^A(\cdot, t)$ are certain arbitrary regular macro-functions, and $h^A(\cdot)$ are the postulated *a priori* V_R -periodic functions (hence depending on l), satisfying for every \mathbf{X} the conditions $h^A(\mathbf{X}) \in O(l)$, $h^A_{,\alpha}(\mathbf{X}) \in O(1)$ as well as the condition $\langle h^A \rangle = 0$. Functions $h^A(\cdot)$ are called micro-shape functions and from the qualitative viewpoint, they determine the investigated class of disturbances in displacements caused by the V_R -periodic structure of the composite. Functions $U_i(\cdot)$, $Q_i^A(\cdot)$ are the basic dynamic variables of the refined macrodynamics being referred to as macro-displacements and macro-internal variables, respectively. By virtue of Eq. (2.1), macro-internal variables Q_i^A describe the aforementioned disturbances in displacements from a quantitative viewpoint. Define by \mathbf{F} a field with components

$$F^i_{\alpha} := \delta^i_{\alpha} + U^i_{,\alpha}$$

which will be called the macro-deformation gradient. Hence every $F^i_{\alpha}(\cdot, t)$, $t \geq 0$, is a certain regular macro-function. In the framework of the refined macrodynamics the deformation gradient $\nabla \mathbf{p}$ is approximated by $\mathbf{F} + \nabla h^A \mathbf{Q}^A$, [19]. It

follows that the function π_R defined by

$$\pi_R = \pi_R(\mathbf{F}, \mathbf{Q}) := \langle \varepsilon_R(\mathbf{X}, \mathbf{F} + \nabla h^A(\mathbf{X})\mathbf{Q}^A) \rangle, \quad \mathbf{Q} := (\mathbf{Q}^1, \dots, \mathbf{Q}^N)$$

represents an averaged strain energy. Macro-deformation gradients \mathbf{F} and macro-internal variables \mathbf{Q}^A are restricted by the condition

$$\det(\mathbf{F} + \nabla h^A \mathbf{Q}^A) > 0.$$

Let us define $g^A := l^{-1}h^A$; obviously, values of functions g^A satisfy conditions $g^A(\mathbf{X}) \in O(1)$. The field equations for $U_i(\cdot)$, $Q_i^A(\cdot)$ which were obtained in [17], after neglecting the body forces, can be written down in the form

$$(2.2) \quad S_{R,\alpha}^{i\alpha} - \langle \varrho_R \rangle \ddot{U}^i = 0, \quad H_R^{Ai} + l^2 \langle \varrho_R g^A g^B \rangle \ddot{Q}^{Bi} = 0,$$

where

$$(2.3) \quad S_R^{i\alpha} = \frac{\partial \pi_R}{\partial F_{i\alpha}}, \quad H_R^{Ai} = \frac{\partial \pi_R}{\partial Q_i^A}.$$

Fields $S_R^{i\alpha}$ and H_R^{Ai} are called the Piola - Kirchhoff macro-stresses and the micro-dynamic forces (related to Ω_R), respectively. In the natural configuration, i.e. for $\mathbf{F} = \mathbf{1}$ and $\mathbf{Q} = \mathbf{0}$, the macro-stresses $S_R^{i\alpha}$ and micro-dynamic forces H_R^{Ai} have to be equal to zero. If this condition is not satisfied by the derivatives of $\langle \varepsilon_R \rangle$ with respect to \mathbf{F} and \mathbf{Q} then the strain energy function π_R in Eqs.(2.3) has to be assumed in the form

$$(2.4) \quad \pi_R = \pi_R(\mathbf{F}, \mathbf{Q}) := \langle \varepsilon_R(\mathbf{X}, \mathbf{F} + \nabla h^A(\mathbf{X})\mathbf{Q}^A) \rangle - \lambda_R^{i\alpha} (F_{i\alpha} - \delta_{i\alpha}) - \mu_R^{Ai} Q_i^A,$$

where

$$(2.5) \quad \lambda_R^{i\alpha} := \left. \frac{\partial \langle \varepsilon_R \rangle}{\partial F_{i\alpha}} \right|_{\mathbf{F}=\mathbf{1}, \mathbf{Q}=\mathbf{0}}, \quad \mu_R^{Ai} := \left. \frac{\partial \langle \varepsilon_R \rangle}{\partial Q_i^A} \right|_{\mathbf{F}=\mathbf{1}, \mathbf{Q}=\mathbf{0}}.$$

Formula (2.4) defines the macro-strain energy function related to the natural configuration of the body.

Let Γ_R be a part of $\partial\Omega_R$ on which surface tractions s_R^i (averaged over the surface area) are known. The related boundary conditions are given by

$$(2.6) \quad S_{R\alpha}^{i\alpha} n_{R\alpha} = s_R^i \quad \text{on } \Gamma_R$$

with \mathbf{n}_R as a unit outward normal to $\partial\Omega_R$. It will be also assumed that on $\partial\Omega_R \setminus \Gamma_R$, values U_i^0 of macro-displacements are prescribed:

$$(2.7) \quad U_i = U_i^0 \quad \text{on } \partial\Omega_R \setminus \Gamma_R.$$

Equations (2.2), (2.3) and boundary conditions (2.6), (2.7) hold for every $t > 0$ and together with initial conditions for $U_i, \dot{U}_i, Q_i^A, \dot{Q}_i^A$, describe a certain boundary-value problem formulated in the framework of the refined macro-dynamics of a highly-elastic micro-periodic body and for a class of motions given by (2.1). The main feature of the refined macrodynamics is that the above problem takes into account the effect of the microstructure length-parameter l on the dynamic behaviour of the composite. It has to be emphasized that a solution to this problem has a physical sense only if $U_i(\cdot, t), Q_i^A(\cdot, t)$ are regular macro-functions for every $t \geq 0$. For more detailed information the reader is referred to the references given in Introduction.

3. Analysis

Let us assume that a certain static deformation of the composite described by Eq.(2.1) is known, where the fields $U_i = U_i(\mathbf{X}), Q_i^A = Q_i^A(\mathbf{X}), \mathbf{X} \in \Omega_R$ are constant in time and hence satisfy in Ω_R the field equations

$$(3.1) \quad \left(\frac{\partial \pi_R(\mathbf{F}(\mathbf{X}), \mathbf{Q}(\mathbf{X}))}{\partial F_{i\alpha}} \right)_{,\alpha} = 0, \quad \frac{\partial \pi_R(\mathbf{F}(\mathbf{X}), \mathbf{Q}(\mathbf{X}))}{\partial Q_i^A} = 0, \quad \mathbf{X} \in \Omega_R,$$

and fulfil on $\partial\Omega_R$ the time-independent boundary conditions of the form (2.6), (2.7); in (3.1) $\mathbf{F}(\mathbf{X}) = \mathbf{1} + \nabla \mathbf{U}(\mathbf{X})$. Every static deformation of the composite, defined by a pair $\mathbf{E} = (\mathbf{U}(\cdot), \mathbf{Q}(\cdot))$ satisfying Eqs. (3.1), will be referred to as the equilibrium state. In order to investigate the stability of the above equilibrium state, the line of approach described in [10] will be applied. To this end let us assume that on the static deformation represented by a displacement field $u_i(\mathbf{X}) = U_i(\mathbf{X}) + h^A(\mathbf{X})Q_i^A(\mathbf{X}), \mathbf{X} \in \Omega_R$, a small deformation is superimposed, given by $\varepsilon' u_i(\mathbf{X}, t) = \varepsilon[U_i(\mathbf{X}, t) + h^A(\mathbf{X})'Q_i^A(\mathbf{X}, t)], t \geq 0$, where ε is a small parameter, the squares and higher powers of which will be neglected as compared to ε , and where $'U_i(\cdot, t), 'Q_i^A(\cdot, t)$ are arbitrary regular macro-functions. Using Eqs.(2.2), (2.3), (2.6), (2.7) and denoting

$$(3.2) \quad \begin{aligned} A_R^{i\alpha j\beta} &:= \frac{\partial^2 \pi_R(\mathbf{F}(\mathbf{X}), \mathbf{Q}(\mathbf{X}))}{\partial F_{i\alpha} \partial F_{j\beta}}, & B_R^{Aji\alpha} &:= \frac{\partial^2 \pi_R(\mathbf{F}(\mathbf{X}), \mathbf{Q}(\mathbf{X}))}{\partial F_{i\alpha} \partial Q_j^A}, \\ C_R^{ABij} &:= \frac{\partial^2 \pi_R(\mathbf{F}(\mathbf{X}), \mathbf{Q}(\mathbf{X}))}{\partial Q_i^A \partial Q_j^B}, & \mathbf{X} \in \Omega_R, \end{aligned}$$

after simple manipulations we obtain the linearized homogeneous field equations for $'U_i, 'Q_i^A$, which have to be satisfied in $\Omega_R \times (0, \infty)$:

$$(3.3) \quad \begin{aligned} &\left(A_R^{i\alpha j\beta} 'U_{j,\beta} + B_R^{Bji\alpha} 'Q_j^B \right)_{,\alpha} - \langle \varrho_R \rangle ' \ddot{U}^i = 0, \\ &l^2 \langle \varrho_R g^A g^B \rangle ' \ddot{Q}_i^B + C_R^{ABij} 'Q_j^B + B_R^{Aij\alpha} 'U_{j,\alpha} = 0, \end{aligned}$$

together with the homogeneous boundary conditions:

$$(3.4) \quad \begin{aligned} (A_R^{i\alpha j\beta} {}'U_{j,\beta} + B_R^{Bji\alpha} {}'Q_j^B) n_{R\alpha} &= 0 && \text{on } \Gamma_R \times (0, \infty), \\ {}'U_i &= 0 && \text{on } \partial\Omega_R \setminus \Gamma_R \times (0, \infty). \end{aligned}$$

From the definitions (3.2) and since $\mathbf{F} = \mathbf{1} + \nabla \mathbf{U}$, it follows that solutions $'U_i, {}'Q_i^A$ to the boundary-value problems described by Eqs. (3.3), (3.4) depend on the known static deformation represented by the equilibrium state $\mathbf{E} = (\mathbf{U}(\cdot), \mathbf{Q}(\cdot))$. At the same time, every pair (\mathbf{F}, \mathbf{Q}) satisfying the last of Eqs. (3.1) will be referred to as the local equilibrium state. Obviously, if a composite is in the equilibrium state $(\mathbf{U}(\cdot), \mathbf{Q}(\cdot))$ then every $(\mathbf{F}(\mathbf{X}), \mathbf{Q}(\mathbf{X}))$, $\mathbf{X} \in \Omega_R$, represents a certain local equilibrium state (but not conversely).

Now we shall pass to the analysis of some special cases.

First, let us assume that the superimposed deformations are time-independent, i.e.:

$$'U_i = {}'U_i(\mathbf{X}), \quad {}'Q_i^A = {}'Q_i^A(\mathbf{X}), \quad \mathbf{X} \in \Omega_R.$$

Under this assumption two special cases of instability can take place.

CASE 1.1. Let for every $\mathbf{X} \in \Omega_R$ the linear transformation $\mathbf{R}^{3N} \rightarrow \mathbf{R}^{3N}$ given by C_R^{ABij} be invertible for the known equilibrium state $\mathbf{E} = (\mathbf{U}(\cdot), \mathbf{Q}(\cdot))$. In this case the macro-internal variables $'Q_i^A$ can be eliminated from Eqs. (3.3), (3.4) and we arrive at

$$(3.5) \quad \begin{aligned} (N_R^{i\alpha j\beta} {}'U_{j,\beta})_{,\alpha} &= 0 && \text{in } \Omega_R, \\ N_R^{i\alpha j\beta} {}'U_{j,\beta} n_{R\alpha} &= 0 && \text{on } \Gamma_R, \\ {}'U_i &= 0 && \text{on } \partial\Omega_R \setminus \Gamma_R, \end{aligned}$$

where we have denoted

$$N_R^{i\alpha j\beta} := A_R^{i\alpha j\beta} - B_R^{Aki\alpha} D_R^{ABkl} B_R^{Blj\beta}$$

and where D_R^{ABkl} determines the linear transformation $\mathbf{R}^{3N} \rightarrow \mathbf{R}^{3N}$ inverse to that given by C_R^{ABkl} . If there exist non-trivial solutions to Eqs. (3.5) then the body in the equilibrium state $\mathbf{E} = (\mathbf{U}(\cdot), \mathbf{Q}(\cdot))$ is assumed to have a hidden macro-instability, [10], and we deal with a *bifurcation of the equilibrium state E*. Moreover, if $\Gamma_R = \emptyset$ then we arrive at the problem of the *internal macro-instability* investigated by BIOT [1, 2] as the internal buckling.

CASE 1.2. Now assume that under the known equilibrium state, a linear transformation $\mathbf{R}^{3N} \rightarrow \mathbf{R}^{3N}$ determined by C_R^{ABkl} is singular for some local equilibrium state $(\mathbf{F}(\mathbf{X}), \mathbf{Q}(\mathbf{X}))$. In this case the body at the point \mathbf{X} is said to be in the state of a hidden micro-instability and we deal with a *bifurcation of the local equilibrium state (F(X), Q(X))*. Moreover if $\mathbf{F} = \text{const}$, $\mathbf{Q} = \text{const}$ in Ω_R and $\Gamma_R = \emptyset$,

then Eqs.(3.3) are satisfied by $'U_i = 0, 'Q_i^A = \text{const}$ in Ω_R and we deal with what can be called *the internal micro-instability*.

Second, let us assume that the superimposed motion is given by

$$'U_i(\mathbf{X}, t) = \bar{U}_i(\mathbf{X})e^{i\omega t}, \quad 'Q_i^A(\mathbf{X}, t) = \bar{Q}_i^A(\mathbf{X})e^{i\omega t},$$

where ω is a certain complex number. Substituting the right-hand sides of the above formulae into Eqs. (3.3), (3.4) we obtain for \bar{U}_i, \bar{Q}_i^A the following system of equations

$$(3.6) \quad \begin{aligned} & \left(A_R^{i\alpha j\beta} \bar{U}_{j,\beta} + B_R^{Aj i\alpha} \bar{Q}_j^A \right)_{,\alpha} - \langle \varrho_R \rangle \omega^2 \bar{U}^i = 0, \\ & \left(C_R^{ABij} - l^2 \langle \varrho_R g^A g^B \rangle \delta^{ij} \omega^2 \right) \bar{Q}_j^B + B_R^{Aij\alpha} \bar{U}_{j,\alpha} = 0, \quad \text{in } \Omega_R, \end{aligned}$$

together with the boundary conditions

$$(3.7) \quad \begin{aligned} & \left(A_R^{i\alpha j\beta} \bar{U}_{j,\beta} + B_R^{Aj i\alpha} \bar{Q}_j^A \right) n_{R\alpha} = 0 \quad \text{on } \Gamma_R, \\ & \bar{U}_i = 0 \quad \text{on } \partial\Omega_R \setminus \Gamma_R. \end{aligned}$$

It has to be remembered that the eigenvalues ω^2 in Eqs.(3.6) depend on the known equilibrium state $\mathbf{E} = (\mathbf{U}(\cdot), \mathbf{Q}(\cdot))$ since the coefficients in Eqs. (3.6), (3.7) are functions of $\mathbf{F}(\mathbf{X}) = \mathbf{1} + \nabla \mathbf{U}(\mathbf{X})$ and $\mathbf{Q}(\mathbf{X})$, cf. formulae (3.2). The analysis of Eqs.(3.6), (3.7) leads to the so-called dynamic instability, [10]. Two special cases will be considered below.

CASE 2.1. Let us assume that for the known equilibrium state $\mathbf{E} = (\mathbf{U}(\cdot), \mathbf{Q}(\cdot))$ and for every $\mathbf{X} \in \Omega_R$ the linear transformation $\mathbf{R}^{3N} \rightarrow \mathbf{R}^{3N}$ given by $C_R^{ABij} - l^2 \langle \varrho_R g^A g^B \rangle \delta^{ij} \omega^2$ is invertible. Then every inverse transformation can be represented in the form of the asymptotic expansion

$$D_R^{ABkl} + l^2 \omega^2 D_R^{ADik} \langle \varrho_R g^D g^E \rangle D_R^{EBil} + o(l^2).$$

Neglecting terms $o(l^2)$ we can eliminate \bar{Q}_j^B from Eqs.(3.6), (3.7). Defining

$$M_R^{i\alpha j\beta} := B_R^{Aik\alpha} D_R^{ADkl} \langle \varrho_R g^D g^E \rangle D_R^{EBlm} B_R^{Bjm\beta} \langle \varrho_R \rangle^{-1},$$

after some manipulations we arrive at the following system of equations for \bar{U}_i , which have to be satisfied in $\Omega_R \times (0, \infty)$:

$$(3.8) \quad \left(N_R^{i\alpha j\beta} \bar{U}_{j,\beta} \right)_{j,\alpha} + \langle \varrho_R \rangle \left[\bar{U}^i - l^2 \left(M_R^{i\alpha j\beta} \bar{U}_{j,\beta} \right)_{,\alpha} \right] \omega^2 = 0$$

together with the boundary conditions

$$(3.9) \quad \begin{aligned} & \left(N_R^{i\alpha j\beta} - \langle \varrho_R \rangle l^2 \omega^2 M_R^{i\alpha j\beta} \right) \bar{U}_{j,\beta} n_{R\alpha} = 0 \quad \text{on } \Gamma_R \times (0, \infty), \\ & \bar{U}_j = 0 \quad \text{on } [\partial\Omega_R \setminus \Gamma_R] \times (0, \infty). \end{aligned}$$

Following [10] we shall assume that if $\text{Im } \omega \geq 0$ then the equilibrium state $\mathbf{E} = (\mathbf{U}(\cdot), \mathbf{Q}(\cdot))$ is stable. If in the vicinity of \mathbf{E} there exists a passage from $\text{Im } \omega \geq 0$ to $\text{Im } \omega < 0$, then we deal with the loss of the *macro-vibrational stability* (provided that $\text{Re } \omega \neq 0$) or the loss of the *macro-static stability* (if $\text{Re } \omega = 0$) in this state. Moreover, if $\Gamma_R = \emptyset$ then it is the loss of the *internal macro-vibrational* or *macro-static stability*, respectively.

CASE 2.2. Assume that for the known equilibrium state values ω^2 are the generalized eigenvalues given by $(C_R^{ABij} - l^2 \omega^2 \langle \rho_R g^A g^B \rangle \delta^{ij}) \bar{Q}_j^B = 0$ for some local equilibrium state $(\mathbf{F}(\mathbf{X}), \mathbf{Q}(\mathbf{X}))$. If $\text{Im } \omega$ attains a negative value in this local equilibrium state then we shall deal with the loss of the *micro-vibrational stability* (for $\text{Re } \omega \neq 0$) or the *micro-static stability* (for $\text{Re } \omega = 0$). Moreover if $\mathbf{F} = \text{const}$, $\mathbf{Q} = \text{const}$, in Ω_R and $\Gamma_R = \emptyset$ then Eqs. (3.6) are satisfied by $'U_i = 0$, $'Q_i^A = \text{const}$ in Ω_R and we arrive at the problem of the loss of *internal micro-vibrational* or *micro-static stability*, respectively.

All the aforementioned cases of instability can be referred to as the local loss of stability. However, for micro-periodic highly-elastic materials we can also deal with the special case of a non-local instability described below.

CASE 3. Let us assume that for a certain $\mathbf{X} \in \Omega_R$ there exists the macro-deformation gradient $\mathbf{F}(\mathbf{X})$ for which the last of Eqs. (3.1) has more than one solution \mathbf{Q} satisfying together with \mathbf{F} condition $\det(\mathbf{F} + \nabla h^A(\mathbf{Z})\mathbf{Q}^A) > 0$ for every $\mathbf{Z} \in V_R + \mathbf{X}$. In this case we deal with the non-local micro-instability. This kind of instability can be also referred to as the material instability strictly related to the micro-periodic heterogeneous structure of the composite body.

Summing up, the stability analysis for highly-elastic micro-periodic composites leads to the following three types of stability:

1. *Local macro-stability* described by Cases 1.1 and 2.1, which can be investigated similarly to the instability of homogeneous body.

2. *Local micro-stability* described by Cases 1.2 and 2.2 related to the investigations of the linear transformation given by $C_R^{i\alpha j\beta}(\mathbf{F}, \mathbf{Q})$.

3. *Non-local micro-stability* described by Case 3, related to the analysis of the last of Eqs. (3.1).

The problem of the non-local macro-stability is not investigated in this contribution. It has to be emphasized that the concept of the micro-stability is characteristic for composite micro-periodic bodies subjected to finite deformations.

4. Analysis: incompressible bodies

The refined macrodynamics of micro-periodic composites made of highly-elastic incompressible constituents will take as a starting point the averaged incompressibility condition

$$(4.1) \quad \langle \det(\mathbf{F} + \nabla h^A(\mathbf{X})\mathbf{Q}^A) \rangle - 1 = 0.$$

It has to be emphasized that in the framework of the proposed macro-model, the exact incompressibility condition $\det(\mathbf{F} + \nabla h^A(\mathbf{X})\mathbf{Q}^A) - 1 = 0$ may not be satisfied at every point \mathbf{X} of Ω_R . Equation (4.1) can be also written in the explicit form

$$(4.2) \quad \det \mathbf{F} + \frac{1}{6} \varepsilon^{ijk} \varepsilon^{\alpha\beta\gamma} \left(3 \langle h^A_{,\alpha} h^B_{,\beta} \rangle Q^A_i Q^B_j F_{k\gamma} + \langle h^A_{,\alpha} h^B_{,\beta} h^C_{,\gamma} \rangle Q^A_i Q^B_j Q^C_k \right) - 1 = 0,$$

where ε^{ijk} , $\varepsilon^{\alpha\beta\gamma}$ are the Ricci symbols.

In many special problems the analysis can be confined to a class of motions (2.1) in which all micro-shape functions satisfy the conditions:

$$(4.3) \quad \begin{aligned} \langle h^A_{,\alpha} h^B_{,\beta} \rangle &= 0 & \text{if } \alpha \neq \beta, \\ \langle h^A_{,\alpha} h^B_{,\beta} h^C_{,\gamma} \rangle &= 0 & \text{if } \alpha \neq \beta \neq \gamma \neq \alpha. \end{aligned}$$

This situation is typical for many disturbances investigated in dynamics of composite materials. In the simplest case relations (4.3) hold if every micro-shape function $h^A(\cdot)$ depends exclusively on one arbitrary material coordinate X^α . Under (4.3) the averaged incompressibility condition (4.2) reduces to the following one

$$(4.4) \quad \det \mathbf{F} - 1 = 0$$

being independent of macro-internal variables Q_i^A . The above condition represents the internal constraints imposed on the class of motions determined by Eqs.(2.1). Introducing the concept of a macro-pressure $p_R = p_R(\mathbf{X})$ as a Lagrange multiplier related to Eq.(4.4), bearing in mind definitions (2.5) and modifying Eq.(2.4) to the form

$$(4.5) \quad \pi_R = \pi_R(\mathbf{F}, \mathbf{Q}) := \langle \varepsilon_R(\mathbf{X}, \mathbf{F} + \nabla h^A(\mathbf{X})\mathbf{Q}^A) \rangle - \lambda_R^{i\alpha} (F_{i\alpha} - \delta_{i\alpha}) - \mu_R^{A\alpha} Q_i^A + p_R(\det \mathbf{F} - 1),$$

we shall assume that the equilibrium equations (3.1) holds also for incompressible bodies (in the averaged sense explained above).

Summing up, under definitions (4.5), (2.5) and bearing in mind that $\mathbf{F} = \mathbf{I} + \nabla \mathbf{U}$, the equilibrium equations of the form (3.1) together with Eq.(4.4) lead to a system of equations for macro-displacements \mathbf{U} , macro-internal variables \mathbf{Q}^A and a macro-pressure p_R . This result holds true under conditions (4.3). If the above conditions do not hold then the averaged incompressibility condition has to be taken in its general form (4.2), and in Eq.(4.5) the term $\det \mathbf{F} - 1$ has to be replaced by the left-hand side of Eq.(4.2).

The stability analysis for incompressible bodies has to be carried out similarly to that of the compressible bodies described in Sec.3. Apart from the superimposed small motions $\varepsilon(U_i + h^A Q_i^A)$, also a small excess of a macro-pressure

$\varepsilon' p_R$ has to be superimposed on p_R . Hence in the incremental equations, which for compressible bodies were given by Eqs. (3.3)–(3.9), we deal with terms involving $'p_R$ and with the incremental form of Eq. (4.4). Under notation $L_R := F^{-1}$ this equation is given by

$$L_R^{i\alpha} 'U_{i,\alpha} = 0.$$

The general line of approach to the stability analysis for incompressible composites, outlined in this section, will be illustrated by a simple example in the subsequent section of the paper.

5. Example

The general results obtained in this contribution will be now illustrated by the micro-stability analysis for a laminated body made of two perfectly bonded incompressible isotropic rubber-like materials. The scheme of the laminate is shown in the left-hand side of Fig. 1. Moreover, every lamina is assumed to be

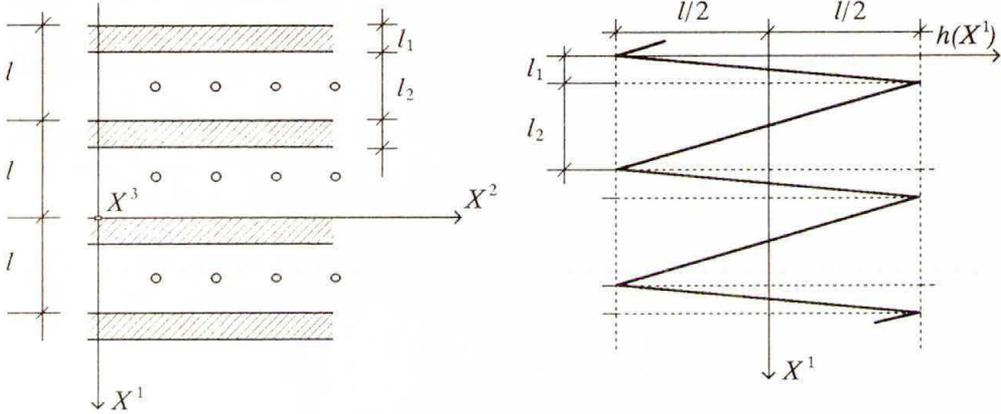


FIG. 1.

reinforced by a system of periodically distributed inextensible fibres parallel to the X^3 -axis. Let the body be subjected to finite deformations caused by the uniform axial macro-strains along the coordinate axes. Using (3.1), the class of displacement fields under consideration will be expected in the form

$$\begin{aligned}
 (5.1) \quad u_1 &= U_1(X^1) + h^1(X^1)Q_1^1 + h^3(X^2)Q_1^3, \\
 u_2 &= U_2(X^2) + h^2(X^1)Q_2^2, \\
 u_3 &= 0,
 \end{aligned}$$

where

$$U_1(X^1) = (F_{11} - 1)X^1, \quad U_2(X^2) = (F_{22} - 1)X^2,$$

and (for the time being) F_{11} , F_{22} , Q_1^1 , Q_1^3 , Q_2^2 are constants constituting the system of basic unknowns. We have tacitly assumed that the effect of periodic

inhomogeneity along X^2 -axis on the displacement field is small and can be neglected. That is why a term $h^4(X^2)Q_2^4$ in the second equation of (5.1) was not taken into account; the analysis involving this term is more complicated and will be given separately. The diagram of the micro-shape function $h^1(\cdot)$ is shown on the right-hand side in Fig. 1; we also assume $h^2(X^1) = l \cos(2\pi X^1/l)$ and $h^3(X^2) = l_2 \sin(2\pi X^2/l_2)$, where l_2 is the period of the reinforcement along X^2 -axis. In the problem under consideration $V_R = (-l/2, l/2) \times (-l_2/2, l_2/2)$ and it is assumed that $X^1 = 0$ is the plane of symmetry of the material structure of V_R .

Let us denote the basic unknown variables by means of

$$F_1 := F_{11}, \quad F_2 := F_{22}, \quad Q := Q_1^1, \quad Q_1 = Q_1^3, \quad Q_2 = Q_2^2.$$

It can be shown that the averaged incompressibility condition (4.1) yields

$$(5.2) \quad F_1 F_2 - 1 = 0.$$

Under Eq. (5.2) every quintuplet (F_1, F_2, Q, Q_1, Q_2) represents a certain micro-equilibrium state (now constant throughout the whole body) provided that the last of the equilibrium equations (3.1) holds; the first of these equations is identically satisfied since $S_R^{i\alpha}$ are constant.

As we have stated in Sec. 2, in the framework of the refined macrodynamics the deformation gradient is approximated by $\mathbf{F} + \nabla h^A(\mathbf{X})\mathbf{Q}^A$, where now $\mathbf{X} = (X^1, X^2)$. In the problem under consideration, under extra notations

$$\begin{aligned} d &= d(X^1) := h^1_{,1}(X^1), \\ d_1 &= d_1(X^1) := h^2_{,1}(X^1), \\ d_2 &= d_2(X^2) := h^2_{,2}(X^2), \end{aligned}$$

the deformation gradient matrix is given by

$$\begin{bmatrix} F_1 + dQ & d_2 Q_1 & 0 \\ d_1 Q_2 & F_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and for every $\mathbf{X} \in \bar{V}_R := [-l/2, l/2] \times [-l_2/2, l_2/2]$ has to satisfy conditions

$$(5.3) \quad \begin{aligned} F_1 + d(\mathbf{X})Q &> 0, & F_2 &> 0, \\ F_1 F_2 + d(\mathbf{X})F_2 Q - d_1(\mathbf{X})d_2(\mathbf{X})Q_1 Q_2 &> 0. \end{aligned}$$

The components $c_{\alpha\beta}$ of the deformed body metric tensor are given by the matrix

$$\begin{bmatrix} (F_1 + dQ)^2 + (d_1 Q_2)^2 & d_2(F_1 + dQ)Q_1 + d_1 F_2 Q_2 & 0 \\ d_2(F_1 + dQ)Q_1 + d_1 F_2 Q_2 & F_2^2 + (d_2 Q_1)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the strain invariants I_1, I_2, I_3 are equal to

$$\begin{aligned} I_1 &= \delta^{\alpha\beta} c_{\alpha\beta} = 1 + F_1^2 + F_2^2 + 2dF_1Q + (dQ)^2 + (d_1Q_1)^2 + (d_2Q_2)^2, \\ I_2 &= I_3 \delta_{\alpha\beta} c^{\alpha\beta} = 1 + F_1^2 + F_2^2 + 2d(F_1 + F_2)Q + (dQ)^2(1 + F_2^2) + (d_2Q_1)^2 \\ &\quad + (d_1Q_2)^2 + (d_1d_2Q_1Q_2)^2 - 2d_1d_2Q_1Q_2 - 2dd_1d_2F_2Q_1Q_2, \\ I_3 &= \det c_{\alpha\beta} = [(F_1 + dQ)F_2 - d_1d_2Q_1Q_2]^2. \end{aligned}$$

It has to be emphasized that in the applied approach, the local incompressibility condition $\sqrt{I_3} - 1 = 0$ does not hold and we deal exclusively with the averaged form of this condition, given by Eq. (4.1) which now reduces to Eq. (5.2).

The strain energy function for rubber-like materials will be assumed in the known form

$$\varepsilon_R = C(I_1 - 3) + D(I_2 - 3),$$

where the material moduli C, D are now l -periodic functions of X^1 , attaining different values in the adjacent laminae. Due to the presence of a reinforcement we shall also treat C, D as l_2 -periodic functions of X^2 . Hence C and D as well as the invariants I_1, I_2 are V_R -periodic functions of $\mathbf{X} = (X^1, X^2)$. The formula (4.5) for the macro-strain energy function of an incompressible isotropic material is given by

$$\pi_R = \langle \varepsilon_R(\mathbf{X}, I_1(\mathbf{X}), I_2(\mathbf{X})) \rangle - \lambda_R^{i\alpha} (F_{i\alpha} - \delta_{i\alpha}) - \mu_R^{Ai} Q_i^A + p_R(\det \mathbf{F} - 1),$$

where the averaging operation has to be carried out with respect to \mathbf{X} , and $\lambda_R^{i\alpha}, \mu_R^{Ai}$ are defined by Eqs. (2.5). After some calculations we obtain

$$\begin{aligned} \pi_R &= \langle C + D \rangle (F_1^2 + F_2^2 - 2) + 2[\langle (C + D)d \rangle F_1 + \langle Dd \rangle F_2] Q \\ &\quad + [\langle (C + D)d^2 \rangle + \langle Dd^2 \rangle F_2^2] Q^2 + \langle (C + D)d_2^2 \rangle Q_1^2 \\ &\quad + \langle (C + D)d_1^2 \rangle Q_2^2 + \langle D(d_1d_2)^2 \rangle (Q_1Q_2)^2 \\ &\quad - 2[\langle Dd_1d_2 \rangle + \langle Ddd_1d_2 \rangle F_2Q] Q_1Q_2 - \langle C + D \rangle (F_1 + F_2 - 2) \\ &\quad - 2[\langle (C + D)d \rangle + \langle Dd \rangle] Q + p_R(F_1F_2 - 1). \end{aligned}$$

Under notations

$$\begin{aligned} \alpha &:= \langle (C + D)d^2 \rangle, & \alpha_1 &:= \langle (C + D)d_1^2 \rangle, & \alpha_2 &:= \langle (C + D)d_2^2 \rangle, \\ \beta &:= \langle D(d_1d_2)^2 \rangle, & \gamma &:= \langle Dd^2 \rangle, & \phi &:= \langle Ddd_1d_2 \rangle, \\ \mu &:= \langle (C + D)d \rangle, & \nu &:= \langle Dd \rangle, & \chi &:= \langle Dd_1d_2 \rangle \end{aligned}$$

and setting

$$F := F_2 = (F_1)^{-1},$$

the second of the equilibrium equations (3.1) takes the form

$$(5.4) \quad \begin{aligned} (\alpha + \gamma F^2)FQ - \phi F^2 Q_1 Q_2 + (1 - F)(\mu - \nu F) &= 0, \\ (\alpha_1 Q_2 + \beta)Q_2 Q_1^2 - (\chi + \phi FQ)Q_1 &= 0, \\ (\alpha_2 Q_1 + \beta)Q_1 Q_2^2 - (\chi + \phi FQ)Q_2 &= 0. \end{aligned}$$

At the beginning let us consider two special cases.

First, assume that the laminae are not reinforced. In this case $C(\cdot)$ and $D(\cdot)$ are independent of X^2 and

$$\begin{aligned} \phi &:= \langle Ddd_1 d_2 \rangle = \langle Ddd_1 \rangle \langle d_2 \rangle = 0, \\ \chi &:= \langle Dd_1 d_2 \rangle = \langle Dd_1 \rangle \langle d_2 \rangle = 0, \end{aligned}$$

because of $\langle d_2 \rangle = 0$. In this case there exists the unique solution to Eqs. (5.4) given by

$$(5.5) \quad Q = \frac{(F - 1)(\mu - \nu F)}{(\alpha + \gamma F^2)F}, \quad Q_1 = Q_2 = 0.$$

Second, let the body be homogeneous. Then, apart from conditions $\phi = \chi = 0$, we also obtain $\mu = 0$ and $\nu = 0$. In this case $Q = Q_1 = Q_2 = 0$ and by means of Eqs. (5.1), an arbitrary uniform biaxial strain, given by $F_1 = F^{-1}$, $F_2 = F$, holds for every $F > 0$.

Now we shall pass to the general case of the micro-periodic body under consideration. In order to investigate the bifurcation of a micro-equilibrium state $(F_1, F_2, Q, 0, 0)$ let us assume that $Q_1 = \varepsilon' Q_1$, $Q_2 = \varepsilon' Q_2$, where $\varepsilon \rightarrow 0$. Let us also denote

$$\delta := \chi - \sqrt{\alpha_1 \alpha_2}, \quad \bar{\delta} := \chi + \sqrt{\alpha_1 \alpha_2}$$

and assume that $\delta \bar{\delta} \neq 0$. If $\phi \neq 0$ then the non-zero solutions Q_1, Q_2 to the second and third of Eqs. (5.4) exist either if

$$(5.6) \quad Q = -\frac{\delta}{\phi F}$$

or if

$$(5.7) \quad Q = -\frac{\bar{\delta}}{\phi F}.$$

The two aforementioned conditions will be treated separately.

Substituting the right-hand side of Eq. (5.6) into the first of Eqs. (5.4) (where now $Q_1 Q_2 = \varepsilon^2 {}'Q_1 {}'Q_2 \rightarrow 0$) we arrive at

$$(5.8) \quad (\nu \phi - \gamma \delta)F^2 - \phi(\mu + \nu)F + \mu \phi - \alpha \delta = 0.$$

The above equation together with the condition $F > 0$ represent the solution in which the bifurcation of a micro-equilibrium state $(F^{-1}, F, Q, 0, 0)$, where $Q = -\delta(\phi F)^{-1}$, takes place. Now assume

$$(5.9) \quad \nu\phi - \gamma\delta \neq 0$$

and define

$$(5.10) \quad b := \frac{\phi(\mu + \nu)}{\gamma\delta - \nu\phi}, \quad c := \frac{\alpha\delta - \mu\phi}{\gamma\delta - \nu\phi}.$$

It can be shown that the bifurcation can take place in the following cases:

(i) If $c < 0$ and $1 + b + c < 0$ then there exists one positive root $F = F_E$ of Eq. (5.8) such that $F_E > 1$. In this case the bifurcation occurs under extension of the body along X^2 -axis.

(ii) If $c < 0$ and $1 + b + c < 0$ then there exists one positive root $F = F_C$ of Eq. (5.8) satisfying condition $0 < F_C < 1$ and the bifurcation occurs under compression of the body along X^2 -axis.

(iii) If $c > 0$ and $1 + b + c < 0$ then there exist two positive roots $F = F_C$, $F = F_E$ of Eq. (5.8) related to the compression and extension of the body along X^2 -axis, respectively, i.e., $0 < F_C < 1$ and $F_E > 1$.

(iv) If $\nu\phi - \gamma\delta = 0$ and

$$F = \frac{\mu\phi - \alpha\delta}{\gamma\delta + \mu\phi},$$

then we obtain $F = F_E > 1$ if $(\delta/\phi)(\mu + \nu) > 0$ or $F = F_C$, $0 < F_C < 1$, if $(\delta/\phi)(\mu + \nu) < \min\{0, (\mu/\alpha)(\mu + \nu)\}$.

Let us also observe that since $\delta \neq 0$, $\alpha > 0$ and $\gamma > 0$, then the bifurcation cannot take place in the natural state in which $F = 1$.

If one from the above conditions takes place, then the value of Q for which the bifurcation occurs is determined by Eq. (5.6). The analysis similar to that given above can be carried out if the constant δ will be replaced by the constant $\bar{\delta}$. In this case the value of Q related to the bifurcation state will be determined by Eq. (5.7) and instead of parameters b, c , under condition

$$\nu\phi - \gamma\bar{\delta} \neq 0,$$

we shall introduce the parameters

$$\bar{b} := \frac{\phi(\mu + \nu)}{\gamma\bar{\delta} - \nu\phi}, \quad \bar{c} := \frac{\alpha\bar{\delta} - \mu\phi}{\gamma\bar{\delta} - \nu\phi}.$$

Hence the discussion of cases (i)–(iii) remains unchanged if moduli b, c will be replaced by moduli \bar{b}, \bar{c} , respectively. Similarly, in the case (iv) δ has to be

replaced by $\bar{\delta}$. It means that apart from values F_C, F_E of a macro-deformation gradients for which the bifurcation can take place, we also obtain two other values \bar{F}_C, \bar{F}_E related to the constants $\bar{\delta}, \bar{b}$ and \bar{c} , where $\bar{F}_C \in (0, 1)$, and $\bar{F}_E > 1$.

Now let us investigate the problem of the nonlocal (postbifurcation) microstability. To simplify the calculations let us assume $\alpha_1 = \alpha_2$ and denote $\alpha_0 = \alpha_1 = \alpha_2$. Using this assumption from Eqs. (5.4) we obtain either

$$(5.11) \quad \begin{aligned} (\alpha + \gamma F^2)FQ - \phi(FQ_1)^2 + (1 - F)(\mu - \nu F) &= 0, \\ \alpha_0 Q_1 + \beta Q_1^3 - (\chi + \phi FQ)Q_1 &= 0, \\ Q_2 &= Q_1, \end{aligned}$$

or

$$(5.12) \quad \begin{aligned} (\alpha + \gamma F^2)FQ + \phi(FQ_1)^2 + (1 - F)(\mu - \nu F) &= 0, \\ \alpha_0 Q_1 + \beta Q_1^3 + (\chi + \phi FQ)Q_1 &= 0, \\ Q_2 &= -Q_1. \end{aligned}$$

The two above cases have to be treated separately.

CASE 1. From Eqs. (5.11), apart from the solution

$$(5.13) \quad Q = \frac{(F - 1)(\mu - \nu F)}{(\alpha + \gamma F^2)F}, \quad Q_1 = Q_2 = 0,$$

which holds for every $F > 0$ (and coincides with that given by Eqs. (5.5)), we also obtain two other solutions

$$(5.14) \quad \begin{aligned} Q &= -\frac{(\beta\nu + \alpha_0\phi - \chi\phi)F^2 - \beta(\mu + \nu)F + \beta\mu}{[\alpha\beta + (\beta\gamma - \phi^2)F^2]F}, \\ Q_1^2 &= -\beta\frac{(\nu\phi - \gamma\delta)F^2 - \phi(\mu + \nu)F - \mu\phi + \alpha\delta}{\alpha\beta + (\beta\gamma - \phi^2)F^2}, \quad Q_2 = Q_1, \end{aligned}$$

where we have denoted $\delta := \chi - \alpha_0$.

CASE 2. From Eqs. (5.12), apart from the solution (5.13) which holds for every $F > 0$ we obtain two other solutions

$$(5.15) \quad \begin{aligned} Q &= -\frac{(\beta\nu - \alpha_0\phi - \chi\phi)F^2 - \beta(\mu + \nu)F + \beta\mu}{[\alpha\beta + (\beta\gamma - \phi^2)F^2]F}, \\ Q_1^2 &= \beta\frac{(\nu\phi - \gamma\bar{\delta})F^2 - \phi(\mu + \nu)F - \mu\phi - \alpha\bar{\delta}}{\alpha\beta + (\beta\gamma - \phi^2)F^2}, \quad Q_2 = -Q_1, \end{aligned}$$

where $\bar{\delta} := \chi + \alpha_0$.

Under assumption $\beta\gamma > \phi^2$ and using notations introduced above solutions (5.14) hold, for $F \in (0, F_C]$ and $F \geq F_E$, where $F_C < 1 < F_E$. If $F \in (F_C, F_E)$ then there exists solution given by Eqs. (5.13). At the same time solutions (5.15) hold for $F \in (0, \bar{F}_C]$ and $F \geq \bar{F}_E$, where $\bar{F}_C < 1 < \bar{F}_E$. If $F \in (\bar{F}_C, \bar{F}_E)$ the solution is given by Eqs. (5.13). It means that there can exist two kinds of bifurcations; in the first case after the bifurcation we obtain $Q_1 = Q_2$ and in the second $Q_1 = -Q_2$.

It has to be remembered that all the obtained results have the physical sense if and only if conditions (5.3) hold for every $\mathbf{X} \in \bar{V}_R$.

The micro-bifurcation cannot take place in materials for which either conditions $b^2 < 4c$ and $\bar{b}^2 < 4\bar{c}$ or conditions $b \leq 0$, $c \geq 0$ and $\bar{b} \leq 0$, $\bar{c} \geq 0$ hold. In this case there exist one micro-equilibrium path $(F_1, F_2, Q, \bar{Q}_1, \bar{Q}_2)$ in which $F_2 = F$, $F_1 = F^{-1}$ and Eqs. (5.5) hold for every $F > 0$.

To make the above example more clear from the physical viewpoint we have stated at the beginning of this section that the variables F_1 , F_2 as well as Q_1 , Q_2 , Q_3 are constant throughout the body. However, all investigations given above also hold true if the aforementioned variables are arbitrary regular macro-functions of $\mathbf{X} \in \Omega_R$. In this case we can also take into account the first of Eqs. (3.1) and after that pass to the analysis of the macro-stability of a body.

6. Conclusions

The obtained general relations concerning stability of highly-elastic periodic composites under finite deformations yield the analytical basis for calculations of different special problems. Following the general comments given at the end of Sec.3 we can mention here the problems of macro-stability and those of the local and non-local micro-stability. It can be seen that in the problems of macro-stability, after neglecting the effect of the microstructure length dimension on the dynamic behaviour of the body, the obtained formulae are similar to those of the nonlinear elasticity of homogeneous bodies. Under this approximation terms involving l^2 drop out from Eqs.(3.8), (3.9). Hence the first new result is the investigation of the effect of the microstructure length parameter l on the dynamic macro-stability of the body. The second new result is the existence of the local and non-local micro-stability in highly-elastic composites. This phenomenon is due to the micro-periodic material structure of the body and was illustrated in Sec.5. More general applications of the obtained results are under consideration and will be presented in a separate paper.

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