

Boundary element method to the study of a Stokes flow past an obstacle in a channel

M. KOHR (CLUJ-NAPOCA)

IN THIS PAPER the author gives an integral representation for the stream function and for the vorticity, corresponding to the problem of the Stokes flow past an obstacle in a channel. Using the Green's functions of the biharmonic equation and of the Laplace equation for the infinite horizontal strip, the above problem is reduced to a set of integral equations on the boundary of the obstacle. The boundary element method is used to solve these integral equations. The numerical results are given for the case of a circular obstacle.

1. Introduction

IN THIS PAPER we describe a semi-direct boundary integral method which is used to the study of a two-dimensional Stokes flow in a wind tunnel past a rigid obstacle. To derive the set of boundary integral equations for the stream function and the flow vorticity, we construct the Green function of the biharmonic equation in an arbitrary simply connected domain. As a consequence, we obtain the Green function of the strip or of the half-plane.

The derived integral equations, which are valid in any point of the flow domain, are applied at the boundary of the domain resulting in a system of two scalar Fredholm integral equations on the boundary obstacle only for the stream function and vorticity. In fact, these equations represent the boundary integral formulation of our problem.

It should be noted that G. BEZINE and D. BONNEAU [1] presented an alternative boundary integral representation for the stream function in terms of boundary distributions of the velocity, shear stress, and the normal derivative of the vorticity, corresponding to a two-dimensional Stokes flow. Also, C.J. COLEMAN [3] has developed a semi-direct boundary integral representation in complex variables, using the stream function and the Airy stress function for the study of a plane creeping viscous flow.

Let us remark that a direct boundary-integral method for the solution of Stokes equations in an arbitrary two-dimensional domain was given by J.J.L. HIGDON [7]. He used the fundamental solution of the Stokes equation and he obtained a representation of the flow in terms of the velocity, the pressure and the stress tensor, respectively.

A nice direct method of integral equations was recently proposed by L. DRAGOȘ and A. DINU [4] for the study of a subsonic flow with circulation past thin airfoils

in a wind tunnel. A semi-direct boundary element method in the study of an incompressible flow in a channel was also applied by A. CARABINEANU and A. DINU [2]. They used the stream function.

2. Mathematical formulation

A fluid flow of velocity $\mathbf{U}_\infty = (Uy(l-y), 0)$, is placed between two walls being parallel to the Ox -axis. It is perturbed by the presence of an obstacle Ω_i , with the boundary Γ . We determine the perturbation produced and the hydrodynamic forces acting on the obstacle. We suppose that the walls, denoted by L_1 and L_2 , have the equations:

$$\begin{aligned} L_1 &= \{(x, y) \mid x \in \mathbb{R}, y = 0\}, \\ L_2 &= \{(x, y) \mid x \in \mathbb{R}, y = l\}, \end{aligned}$$

where $l > 0$, and xOy is a Cartesian system of coordinates.

Also, the Reynolds number of the flow, denoted by Re , is supposed to be very small and hence the motion equations can be reduced to the creep equations and continuity equations, respectively (i.e. the Stokes equations):

$$(2.1) \quad \begin{aligned} \operatorname{div} \mathbf{v}(x) &= 0, & x \in D, \\ \operatorname{grad} P(x) - \mu \Delta \mathbf{v}(x) &= 0, & x \in D, \end{aligned}$$

where $\mathbf{v}(u, v)$ is the global fluid velocity, P the global pressure and μ the dynamic viscosity of the fluid. By D we denote the domain of the flow (Fig. 1).

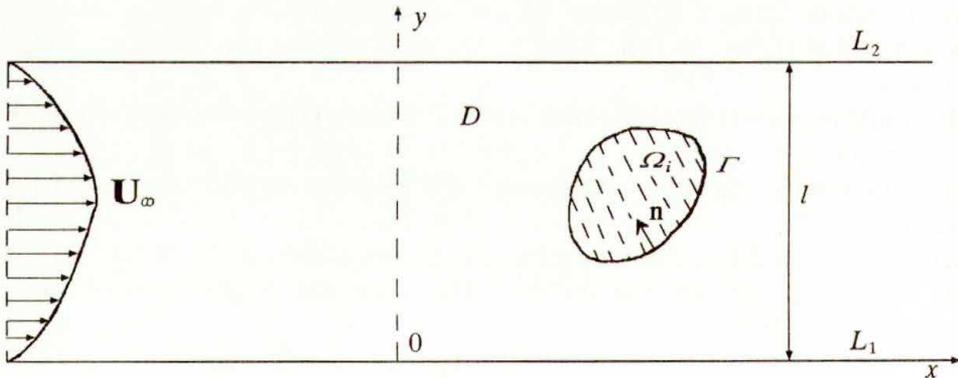


FIG. 1.

Using the stream function ψ , the above equations are reduced to the biharmonic equation

$$(2.2) \quad \Delta^2 \psi = 0$$

with the following boundary conditions:

$$(2.3) \quad \psi|_{L_1} = 0, \quad \psi|_{L_2} = C, \quad \psi|_{\Gamma} = b,$$

where C and b are unknown constants.

We have the following asymptotic conditions at infinity:

$$(2.4) \quad \lim_{|x| \rightarrow \infty} \left(\psi(x, y) - Uy^2 \left(\frac{l}{2} - \frac{y}{3} \right) \right) = 0, \quad \frac{\partial \psi}{\partial x} = 0,$$

$$\frac{\partial \psi}{\partial y} = Uy(l - y), \quad \text{as } |x| \rightarrow \infty.$$

After a simple analysis, we deduce that the rate of the flow in the channel, denoted by Q , is given by

$$(2.5) \quad Q = C.$$

On the other hand, from the boundary condition (2.3)₂, we obtain

$$(2.6) \quad C = Q = \frac{Ul^3}{6}.$$

Let us now denote by ϕ the stream function of the perturbation flow. Using the form of the stream function at infinity, we obtain that the global stream function can be written as:

$$(2.7) \quad \psi(x, y) = Uy^2 \left(\frac{l}{2} - \frac{y}{3} \right) + \phi(x, y).$$

The perturbation will be evaluated from the biharmonic equation

$$(2.8) \quad \Delta^2 \phi(x, y) = 0 \quad \text{in } \Omega,$$

with the boundary conditions:

$$(2.9) \quad \phi|_{L_1} = \phi|_{L_2} = 0,$$

$$\phi|_{\Gamma} = b - Uy^2 \left(\frac{l}{2} - \frac{y}{3} \right) \Big|_{\Gamma}$$

and the asymptotic conditions at infinity

$$(2.10) \quad \lim_{|x| \rightarrow \infty} \phi(x, y) = \lim_{|x| \rightarrow \infty} \text{grad } \phi(x, y) = 0.$$

3. The Green function of the biharmonic equation in an arbitrary simply connected domain

Let D be a simply connected domain in the (z) plane, $z = x + iy$, with the boundary C , and let $w = f(z_0, z)$ be the conformal mapping of the domain D onto the domain $|w| < 1$, in the w plane, such that the fixed point $z_0 \in D$ is mapped in $w = 0$.

We determine the function $G(M_0, M)$, where the points M_0 and M correspond to z_0 and z , with the following conditions:

- a) $\Delta_M^2 G(M_0, M) = 0$, for $M \neq M_0$,
- b) in the neighbourhood of the point M_0 , G has the representation

$$G(M_0, M) = \frac{1}{8\pi} |M_0 M|^2 [\ln |M_0 M| - 1] + g(M_0, M),$$

where the function $g(M_0, M)$ is a biharmonic function with respect to the point M , throughout the domain D ; and

- c) $G(M_0, M) = 0$.

The following theorem determines the function G .

THEOREM. *The function G is given by*

$$(3.1) \quad G(M_0, M) = \frac{1}{8\pi} |z - z_0|^2 \ln |f(z_0, z)|.$$

P r o o f. We prove that the function defined by (3.1) satisfies the conditions a), b), c). Because the function $w = f(z_0, z)$ defines a conformal mapping between D and the unit disc, then it is an analytic function, with $f(z, z_0) \neq 0$ for $z \neq z_0$.

Also the function $\log f(z_0, z) = \ln |f(z_0, z)| + i \arg f(z_0, z)$ is analytic in the domain D , with the exception of the point z_0 . The function $\ln |f(z_0, z)| = \operatorname{Re} \log f(z_0, z)$ is a harmonic function and hence G given by (3.1), satisfy the condition a). Since $f'(z, z_0) \neq 0$ in the domain D including the point $z = z_0$ and $f(z_0, z_0) = 0$, the point z_0 is a first order zero of the function f . Then, in a neighbourhood of this point we have:

$$(3.2) \quad f(z, z_0) = (z - z_0)\varphi(z, z_0),$$

where $\varphi(z, z_0)$ is an analytic function in the respective neighbourhood of z_0 , and $\varphi(z, z_0) \neq 0$. So

$$G(M_0, M) = \frac{1}{8\pi} |z - z_0|^2 [\ln |z - z_0| - 1] + \frac{1}{8\pi} |z - z_0|^2 \ln e|\varphi(z, z_0)|,$$

and the last function is denoted by $g(M_0, M)$. The condition b) is also satisfied.

Since $f(z, z_0)|_C = 1$, from (3.1) follows the condition c).

COROLLARY. The Green function of the biharmonic equation in the domain $\Omega = \{(x, y) \mid x \in \mathbb{R}, 0 < y < l\}$ is given by

$$(3.3) \quad G(M_0, M) = \frac{1}{16\pi} [(x - x_0)^2 + (y - y_0)^2] \ln \frac{\operatorname{ch} \frac{\pi}{l}(x - x_0) - \cos \frac{\pi}{l}(y - y_0)}{\operatorname{ch} \frac{\pi}{l}(x - x_0) - \cos \frac{\pi}{l}(y + y_0)},$$

where $M_0(x_0, y_0)$ and $M(x, y)$ belong to $\overline{\Omega}$.

P r o o f. The conformal mapping of the domain Ω onto the interior of the circle $|w| < 1$, has the form

$$f(z_0, z) = \frac{\exp\left(\frac{\pi}{l}z\right) - \exp\left(\frac{\pi}{l}z_0\right)}{\exp\left(\frac{\pi}{l}z\right) - \exp\left(\frac{\pi}{l}\bar{z}_0\right)}.$$

Performing elementary computations and applying the above theorem, we obtain Eq.(3.3).

4. The integral representation of solution

We remark that the biharmonic equation $\Delta^2\phi = 0$ is equivalent to the following system:

$$(4.1) \quad \begin{aligned} \Delta\phi &= \omega, \\ \Delta\omega &= 0, \end{aligned}$$

where ω represents the vorticity of the perturbation flow.

In the preceding section we have determined the Green function G for the biharmonic operator in the infinite strip $\Omega = \{(x, y) \mid x \in \mathbb{R}, 0 < y < l\}$.

This function satisfies the following equation:

$$(4.2) \quad \Delta_q^2 G(p, q) = \delta(|p - q|), \quad \text{for } 0 < \eta < l,$$

where δ is the Dirac distribution, $p(x, y)$ is a variable point in Ω where the solution is sought, and $q(\xi, \eta)$ is a general point located on the boundary or in the domain Ω . From (3.3) we have

$$(4.3) \quad G(x, y; \xi, \eta) = \frac{1}{16\pi} [(x - \xi)^2 + (y - \eta)^2] \ln \frac{\operatorname{ch} \frac{\pi}{l}(x - \xi) - \cos \frac{\pi}{l}(y - \eta)}{\operatorname{ch} \frac{\pi}{l}(x - \xi) - \cos \frac{\pi}{l}(y + \eta)},$$

$$G(x, y; \xi, \eta) = \frac{1}{8\pi} [(x - \xi)^2 + (y - \eta)^2] \ln \sqrt{\frac{\operatorname{sh}^2 \frac{\pi}{2l}(x - \xi) + \sin^2 \frac{\pi}{2l}(y - \eta)}{\operatorname{sh}^2 \frac{\pi}{2l}(x - \xi) + \sin^2 \frac{\pi}{2l}(y + \eta)}}.$$

The Green function F of the Laplace operator in the strip Ω satisfies the following equation:

$$(4.4) \quad \Delta_q F(p, q) = \delta(|p - q|), \quad \text{for } 0 < \eta < l$$

and is given by (see [2]):

$$(4.5) \quad F(x, y; \xi, \eta) = \frac{1}{4\pi} \ln \frac{\operatorname{ch} \frac{\pi}{l}(x - \xi) - \cos \frac{\pi}{l}(y - \eta)}{\operatorname{ch} \frac{\pi}{l}(x - \xi) - \cos \frac{\pi}{l}(y + \eta)},$$

$$F(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \sqrt{\frac{\operatorname{sh}^2 \frac{\pi}{2l}(x - \xi) + \sin^2 \frac{\pi}{2l}(y - \eta)}{\operatorname{sh}^2 \frac{\pi}{2l}(x - \xi) + \sin^2 \frac{\pi}{2l}(y + \eta)}}.$$

Using Green's identity for the functions ϕ and $\Delta_q G$, ω and G , and for the functions ω and F , we obtain the following integral representations:

$$(4.6) \quad \phi(p) = \int_{\partial D} \left[\phi(q) \frac{\partial(\Delta_q G)}{\partial n_q}(p, q) - \Delta_q G(p, q) \frac{\partial \phi}{\partial n_q}(q) \right] ds_q$$

$$+ \int_{\partial D} \left[\omega(q) \frac{\partial G(p, q)}{\partial n_q} - G(p, q) \frac{\partial \omega(q)}{\partial n_q} \right] ds_q, \quad p \in D,$$

$$\omega(p) = \int_{\partial D} \left[\omega(q) \frac{\partial F}{\partial n_q} - F \frac{\partial \omega}{\partial n_q} \right] ds_q, \quad p \in D,$$

where D is the domain of the flow, exterior to the obstacle Ω_i and enclosed by the walls L_0, L_1 .

By $\partial/\partial n_q$ we denote the differentiation with respect to the outward normal of D , in a point q of the boundary, denoted by ∂D .

We have satisfied the following properties:

$$(4.7) \quad F(x, 0; \xi, \eta) = F(x, l; \xi, \eta) = 0, \quad G(x, 0; \xi, \eta) = G(x, l; \xi, \eta) = 0,$$

$$(4.8) \quad \left. \frac{\partial F}{\partial n_q} \right|_{\partial \Omega} (x, 0; \xi, \eta) = \left. \frac{\partial F}{\partial n_q} \right|_{\partial \Omega} (x, l; \xi, \eta) = \left. \frac{\partial G}{\partial n_q} \right|_{\partial \Omega} (x, 0; \xi, \eta)$$

$$= \left. \frac{\partial G}{\partial n_q} \right|_{\partial \Omega} (x, l; \xi, \eta) = 0$$

and

$$(4.9) \quad \Delta_q G(x, 0; \xi, \eta) = \Delta_q G(x, l; \xi, \eta) = 0.$$

Also, we have

$$(4.9') \quad \frac{\partial \phi}{\partial n_q}(q) = 0 \quad \text{for } \eta = 0 \quad \text{or} \quad \eta = l.$$

Using the above properties and the asymptotic conditions at infinity (2.10), we derive the integral representation of solution, valid in any point of the flow domain:

$$(4.10) \quad \begin{aligned} \phi(p) = & \int_{\Gamma} \left[\omega(q) \frac{\partial G(p, q)}{\partial n_q} - G(p, q) \frac{\partial \omega(q)}{\partial n_q} \right] ds_q \\ & + \int_{\Gamma} \left[\phi(q) \frac{\partial}{\partial n_q} (\Delta_q G) - \Delta_q G(p, q) \frac{\partial \phi(q)}{\partial n_q} \right] ds_q \\ & + \int_{L_1 \cup L_2} \omega(q) \frac{\partial G}{\partial n_q}(p, q), \quad p \in D, \\ \omega(p) = & \int_{\Gamma} \left[\omega(q) \frac{\partial F(p, q)}{\partial n_q} - F(p, q) \frac{\partial \omega}{\partial n_q} \right] ds_q \\ & + \int_{L_1 \cup L_2} \omega(q) \frac{\partial F}{\partial n_q}(p, q), \quad p \in D, \end{aligned}$$

where

$$(4.11) \quad \begin{aligned} \frac{\partial F(p, q)}{\partial n_q} = & \frac{1}{4l} \left[\frac{\text{sh} \frac{\pi}{l}(x - \xi)n_1 - \sin \frac{\pi}{l}(y + \eta)n_2}{\text{ch} \frac{\pi}{l}(x - \xi) - \cos \frac{\pi}{l}(y + \eta)} \right. \\ & \left. - \frac{\text{sh} \frac{\pi}{l}(x - \xi)n_1 + \sin \frac{\pi}{l}(y - \eta)n_2}{\text{ch} \frac{\pi}{l}(x - \xi) - \cos \frac{\pi}{l}(y - \eta)} \right], \\ \frac{\partial G(p, q)}{\partial n_q} = & -\frac{1}{8\pi} [(x - \xi)n_1 + (y - \eta)n_2] \ln \frac{\text{ch} \frac{\pi}{l}(x - \xi) - \cos \frac{\pi}{l}(y - \eta)}{\text{ch} \frac{\pi}{l}(x - \xi) - \cos \frac{\pi}{l}(y + \eta)} \\ & - \frac{1}{16l} [(x - \xi)^2 + (y - \eta)^2] \left[\frac{\text{sh} \frac{\pi}{l}(x - \xi)n_1 - \sin \frac{\pi}{l}(y + \eta)n_2}{\text{ch} \frac{\pi}{l}(x - \xi) - \cos \frac{\pi}{l}(y + \eta)} \right. \\ & \left. - \frac{\text{sh} \frac{\pi}{l}(x - \xi)n_1 + \sin \frac{\pi}{l}(y - \eta)n_2}{\text{ch} \frac{\pi}{l}(x - \xi) - \cos \frac{\pi}{l}(y - \eta)} \right]. \end{aligned}$$

Now, we suppose that the rigid obstacle denoted by Ω_i with $\partial\Omega_i = \Gamma$, is fixed. The physical implication of vanishing of the fluid velocity on the boundary Γ provides that there is no tangential velocity on Γ , hence

$$(4.12) \quad \left. \frac{\partial \psi}{\partial n} \right|_{\Gamma} = 0,$$

and so

$$\left. \frac{\partial \phi}{\partial n} \right|_{\Gamma} = -Uy(l-y) \left. \frac{\partial y}{\partial n} \right|_{\Gamma}.$$

If we use the Green's identity in the domain Ω_i , we obtain that the second integral term in (4.10)₁ is given by

$$\begin{aligned} \int_{\Gamma} \left[\left(b - U\eta^2 \left(\frac{l}{2} - \frac{\eta}{3} \right) \right) \frac{\partial(\Delta G)}{\partial n_q} + U\Delta G \frac{\partial \eta}{\partial n_q} \right] ds_q \\ = -U \int_{\Gamma} \left[(2\eta - l) \frac{\partial G}{\partial n_q}(p, q) - G(p, q) \frac{\partial}{\partial n_q} (2\eta - l) \right] ds_q. \end{aligned}$$

We remark that we must satisfy the boundary conditions (2.9). Using the properties (4.7)–(4.9), it is easy to show that for $p \in D \rightarrow p_0 \in L_1$ or L_2 , we obtain the equality with zero on the two sides of (4.10)₁. Using the Plemelj's formula (see [6]) and the equation (4.10)₂, we deduce the equality: $\omega(p_0) = \omega(p_0)$, for all $p_0 \in L_1$ or L_2 . For $p \rightarrow p_0 \in \Gamma$, from (4.10), we obtain a set of two equations with four unknowns: ω and $(\partial\omega/\partial n_q)$ on Γ , and ω on $L_1 \cup L_2$. Then we impose the following arbitrary condition on the walls:

$$(4.13) \quad \omega(x, 0) = \omega(x, l) = 0, \quad \forall x \in \mathbb{R}.$$

From (4.10)–(4.13), we obtain the following integral representation on the boundary Γ only:

$$(4.14) \quad \begin{aligned} \phi(p) &= \int_{\Gamma} \left[\omega(q) \frac{\partial G(p, q)}{\partial n_q} - G(p, q) \frac{\partial \omega(q)}{\partial n_q} \right] ds_q \\ &\quad - U \int_{\Gamma} \left[(2\eta - l) \frac{\partial G}{\partial n_q}(p, q) - G(p, q) \frac{\partial}{\partial n_q} (2\eta - l) \right] ds_q, \quad p \in \Gamma, \\ \omega(p) &= \int_{\Gamma} \left[\omega(q) \frac{\partial F(p, q)}{\partial n_q} - F(p, q) \frac{\partial \omega(q)}{\partial n_q} \right] ds_q, \quad p \in \Gamma. \end{aligned}$$

The integrals which appear at the right-hand side of Eqs.(4.14) can be understood as a principal value in Cauchy' sense.

Because the fluid pressure P must be 2π -periodic around the obstacle Ω , we require $\int_{\Gamma} \frac{\partial p}{\partial t} ds = 0$, where $\partial/\partial t$ represents the differentiation with respect to the unit tangent vector of Γ . If we use the property that the functions ω and p are harmonically conjugate (see [6]), then we obtain the equation

$$(4.15) \quad \int_{\Gamma} \frac{\partial \omega}{\partial n} ds = 0.$$

5. Discretization of the integral equations

If $p \in \Gamma$, from Eqs. (4.14) and (4.15) we obtain the following Fredholm integral system:

$$(5.1) \quad \begin{aligned} b - Uy^2 \left(\frac{l}{2} - \frac{y}{3} \right) &= \int'_{\Gamma} \left[\omega(q) \frac{\partial G(p, q)}{\partial n_q} - G(p, q) \frac{\partial \omega(q)}{\partial n_q} \right] ds_q \\ &\quad - U \int'_{\Gamma} \left[(2\eta - l) \frac{\partial G}{\partial n_q}(p, q) - G(p, q) \frac{\partial}{\partial n_q} (2\eta - l) \right] ds_q, \\ \frac{1}{2} \omega(p) &= \int'_{\Gamma} \left[\omega(q) \frac{\partial F(p, q)}{\partial n_q} - F(p, q) \frac{\partial \omega(q)}{\partial n_q} \right] ds_q, \\ \int_{\Gamma} \frac{\partial \omega(q)}{\partial n_q} ds_q &= 0, \end{aligned}$$

where the symbol $'$ means the principal value in Cauchy's sense of the integral. For simplicity, this symbol will be omitted.

Our unknowns are the functions ω , $\partial\omega/\partial n$ on Γ and the constant b .

In order to reduce the integral system (5.1) to an algebraic system, we use the collocation method. The contour Γ is approximated by a polygonal line determined by the segments Γ_j ($j = \overline{1, N}$), and it is supposed that the midpoints $M_j(x_j^*, y_j^*)$ of these segments are representative. Assuming the discretization equations (5.1) to be satisfied for $(x, y) = (x_i^*, y_i^*)$, $i = \overline{1, N}$, we obtain the following linear system:

$$(5.2) \quad \begin{aligned} b - Uy_i^* \left(\frac{l}{2} - \frac{y_i^*}{3} \right) &= \sum_{j=1}^N \omega_j A_{ij} + \sum_{j=1}^N \left(\frac{\partial \omega}{\partial n} \right)_j B_{ij} \\ &\quad - U \sum_{j=1}^N (2y_j^* - l) A_{ij} + 2U \sum_{j=1}^N n_{2j} B_{ij}, \end{aligned}$$

$$(5.2) \quad \sum_{j=1}^N \left(\frac{\partial \omega}{\partial n} \right)_j \int_{\Gamma_j} ds_q = 0,$$

[cont.]

$$\frac{1}{2} \omega_i = \sum_{j=1}^N \omega_j C_{ij} + \sum_{j=1}^N \left(\frac{\partial \omega}{\partial n} \right)_j D_{ij}, \quad i = \overline{1, N},$$

where $\omega_i = \omega(x_i^*, y_i^*)$, $\left(\frac{\partial \omega}{\partial n} \right)_i = \frac{\partial \omega}{\partial n}(x_i^*, y_i^*)$, and $\mathbf{n}_2(x_i^*, y_i^*) = (n_1^i, n_2^i)$ is the normal unit vector to the segment Γ_i .

The coefficients of the above system are given by:

$$(5.3) \quad \begin{aligned} A_{ij} &= \frac{1}{8\pi} \int_{\Gamma_j} \left[(x_i^* - \xi) n_1^j + (y_i^* - \eta) n_2^j \right] \ln \frac{\operatorname{ch} \frac{\pi}{l}(x_i^* - \xi) - \cos \frac{\pi}{l}(y_i^* - \eta)}{\operatorname{ch} \frac{\pi}{l}(x_i^* - \xi) - \cos \frac{\pi}{l}(y_i^* + \eta)} ds_q \\ &\quad - \frac{1}{16l} \int_{\Gamma_j} \left[(x_i^* - \xi)^2 + (y_i^* - \eta)^2 \right] \left[\frac{\operatorname{sh} \frac{\pi}{l}(x_i^* - \xi) n_1^j - \sin \frac{\pi}{l}(y_i^* + \eta) n_2^j}{\operatorname{ch} \frac{\pi}{l}(x_i^* - \xi) - \cos \frac{\pi}{l}(y_i^* + \eta)} \right. \\ &\quad \left. - \frac{\operatorname{sh} \frac{\pi}{l}(x_i^* - \xi) n_1^j - \sin \frac{\pi}{l}(y_i^* - \eta) n_2^j}{\operatorname{ch} \frac{\pi}{l}(x_i^* - \xi) - \cos \frac{\pi}{l}(y_i^* - \eta)} \right] ds_q, \\ B_{ij} &= -\frac{1}{16\pi} \int_{\Gamma_j} \left[(x_i^* - \xi)^2 + (y_i^* - \eta)^2 \right] \ln \frac{\operatorname{ch} \frac{\pi}{l}(x_i^* - \xi) - \cos \frac{\pi}{l}(y_i^* - \eta)}{\operatorname{ch} \frac{\pi}{l}(x_i^* - \xi) - \cos \frac{\pi}{l}(y_i^* + \eta)} ds_q, \\ C_{ij} &= \frac{1}{4l} \int_{\Gamma_j} \left[\frac{\operatorname{sh} \frac{\pi}{l}(x_i^* - \xi) n_1^j - \sin \frac{\pi}{l}(y_i^* + \eta) n_2^j}{\operatorname{ch} \frac{\pi}{l}(x_i^* - \xi) - \cos \frac{\pi}{l}(y_i^* + \eta)} \right. \\ &\quad \left. - \frac{\operatorname{sh} \frac{\pi}{l}(x_i^* - \xi) n_1^j - \sin \frac{\pi}{l}(y_i^* - \eta) n_2^j}{\operatorname{ch} \frac{\pi}{l}(x_i^* - \xi) - \cos \frac{\pi}{l}(y_i^* - \eta)} \right] ds_q, \\ D_{ij} &= -\frac{1}{4\pi} \int_{\Gamma_j} \ln \frac{\operatorname{ch} \frac{\pi}{l}(x_i^* - \xi) - \cos \frac{\pi}{l}(y_i^* + \eta)}{\operatorname{ch} \frac{\pi}{l}(x_i^* - \xi) - \cos \frac{\pi}{l}(y_i^* - \eta)} ds_q, \quad i, j = \overline{1, N}. \end{aligned}$$

To evaluate the above integrals, we denote by (x_1^j, y_1^j) and (x_2^j, y_2^j) the coordinates of the ends of segment Γ_j , in the order leaving the inside of the obstacle to the right. Then Γ_j will be parametrized by taking (see for example, [5]):

$$(5.4) \quad x = x_j^* + \frac{x_2^j - x_1^j}{2} t, \quad y = y_j^* + \frac{y_2^j - y_1^j}{2} t, \quad t \in [-1, 1],$$

where $x_j^* = (x_1^j + x_2^j)/2$, $y_j^* = (y_1^j + y_2^j)/2$ are the coordinates of the midpoints of the segment Γ_j , $j = \overline{1, N}$.

From Eqs.(5.4), it follows that $ds = (L/2)dt$, where L is the length of the segment Γ_j , given by

$$(5.5) \quad L = \sqrt{(x_2^j - x_1^j)^2 + (y_2^j - y_1^j)^2}.$$

The coordinates of the unit vector \mathbf{n}^j will be calculated as follows:

$$(5.5') \quad \mathbf{n}^j = \left(\frac{y_2^j - y_1^j}{L}, -\frac{x_2^j - x_1^j}{L} \right).$$

For $i = j$ we obtain:

$$(5.6) \quad \begin{aligned} A_{ii} &= \frac{1}{8\pi} \int_{\Gamma_i} [(x_i^* - \xi)n_1^i + (y_i^* - \eta)n_2^i] \ln \left[\operatorname{ch} \frac{\pi}{l}(x_i^* - \xi) - \cos \frac{\pi}{l}(y_i^* + \eta) \right] ds_q \\ &+ \frac{1}{16l} \int_{\Gamma_i} \left[\frac{\operatorname{sh} \frac{\pi}{2l}(x_i^* - \xi) \operatorname{ch} \frac{\pi}{2l}(x_i^* - \xi)n_1^i - \sin \frac{\pi}{2l}(y_i^* + \eta) \cos \frac{\pi}{2l}(y_i^* + \eta)n_2^i}{\operatorname{sh}^2 \frac{\pi}{2l}(x_i^* - \xi) + \sin^2 \frac{\pi}{2l}(y_i^* + \eta)(y_i^* + \eta)} \right] \\ &\quad \cdot [(x_i^* - \xi)^2 + (y_i^* - \eta)^2] ds_q, \\ B_{ii} &= \frac{1}{16\pi} \int_{\Gamma_i} [(x_i^* - \xi)^2 + (y_i^* - \eta)^2] \ln \left[\operatorname{ch} \frac{\pi}{l}(x_i^* - \xi) - \cos \frac{\pi}{l}(y_i^* + \eta) \right] ds_q \\ &\quad - \frac{1}{32\pi} \int_{-1}^1 t^2 \ln \left[\operatorname{ch} \frac{\pi}{2l}(x_1^i - x_2^i)t - \cos \frac{\pi}{2l}(y_1^i - y_2^i)t \right] dt, \\ C_{ii} &= -\frac{1}{4l} \int_{\Gamma_i} \left[\frac{-n_1^i \operatorname{sh} \frac{\pi}{2l}(x_i^* - \xi) \operatorname{ch} \frac{\pi}{2l}(x_i^* - \xi)n_1}{\operatorname{sh}^2 \frac{\pi}{2l}(x_i^* - \xi) + \sin^2 \frac{\pi}{2l}(y_i^* + \eta)} \right. \\ &\quad \left. + \frac{n_2^i \sin \frac{\pi}{2l}(y_i^* + \eta) \cos \frac{\pi}{2l}(y_i^* + \eta)}{\operatorname{sh}^2 \frac{\pi}{2l}(x_i^* - \xi) + \sin^2 \frac{\pi}{2l}(y_i^* + \eta)} \right] ds_q, \\ D_{ii} &= -\frac{1}{2\pi} \int_{\Gamma_i} \ln \sqrt{\operatorname{sh}^2 \frac{\pi}{2l}(x_i^* - \xi) + \sin^2 \frac{\pi}{2l}(y_i^* + \eta)} ds_q \\ &\quad + \frac{1}{2\pi} L \ln \frac{\pi}{2l} - \frac{L}{2\pi} + \frac{L \ln \frac{L}{2}}{2\pi}. \end{aligned}$$

The coefficients (5.6) may be computed numerically, using the same technique as for the coefficients (5.2)–(5.3).

6. Numerical results

From (4.10), we can obtain the discretization form of the total stream function ψ in any point of the domain D :

$$(6.1) \quad \psi(p) = Uy^2 \left(\frac{l}{2} - \frac{y}{3} \right) + \sum_{j=1}^N \omega_j \int_{\Gamma_j} \frac{\partial G(p, q)}{\partial n_q} ds_q \\ - \sum_{j=1}^N \left(\frac{\partial \omega}{\partial n} \right)_j \int_{\Gamma_j} G(p, q) ds_q - U \sum_{j=1}^N (2y_j^* - l) \int_{\Gamma_j} \frac{\partial G}{\partial n_q}(p, q) ds_q \\ + U \sum_{j=1}^N n_{2j} \int_{\Gamma_j} G(p, q) ds_q, \quad p \in D.$$

Numerical computations of the method were performed for a fixed circular obstacle. It was considered that the circle had the center (X_0, Y_0) , $0 < Y_0 < l$ and the radius a . The maximum value chosen for N was 60. Also, we supposed that the segments Γ_j were of the same length.

The test of the method is given for the drag coefficient C_D , defined by:

$$(6.2) \quad C_D = \frac{1}{\frac{1}{2} \rho a^2 U^2 l^4} \int_{\Gamma} \{ \sigma_{nn}(q) \cos \theta(q) - \sigma_{tn}(q) \sin \theta(q) \} ds_q,$$

where $\theta(q)$ is the angle between the unit normal vector $\mathbf{n}(q)$ to the boundary Γ , and the positive Ox -axis. Symbols

$$(6.3) \quad \sigma_{nn} = -P - 2 \frac{\partial^2 \psi}{\partial t \partial n}, \quad \sigma_{tn} = - \frac{\partial^2 \psi}{\partial n^2} - \frac{\partial^2 \psi}{\partial t^2}, \quad \sigma_{tt} = -P + \frac{\partial^2 \psi}{\partial t \partial n}$$

denote the components of the stress tensor referred to the (\mathbf{t}, \mathbf{n}) axes.

From $(2.1)_1$ and using the following property (see [8]):

$$(6.4) \quad \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial s^2} - \frac{1}{a} \frac{\partial \psi}{\partial n},$$

we obtain the drag coefficient C_D in the form:

$$(6.5) \quad C_D = \frac{2}{\rho a^2 U^2 l^4} \int_{\Gamma} \eta \left(\frac{\partial \omega}{\partial n}(q) - \frac{1}{a} \omega(q) \right) ds_q - \frac{2Y_0}{\rho a^3 U^2} \int_{\Gamma} \omega(q) ds_q.$$

If we assume $\rho = 0.8$, $a = 1$, $l = 4$, $U = 1$, $X_0 = 0$, $Y_0 = 2$, then for $47 \leq N \leq 60$, we obtain the same value for the drag coefficient: $C_D = 7.8537$. Also, if we choose $\rho = 0.8$, $a = 1$, $l = 4$, $U = 2$, $X_0 = 0$, $Y_0 = 2$, and

$47 \leq N \leq 60$, it follows that $C_D = 3.92699$. These remarks show the extremely rapid convergence of the results when the number N of discretization elements increases. In the first case the constant b is equal to 0.5, for all $N \geq 47$, and in the last case b is equal to 0.3.

The Table 1 gives the values of the drag coefficients as the function of the velocity U , when $a = 1, l = 4, X_0 = 0, Y_0 = 2$. We observe that if the Reynolds number ($Re = (\rho a U l^2) / \mu$, ρ, μ are supposed to be fixed) increases, then the drag coefficient C_D decreases.

Table 1.

U	N	Drag coefficient C_D
1	60	7.8537861
1.5	60	5.2352625
2	60	3.9269824
4	60	0.4908738

Table 2.

ρ	N	Drag coefficient C_D
0.8	60	7.8537861
0.6	60	10.4717147
0.5	60	12.5660577

For $U = 1, a = 1, l = 4, X_0 = 0$ and $Y_0 = 2$, respectively, the Table 2 gives the values of the coefficient C_D for some values of the density ρ . Finally, Figs. 2 and 3 represent the spectrum of the flow in the case $a = 1, l = 4, X_0 = 0, Y_0 = 2$ and $Y_0 = 2.5$, respectively.

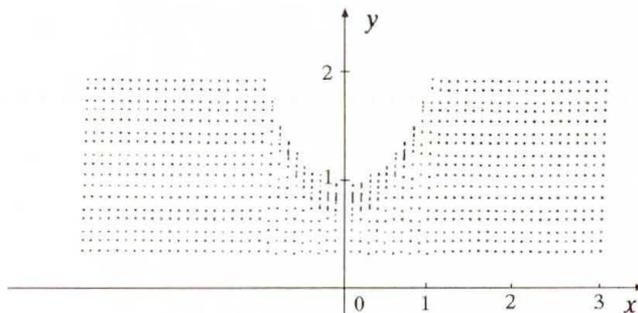


FIG. 2.

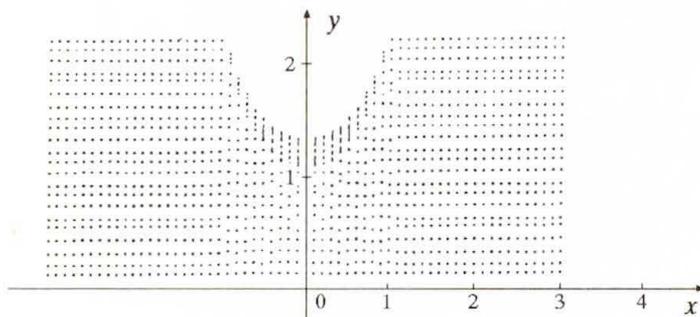


FIG. 3.

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“BABES-BOLYAI” UNIVERSITY
FACULTY OF MATHEMATICS,

1. M. Kogalniceanu str., 3400 CLUJ-NAPOCA, ROMANIA.

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