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A boundary integral equations method for asymmetric Stokes flow between two parallel planes

M. KOHR (CLUJ-NAPOCA)

IN THIS PAPER we apply a direct boundary integral equations method to Stokes flow past a smooth obstacle, occurring between two parallel planes. The problem is formulated exactly as a system of linear Fredholm integral equations of the second kind, over the surface of the obstacle. It is shown that this system has a unique continuous solution when the boundary of the particle is a Lyapunov surface and the velocity distributions on the same boundary is continuous. The numerical results are obtained by a standard boundary element method.

1. Introduction

THE MOTION OF A BODY of a simple shape in a viscous fluid between parallel planar boundaries, has been the subject of many studies. For example, P. GANATOS, R. PFEFFER and S. WEINBAUM (see [7]) gave a numerical method of analysis for the motion of an asymmetric Stokes flow between parallel planes induced by the rotary or translatory motion of a sphere. Also, a special case of the flow due to the rotation of a sphere, considered by the above mentioned authors, was investigated by W.W. HACKBORN (see [8]). This author presented an analytical method for the asymmetric Stokes flow between parallel planes due to a three-dimensional rotlet whose axis is supposed to be parallel to the planes. Using the periodic Green functions, C. POZRIKIDIS (see [14]) investigated the creeping flows in two-dimensional channels. L. DRAGOŞ and A. DINU (see [2, 3]) determined a direct boundary integral method for subsonic flows with circulation in two-dimensional channels.

The aim of this paper is to give a boundary integral method for an asymmetric Stokes flow between two parallel planes and in the presence of a rigid obstacle. The corresponding Green's functions are found. By using these functions, the velocity field is determined as a sum of a single-layer potential with a double-layer potential. The properties of the double-layer and single-layer operators secure the existence and uniqueness results of the solution.

2. Mathematical formulation

The configuration of an asymmetric Stokes flow of a viscous incompressible fluid, induced by the slow motion with the velocity $\mathbf{U}_0$, of an arbitrary rigid particle Ω^1, between two parallel, rigid planes $\mathcal{P}_0$ and $\mathcal{P}_1$, is illustrated in the Fig. 1. We suppose that S, surface of the particle, is a Lyapunov surface (see [9]).

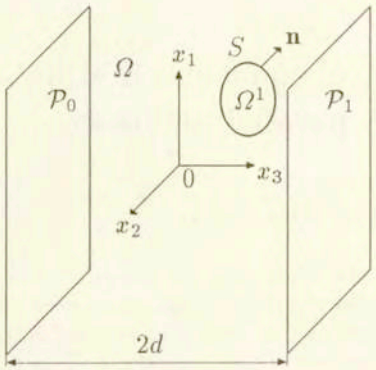

FIG. 1.

Let $\mathbf{U}_\infty$ and p_∞ be the velocity and pressure fields of an undisturbed Stokes flow in the domain with the boundaries $\mathcal{P}_0$ and $\mathcal{P}_1$. These planes $\mathcal{P}_0$, $\mathcal{P}_1$ have the dimensionless equations

$$\mathcal{P}_0 = \{(x_1, x_2, x_3) \in \mathbf{R}^3 |\, x_3 = -d\}, \qquad \mathcal{P}_1 = \{(x_1, x_2, x_3) \in \mathbf{R}^3 |\, x_3 = d\}, \tag{2.1}$$

where d is a positive constant. Also, the velocity $\mathbf{U}_\infty$ and the pressure p_∞ satisfy the following Stokes system of dimensionless equations

$$\Delta \mathbf{U}_\infty(x) - \nabla p_\infty(x) = 0, \qquad \text{for} \quad x = (x_1, x_2, x_3) \in \mathbf{R}^3, \quad |x_3| < d, \tag{2.2}$$

$$\nabla \cdot \mathbf{U}_\infty(x) = 0, \qquad \text{for} \quad x = (x_1, x_2, x_3) \in \mathbf{R}^3, \quad |x_3| < d, \tag{2.3}$$

with the nonslip boundary condition

$$\mathbf{U}_\infty(x) = 0, \qquad \text{for} \quad x \in \mathcal{P}_0 \ \text{ or } \ \mathcal{P}_1 \qquad (\text{i.e.} \quad x_3 = \pm d), \tag{2.4}$$

where Δ and ∇ are three-dimensional Laplace and the gradient operators.

Let $\mathbf{u}_1$, p_1 be the velocity field and the pressure, respectively, of the total flow, which results from the presence of the given obstacle. We denote by $\mathbf{u} = \mathbf{u}_1 - \mathbf{U}_\infty$, $p = p_1 - p_\infty$, the velocity and pressure fields of the disturbed flow. If we suppose that the Reynolds number of the flow $(\mathbf{u}_1, p_1)$ is very small, then the velocity $\mathbf{u}$ and the pressure p satisfy, as a first approximation, the following Stokes equations

$$\Delta \mathbf{u}(x) - \nabla p(x) = 0, \qquad \text{for} \quad x \in \Omega, \tag{2.5}$$

$$\nabla \cdot \mathbf{u}(x) = 0, \qquad \text{for} \quad x \in \Omega. \tag{2.6}$$

Here Ω is the domain exterior to the obstacle, with the boundaries S, $\mathcal{P}_0$ and $\mathcal{P}_1$.

The velocity field satisfies the nonslip boundary conditions

$$\mathbf{u}(x) = \mathbf{U}_0(x) - \mathbf{U}_\infty(x), \qquad \text{for} \quad x \in S, \tag{2.7}$$

$$\mathbf{u}(x) = 0, \qquad \text{for} \quad x \in \mathcal{P}_0 \ \text{ or } \ \mathcal{P}_1 \qquad (\text{i.e.} \quad x_3 = \pm d), \tag{2.8}$$

as well as the following conditions at infinity

$$\mathbf{u}(x) \to 0, \qquad p(x) \to 0, \qquad \text{as} \qquad |x| \to \infty, \tag{2.9}$$

where $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the usual Euclidean distance in $\mathbf{R}^3$, between the point x and the origin O of the fixed orthogonal system $Ox_1x_2x_3$.

3. Construction of Green's functions

Let $\mathbf{G}(G_{ij})$ and $\mathbf{q}(q_i)$ be the Green tensor and pressure vector of the Stokes equations (2.5), (2.6), in the infinite domain, with $\mathcal{P}_0$ and $\mathcal{P}_1$ as boundaries. Additionally, the Green function $\mathbf{G}$ becomes zero when its pole is located on any of the walls. Thus, the next equations and conditions are satisfied

$$\Delta_y G_{ij}(x,y) - \frac{\partial q_j}{\partial y_i}(x,y) = -\delta_{ij}\delta(x-y), \tag{3.1}$$

$$\sum_{i=1}^{3} \frac{\partial G_{ij}}{\partial y_i}(x,y) = 0, \qquad \text{for} \quad y = (y_1, y_2, y_3) \in \mathbf{R}^3, \quad |y_3| < d, \tag{3.2}$$

$$G_{ij}(x,y) = 0, \qquad \text{for} \quad y_3 = \pm d, \tag{3.3}$$

$$G_{ij}(x,y) \to 0, \qquad q_i(x,y) \to 0, \qquad \text{as} \quad |y| \to \infty, \tag{3.4}$$

where δ is the Dirac distribution and $\delta_{ij} = 1$, for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

The Green's functions $\mathbf{G}$ and $\mathbf{q}$ are determined in the following form:

$$\mathbf{G}(x,y) = \mathbf{E}(x-y) + \mathbf{D}(x,y), \qquad \mathbf{q}(x,y) = \mathbf{e}(x-y) + \mathbf{d}(x,y), \tag{3.5}$$

where $\mathbf{E}(E_{ij})$, $\mathbf{e}(e_i)$ are the fundamental solutions of the Stokes equations in the whole space and $\mathbf{D}(D_{ij})$, $\mathbf{d}(d_i)$ represent complementary functions, such that the null conditions (3.3) are satisfied.

In fact, $\mathbf{E}$ and $\mathbf{e}$ solve the following Stokes problem:

$$\Delta_y E_{ij}(x-y) - \frac{\partial e_j}{\partial y_i}(x-y) = -\delta_{ij}\delta(x-y), \tag{3.6}$$

$$\sum_{i=1}^{3} \frac{\partial E_{ij}}{\partial y_i}(x-y) = 0, \tag{3.7}$$

$$E_{ij}(x-y) \to 0, \qquad e_i(x-y) \to 0, \qquad \text{as} \quad |y| \to \infty. \tag{3.8}$$

The functions $\mathbf{D}$ and $\mathbf{d}$ are solutions of the Stokes equations and the conditions given below:

$$\Delta_y D_{ij}(x,y) - \frac{\partial d_j}{\partial y_i}(x,y) = 0, \qquad \text{for} \quad y \in \mathbf{R}^3, \quad |y_3| < d, \quad i,j = \overline{1,3}, \tag{3.9}$$

$$\sum_{i=1}^{3} \frac{\partial D_{ij}}{\partial y_i}(x,y) = 0, \qquad \text{for} \quad y \in \mathbf{R}^3, \quad |y_3| < d, \tag{3.10}$$

$$D_{ij}(x,y) = -E_{ij}(x-y), \qquad \text{for} \quad y \in \mathbf{R}^3, \quad y_3 = \pm d, \quad i,j = \overline{1,3}, \tag{3.11}$$

$$D_{ij}(x,y) \to 0, \qquad d_i(x,y) \to 0, \qquad \text{as} \quad |y| \to \infty. \tag{3.12}$$

The fundamental solutions $\mathbf{E}$ and $\mathbf{e}$ are given by (see [1])

$$E_{ij}(x-y) = \frac{1}{8\pi}\left[\frac{1}{|x-y|} + \frac{(x_i-y_i)(x_j-y_j)}{|x-y|^3}\right], \qquad i,j = \overline{1,3}, \tag{3.13}$$

$$e_i(x-y) = -\frac{1}{4\pi}\cdot\frac{x_i-y_i}{|x-y|^3}, \qquad i = \overline{1,3}. \tag{3.14}$$

We seek the regular parts $\mathbf{D}$ and $\mathbf{d}$ of $\mathbf{G}$ and $\mathbf{q}$ in the following form

$$D_{ij}(x,y) = \int\limits_{z_3=-d} E_{ik}(z-y)\alpha_k^{(j)}(x,z)\,ds_z + \int\limits_{z_3=d} E_{ik}(z-y)\beta_k^{(j)}(x,z)\,ds_z\,, \tag{3.15}$$

$$d_{ij}(x,y) = \int\limits_{z_3=-d} e_k(z-y)\alpha_k^{(j)}(x,z)\,ds_z + \int\limits_{z_3=d} e_k(z-y)\beta_k^{(j)}(x,z)ds_z\,. \tag{3.16}$$

It is easy to see that the functions $\mathbf{D}$ and $\mathbf{d}$ satisfy the Stokes equations (3.9), (3.10) and the asymptotic conditions (3.12). The unknown densities $\alpha_k^{(j)}$, $\beta_k^{(j)}$, $k,j = \overline{1,3}$ will be solutions of the following integral equations

$$\int\limits_{\mathbf{R}^2} E_{ik}(z_1-y_1, z_2-y_2, -d-y_3)\alpha_k^{(j)}(x,z)\,dz_1\,dz_2 + \int\limits_{\mathbf{R}^2} E_{ik}(z_1-y_1, z_2-y_2, d-y_3)\beta_k^{(j)}(x,z)\,dz_1\,dz_2 = -E_{ij}(x-y), \tag{3.17}$$

$$\text{for} \quad y \in \mathbf{R}^3, \quad y_3 = \pm d.$$

In order to solve this integral system, we apply the Fourier transform $\mathcal{F}$, with respect to the variables $y' = (y_1, y_2)$ (see [18])

$$\mathcal{F}\phi(\xi) = \frac{1}{(2\pi)^2}\int\limits_{\mathbf{R}^2} e^{-i\xi\cdot y'}\phi(y')\,dy_1\,dy_2, \qquad \xi = (\xi_1,\xi_2) \in \mathbf{R}^2, \tag{3.18}$$

$$\mathcal{F}^{-1}\psi(y') = \frac{1}{(2\pi)^2}\int\limits_{\mathbf{R}^2} e^{i\xi\cdot y'}\psi(\xi)\,d\xi_1\,d\xi_2, \qquad y' = (y_1,y_2) \in \mathbf{R}^2, \tag{3.19}$$

$\mathcal{F}^{-1}$ being the inverse Fourier transform.

From (3.15) and (3.16), we obtain

$$\mathcal{F}D_{ij}(\xi;x,y_3) = \mathcal{F}E_{ik}(\xi;0,-d-y_3)\mathcal{F}\alpha_k^{(j)}(\xi;x) + \mathcal{F}E_{ik}(\xi;0,d-y_3)\mathcal{F}\beta_k^{(j)}(\xi;x), \qquad i,j=\overline{1,3}, \tag{3.20}$$

$$\mathcal{F}d_j(\xi;x,y_3) = \mathcal{F}e_k(\xi;0,-d-y_3)\mathcal{F}\alpha_k^{(j)}(\xi;x) + \mathcal{F}e_k(\xi;0,d-y_3)\mathcal{F}\beta_k^{(j)}(\xi;x), \qquad j=\overline{1,3}. \tag{3.21}$$

On the other hand, from (3.17) we obtain the following linear system with the unknowns $\mathcal{F}\alpha_k^{(j)}(\xi;x)$ and $\mathcal{F}\beta_k^{(j)}(\xi;x)$, $k,j=\overline{1,3}$:

$$\mathcal{F}E_{ik}(\xi;0,0)\mathcal{F}\alpha_k^{(j)}(\xi;x) + \mathcal{F}E_{ik}(\xi;0,2d)\mathcal{F}\beta_k^{(j)}(\xi;x) = -\mathcal{F}E_{ij}(\xi;x',x_3+d), \tag{3.22}$$

$$\mathcal{F}E_{ik}(\xi;0,-2d)\mathcal{F}\alpha_k^{(j)}(\xi;x) + \mathcal{F}E_{ik}(\xi;0,0)\mathcal{F}\beta_k^{(j)}(\xi;x) = -\mathcal{F}E_{ij}(\xi;x',x_3-d), \tag{3.23}$$

$$i,j=\overline{1,3}, \qquad x'=(x_1,x_2)\in\mathbf{R}^2.$$

It is convenient to write $\mathbf{E}$ and $\mathbf{e}$ as follows (see [5]):

$$\mathbf{E}(x-y) = \frac{1}{8\pi}[\mathbf{I}\Delta_y\phi(x-y) - \nabla_y\nabla_y\phi(x-y)], \tag{3.24}$$

$$\mathbf{e}(x-y) = -\frac{1}{8\pi}\nabla_y\Delta_y\phi(x-y), \tag{3.25}$$

where

$$\phi(x-y) = |x-y| = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}. \tag{3.26}$$

By using the following properties of the Fourier transform (see [18])

$$\nabla^{(m)}(\mathcal{F}v) = \mathcal{F}((-ix)^m v), \qquad \mathcal{F}(\nabla^{(m)}v) = (i\xi)^m\mathcal{F}v, \tag{3.27}$$

with $m=\sum_{i=1}^{2} m_i$, $x^m = x_1^{m_1}x_2^{m_2}$ and $\xi^m = \xi_1^{m_1}\xi_2^{m_2}$, we obtain

$$\mathcal{F}\mathbf{E}(\xi;x',x_3-y_3) = -\frac{1}{4}e^{-i\xi\cdot x'}e^{-|\xi||x_3-y_3|}\cdot\mathbf{A}, \tag{3.28}$$

where the matrix $\mathbf{A}$ has the form (see, also [5])

$$\mathbf{A} = \begin{pmatrix} -\frac{2}{|\xi|}+\frac{\xi_1^2}{|\xi|^3}+|x_3-y_3|\frac{\xi_1^2}{|\xi|^2} & \frac{\xi_1\xi_2}{|\xi|^3}+|x_3-y_3|\frac{\xi_1\xi_2}{|\xi|^2} & -i(x_3-y_3)\frac{\xi_1}{|\xi|} \\ \frac{\xi_1\xi_2}{|\xi|^3}+|x_3-y_3|\frac{\xi_1\xi_2}{|\xi|^2} & -\frac{2}{|\xi|}+\frac{\xi_2^2}{|\xi|^3}+|x_3-y_3|\frac{\xi_2^2}{|\xi|^2} & -i(x_3-y_3)\frac{\xi_2}{|\xi|} \\ -i(x_3-y_3)\frac{\xi_1}{|\xi|} & -i(x_3-y_3)\frac{\xi_2}{|\xi|} & -\frac{1}{|\xi|}-|x_3-y_3| \end{pmatrix}. \tag{3.29}$$

In a similar manner, we obtain

$$\mathcal{F}\mathbf{e}(\xi; x', x_3 - y_3) = -\frac{1}{2} e^{-i\xi\cdot x'} e^{-|\xi|(x_3-y_3)} \left(\frac{i\xi_1}{|\xi|} \quad \frac{i\xi_2}{|\xi|} \quad \operatorname{sgn}(x_3 - y_3) \right). \tag{3.30}$$

We denote by $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{H}$ and $\mathbf{Q}$ the following matrices:

$$\mathbf{B} = -\frac{1}{4} \begin{pmatrix} -\frac{2}{|\xi|} + \frac{\xi_1^2}{|\xi|^3} & \frac{\xi_1\xi_2}{|\xi|^3} & 0 \\ \frac{\xi_1\xi_2}{|\xi|^3} & -\frac{2}{|\xi|} + \frac{\xi_2^2}{|\xi|^3} & 0 \\ 0 & 0 & -\frac{1}{|\xi|} \end{pmatrix}, \tag{3.31}$$

$$\mathbf{C} = -\frac{1}{4} e^{-2|\xi|} \begin{pmatrix} -\frac{2}{|\xi|} + \frac{\xi_1^2}{|\xi|^3} + 2d\frac{\xi_1^2}{|\xi|^2} & \frac{\xi_1\xi_2}{|\xi|^3} + 2d\frac{\xi_1\xi_2}{|\xi|^2} & -2di\frac{\xi_1}{|\xi|} \\ \frac{\xi_1\xi_2}{|\xi|^3} + 2d\frac{\xi_1\xi_2}{|\xi|^2} & -\frac{2}{|\xi|} + \frac{\xi_2^2}{|\xi|^3} + 2d\frac{\xi_2^2}{|\xi|^2} & -2di\frac{\xi_2}{|\xi|} \\ -2di\frac{\xi_1}{|\xi|} & -2di\frac{\xi_2}{|\xi|} & -\frac{1}{|\xi|} - 2d \end{pmatrix}, \tag{3.32}$$

$$\mathbf{D} = -\frac{1}{4} e^{-2|\xi|} \begin{pmatrix} -\frac{2}{|\xi|} + \frac{\xi_1^2}{|\xi|^3} + 2d\frac{\xi_1^2}{|\xi|^2} & \frac{\xi_1\xi_2}{|\xi|^3} + 2d\frac{\xi_1\xi_2}{|\xi|^2} & 2di\frac{\xi_1}{|\xi|} \\ \frac{\xi_1\xi_2}{|\xi|^3} + 2d\frac{\xi_1\xi_2}{|\xi|^2} & -\frac{2}{|\xi|} + \frac{\xi_2^2}{|\xi|^3} + 2d\frac{\xi_2^2}{|\xi|^2} & 2di\frac{\xi_2}{|\xi|} \\ 2di\frac{\xi_1}{|\xi|} & 2di\frac{\xi_2}{|\xi|} & -\frac{1}{|\xi|} - 2d \end{pmatrix}, \tag{3.33}$$

$$\mathbf{H} = -\frac{1}{4} e^{-i\xi\cdot x'} e^{-|\xi|(x_3+1)} \cdot \begin{pmatrix} -\frac{2}{|\xi|} + \frac{\xi_1^2}{|\xi|^3} + (x_3+d)\frac{\xi_1^2}{|\xi|^2} & \frac{\xi_1\xi_2}{|\xi|^2}\left(\frac{1}{|\xi|} + x_3 + d\right) & -i(x_3+d)\frac{\xi_1}{|\xi|} \\ \frac{\xi_1\xi_2}{|\xi|^2}\left(\frac{1}{|\xi|} + x_3 + d\right) & -\frac{2}{|\xi|} + \frac{\xi_2^2}{|\xi|^2}\left(\frac{1}{|\xi|} + x_3 + d\right) & -i(x_3+d)\frac{\xi_2}{|\xi|} \\ -i(x_3+d)\frac{\xi_1}{|\xi|} & -i(x_3+d)\frac{\xi_2}{|\xi|} & -\frac{1}{|\xi|} - (x_3+d) \end{pmatrix}, \tag{3.34}$$

$$\mathbf{Q} = -\frac{1}{4} e^{-i\xi\cdot x'} e^{-|\xi|(x_3-1)} \cdot \begin{pmatrix} -\frac{2}{|\xi|} + \frac{\xi_1^2}{|\xi|^2}\left(\frac{1}{|\xi|} - x_3 + d\right) & \frac{\xi_1\xi_2}{|\xi|^2}\left(\frac{1}{|\xi|} - x_3 + d\right) & -i(x_3-d)\frac{\xi_1}{|\xi|} \\ \frac{\xi_1\xi_2}{|\xi|^2}\left(\frac{1}{|\xi|} - x_3 + d\right) & -\frac{2}{|\xi|} + \frac{\xi_2^2}{|\xi|^2}\left(\frac{1}{|\xi|} - x_3 + d\right) & -i(x_3-d)\frac{\xi_2}{|\xi|} \\ -i(x_3-d)\frac{\xi_1}{|\xi|} & -i(x_3-d)\frac{\xi_2}{|\xi|} & -\frac{1}{|\xi|} + x_3 - d \end{pmatrix}. \tag{3.35}$$

Using the above notations, the linear system of equations (3.22), (3.23), becomes

$$\mathbf{B}\cdot\mathbf{a}^i + \mathbf{C}\cdot\mathbf{b}^i = \mathbf{H}, \qquad \mathbf{D}\cdot\mathbf{a}^i + \mathbf{B}\cdot\mathbf{b}^i = \mathbf{Q}, \quad i = \overline{1,3}, \tag{3.36}$$

where

$$\mathbf{a}^i = \begin{pmatrix} \mathcal{F}\alpha_1^{(i)}(\xi;x) \\ \mathcal{F}\alpha_2^{(i)}(\xi;x) \\ \mathcal{F}\alpha_3^{(i)}(\xi;x) \end{pmatrix}, \qquad \mathbf{b}^i = \begin{pmatrix} \mathcal{F}\beta_1^{(i)}(\xi;x) \\ \mathcal{F}\beta_2^{(i)}(\xi;x) \\ \mathcal{F}\beta_3^{(i)}(\xi;x) \end{pmatrix}, \qquad i = \overline{1,3}. \tag{3.37}$$

From (3.20) and (3.21), we obtain

$$D_{ij}(x,y) = \int_{\mathbf{R}^2} e^{i\xi\cdot y'}\Big[\mathcal{F}E_{ik}(\xi;0,-d-y_3)a_k^j + \mathcal{F}E_{ik}(\xi;0,d-y_3)b_k^j\Big]dy_1dy_2\,, \tag{3.38}$$

$$e_j(x,y) = \int_{\mathbf{R}^2} e^{i\xi\cdot y'}\Big[\mathcal{F}e_k(\xi;0,-d-y_3)a_k^j + \mathcal{F}e_k(\xi;0,d-y_3)b_k^j\Big]\,dy_1dy_2\,, \tag{3.39}$$

$i,j = \overline{1,3}$.

4. Integral representation of solution

The Green function $\mathbf{G}$ satisfies the condition (3.3). When the pole x approaches the point y of $\mathcal{P}_0$ or $\mathcal{P}_1$, $\mathbf{G}$ becomes singular. On the other hand, this function must vanish, due to (3.3). Thus, we obtain the following condition

$$\mathbf{G}(x,y) = 0, \quad \text{for} \quad x \in \mathcal{P}_0 \ \text{ or } \ \mathcal{P}_1 \qquad (\text{i.e. for } \ x_3 = \pm d), \quad \forall\, y \in \mathbf{R}^3, \quad |y_3| < d. \tag{4.1}$$

As a consequence of (4.1) and from the Stokes equations (3.1), we deduce

$$\mathbf{q}(x,y) = 0, \quad \text{for} \quad x \in \mathcal{P}_0 \ \text{ or } \ \mathcal{P}_1, \qquad \forall\, y \in \mathbf{R}^3, \quad |y_3| < d. \tag{4.2}$$

Using Green's function $\mathbf{G}$ and the pressure vector $\mathbf{q}$, we determine the stress tensor $\mathbf{T}(T_{ijk})$, given by

$$T_{ijk}(x,y) = -q_j(x,y)\delta_{ik} + \frac{\partial G_{ij}}{\partial y_k}(x,y) + \frac{\partial G_{kj}}{\partial y_i}(x,y). \tag{4.3}$$

By using the properties (4.1) and (4.2), we obtain

$$T_{ijk}(x,y) = 0, \quad \text{for} \quad x \in \mathcal{P}_0 \ \text{ or } \ \mathcal{P}_1, \quad i,j,k = \overline{1,3}. \tag{4.4}$$

Hence, the Green function $\mathbf{G}$, the associated pressure vector $\mathbf{q}$ and the stress tensor $\mathbf{T}$, vanish when the pole of the Green function is located on the planes $\mathcal{P}_0$ and $\mathcal{P}_1$. Additionally, we have the following asymptotic properties

$$T_{ijk}(x,y) \to 0, \qquad \text{as} \quad |x| \to \infty. \tag{4.5}$$

The pressure vector $\mathbf{q} = \mathbf{q}(y, x)$ satisfies the following continuity equation (see [10]):

$$\sum_{i=1}^{3} \frac{\partial q_i}{\partial y_i}(y, x) = 0, \qquad \text{for} \quad x \neq y, \quad |y_3| < d, \tag{4.6}$$

and, hence, it can be considered as an acceptable solution for the equations of the Stokes flow, due to a point source placed at x.

Let $P = P(x, y)$ be the pressure associated with the velocity $\mathbf{q}(y, x)$. Then we have the following equations:

$$\Delta_y q_i(y, x) - \frac{\partial P}{\partial y_i}(x, y) = 0, \qquad \text{for} \quad x \neq y, \quad |y_3| < d, \quad i = \overline{1,3}, \tag{4.7}$$

with the boundary and asymptotic conditions given below

$$q_i(y, x) = 0, \qquad \text{for} \quad y \in \mathcal{P}_0 \ \text{ or } \ \mathcal{P}_1, \tag{4.8}$$

$$q_i(y, x) \to 0, \qquad P(x, y) \to 0, \quad \text{as} \quad |y| \to \infty. \tag{4.9}$$

The pressure tensor $\mathbf{P}(P_{ij})$, associated to the stress tensor $\mathbf{T}$, is given by the following equalities:

$$P_{ij}(y, x) = -P(y, x)\delta_{ij} + \frac{\partial q_i}{\partial y_j}(y, x) + \frac{\partial q_j}{\partial y_i}(y, x), \qquad i, j = \overline{1,3}. \tag{4.10}$$

Now, we can consider the Stokes flow $(\mathbf{u}, p)$ written in the form

$$u_i(x) = \int_S T_{jik}(x, y) n_k(y) \varphi_j(y) d\sigma_y + \int_S G_{ij}(y, x) \varphi_j(y) d\sigma_y, \tag{4.11}$$

$$p(x) = \int_S P_{jk}(y, x) n_k(y) \varphi_j(y) d\sigma_y + \int_S q_j(y, x) \varphi_j(y) d\sigma_y, \tag{4.12}$$

where $\mathbf{n}(n_1, n_2, n_3)$ denotes the unit outward normal vector to Ω^1, and $d\sigma_y$ denotes differentiation of the surface element of S, with respect to the point y.

From (3.1) – (3.4), (4.1) – (4.10) we deduce that $\mathbf{u}$ and p, given by (4.11), (4.12) satisfy the Stokes equations (2.5), (2.6) in Ω and Ω^1, respectively. Also, the null conditions (4.9) and the asymptotic conditions (2.9) are fulfilled.

In order to determine the unknown density $\boldsymbol{\varphi} : S \to \mathbf{R}^3$, $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$, we use a set of properties specified below. For this end, we consider the following double-layer and single-layer potentials:

$$V_i^1 \boldsymbol{\varphi}(x) = \int_S T_{jik}(x, y) n_k(y) \varphi_j(y) \, d\sigma_y, \quad i = \overline{1,3}, \quad x \in \Omega \cup \Omega^1, \tag{4.13}$$

$$V_i^2 \boldsymbol{\varphi}(x) = \int_S G_{ij}(y, x) \varphi_j(y) \, d\sigma_y, \quad i = \overline{1,3}, \quad x \in \Omega \cup \Omega^1, \tag{4.14}$$

where the function $\boldsymbol{\varphi}$ belongs to the class of functions continuous on S.

PROPERTY 1. The single-layer potential $V_i^2\varphi$, $i = \overline{1,3}$, is a function continuous across the surface S, i.e.

$$\lim_{\substack{x' \to x \in S \\ x' \in \Omega}} V_i^2\varphi(x') = \lim_{\substack{x' \to x \in S \\ x' \in \Omega^1}} V_i^2\varphi(x'), \quad \forall\, x \in S. \tag{4.15}$$

PROPERTY 2. (see [10, 13]). The double-layer potential $V_i^1\varphi$, $i = \overline{1,3}$, has two different values on the two sides of S, given by

$$\lim_{x' \to x \in S} V_i^1\varphi(x') = \pm\frac{1}{2}\varphi_i(x) + \overset{PV}{\int_S} T_{jik}(x,y)n_k(y)\varphi_j(y)\,d\sigma_y\,, \quad \forall\, x \in S, \tag{4.16}$$

where the plus sign is applied for the external side of S (in the direction of the unit normal vector $\mathbf{n}$) and the minus sign is applied for the internal side of S. Symbol PV means the principal value of the integral.

We remark that the kernels $T_{jik}(x,y)n_k(y)$ and $G_{ij}(x,y)$ of the double-layer operators and single-layer operators, respectively, become singular when the point y of S approaches the point x of S. If S is a Lyapunov surface, then the kernels become weakly singular.

In fact, the kernel $K_{ij}(x,y) = T_{jik}(x,y)n_k(y)$, can be written as

$$K_{ij}(x,y) = -\frac{3}{4\pi r^2}\cdot\frac{\partial r}{\partial(x_i - y_i)}\cdot\frac{\partial r}{\partial(x_j - y_j)}\cdot\frac{\partial r}{\partial n_y} + K_{ij}^*(x,y), \tag{4.17}$$

where K_{ij}^* is a continuous function. Also,

$$\left|\frac{\partial r}{\partial n_y}\right| < \lambda r^\alpha,$$

where $\lambda > 0$, $0 < \alpha \leq 1$ is the Lyapunov constant (see [12]), and $r = |x - y|$. Hence, the first term of the right-hand side of the above equality, behaves as $r^{\alpha-2}$ for $y \to x$, and the kernel K_{ij} is weakly singular.

In an analogous way we can show that the kernel of the single layer operator is weakly singular.

In this case the operators (4.13) and (4.14) are linear and compact on the space of continuous functions on S (see [11, 12]).

The stress tensor $\mathbf{S}^0$ of a flow $(\mathbf{v}^0, p^0)$ has the following components

$$T_{ij}(x) = -p^0(x)\delta_{ij} + \frac{\partial v_i^0}{\partial x_j}(x) + \frac{\partial v_j^0}{\partial x_i}(x), \quad i = \overline{1,3}. \tag{4.18}$$

and the stress field $\mathbf{T}$ of the flow is given by $\mathbf{T} = \mathbf{S}^0\mathbf{n}$.

PROPERTY 3. (see [5]). The stress field $\mathbf{T}^1(T_1^1, T_2^1, T_3^1)$, corresponding to the single-layer potentials (4.14) and the associated pressure, on the two sides of S, has two different values. On the external side of S, we have

$$\lim_{\substack{x' \to x \in S \\ x' \in \Omega}} T_i^1(x') = -\frac{1}{2}\varphi_i(x) + n_k(x) \overset{PV}{\int_S} T_{ijk}(y,x)\varphi_j(y)\, ds_y\,, \quad i = \overline{1,3}. \tag{4.19}$$

On the internal side of S, we have

$$\lim_{\substack{x' \to x \in S \\ x' \in \Omega^1}} T_i^1(x') = \frac{1}{2}\varphi_i(x) + n_k(x) \overset{PV}{\int_S} T_{ijk}(y,x)\varphi_j(y)\, ds\,, \quad i = \overline{1,3}. \tag{4.20}$$

PROPERTY 4. The stress field $\mathbf{T}^2(T_1^2, T_2^2, T_3^2)$, corresponding to the double-layer potentials (4.13) and the associated pressure, and having well-defined limiting value at points of S, has the same value on both sides of S. Thus, we have

$$\lim_{\substack{x' \to x \in S \\ x' \in \Omega}} T_i^2(x') = \lim_{\substack{x' \to x \in S \\ x' \in \Omega^1}} T_i^2(x'), \qquad \forall\, x \in S, \quad i = \overline{1,3}. \tag{4.21}$$

P r o o f. Let $\mathbf{w}$ be the Stokes velocity defined by the single-layer potentials (4.14). Hence, we have

$$\mathbf{w}(x) = \int_S \mathbf{G}(y,x)\boldsymbol{\varphi}(y)\, ds_y, \qquad p^0(x) = \int_S \mathbf{q}(y,x)\cdot\boldsymbol{\varphi}(y)\, ds_y, \quad x \in \Omega \cup \Omega^1, \tag{4.22}$$

where p^0 denotes the corresponding pressure of $\mathbf{w}$.

Let $\mathbf{w}^1$, $\mathbf{w}^2$ be smooth and solenoidal functions in Ω (i.e. $\nabla\cdot\mathbf{w}^i(x) = 0$, $x \in \Omega$, $i \in \{1,2\}$), and p^1, p^2, be two smooth, scalar functions in Ω. By applying the Green formula, we obtain

$$\begin{aligned} \int_\Omega \left\{ w_i^1(y)\left[\Delta w_i^2(y) - \frac{\partial p^2}{\partial y_i}(y)\right] - w_i^2(y)\left[\Delta w_i^1(y) - \frac{\partial p^1}{\partial y_i}(y)\right]\right\} dy \\ = -\int_{\partial\Omega} \left\{ w_i^1(y)\left(-p^2(y)\delta_{ij} + \frac{\partial w_i^2}{\partial y_j}(y) + \frac{\partial w_j^2}{\partial y_i}(y)\right) n_j(y) \right. \\ \left. - w_i^2(y)\left(-p^1(y)\delta_{ij} + \frac{\partial w_i^1}{\partial y_j}(y) + \frac{\partial w_j^2}{\partial y_i}(y)\right) n_j(y)\right\} d\sigma_y\,, \end{aligned} \tag{4.23}$$

where $\partial\Omega$ denotes the boundary of the domain Ω and the unit normal vector $\mathbf{n}$ is directed inwardly to the domain Ω.

If we consider in the above equality $\mathbf{w}^1 = \mathbf{w}$, $p^1 = p^0$, and $\mathbf{w}^2(y) = \mathbf{G}(x,y)\boldsymbol{\alpha}$, $p^2(y) = \mathbf{q}(x,y)\boldsymbol{\alpha}$, $\boldsymbol{\alpha}$ being an arbitrary constant vector, then we obtain the following identity

$$w_i(x) = -\int_S G_{ji}(x,y)T_{jk}^{2+}(\mathbf{w})(y)n_k(y)\,d\sigma_y + \int_S T_{jik}(x,y)n_k(y)w_j^+(y)\,d\sigma_y\,, \qquad \forall\, x \in \Omega, \quad i = \overline{1,3}, \tag{4.24}$$

where $T_{jk}^{2+}(\mathbf{w})(y)n_k(y) = T_j^{2+}(\mathbf{w})(y)$, $j = \overline{1,3}$, represent the component of the stress field $\mathbf{T}^2$, associated with the velocity $\mathbf{w}$. The superscript $+$ denotes the limit from Ω, at the point y of S.

In an analogous manner, we obtain the following identity

$$w_i(x) = \int_S G_{ji}(x,y)T_j^{2-}(\mathbf{w})(y)d\sigma_y - \int_S T_{jik}(x,y)n_k(y)w_j^-(y)d\sigma_y\,, \qquad \forall\, x \in \Omega^1, \quad i = \overline{1,3}, \tag{4.25}$$

where the superscript $-$ denotes the limit at the point y, approached from Ω^1. By using the jump formulas (4.16), we deduce

$$w_i^+(x) = -2\int_S G_{ji}(x,y)f_j^+(y)\,d\sigma_y + 2\,PV\int_S T_{jik}(x,y)n_k(y)w_j^+(y)\,d\sigma_y\,, \tag{4.26}$$

$$w_i^-(x) = 2\int_S G_{ji}(x,y)f_j^-(y)\,d\sigma_y - 2\,PV\int_S T_{jik}(x,y)n_k(y)w_j^-(y)\,d\sigma_y\,, \tag{4.27}$$

$\forall\, x \in S$, with $f_j^{\pm}(y) = T_{jk}^{2\pm}(\mathbf{w})(y)n_k(y)$.

From (4.26) and (4.27), we obtain

$$w_i^+(x) + w_i^-(x) = -2\int_S G_{ji}(x,y)(f_j^+(y) - f_j^-(y))\,d\sigma_y + 2\,PV\int_S T_{jik}(x,y)n_k(y)(w_j^+(y) - w_j^-(y))\,d\sigma_y\,, \qquad \forall\, x \in S, \quad i = \overline{1,3}. \tag{4.28}$$

The jump properties (4.16) imply

$$w_i^+(x) + w_i^-(x) = 2\,PV\int_S T_{jik}(x,y)n_k(y)\varphi_j(y)\,d\sigma_y, \qquad \forall\, x \in S, \quad i = \overline{1,3}, \tag{4.29}$$

$$w_j^+(x) - w_j^-(x) = \varphi_j(y), \qquad \forall\, x \in S, \quad j = \overline{1,3}. \tag{4.30}$$

Therefore, the above properties (4.28) – (4.30) give

$$\int_S G_{ji}(x,y)(f_j^+(y) - f_j^-(y))\, d\sigma_y = 0, \qquad \forall\, x \in S, \quad i = \overline{1,3}. \tag{4.31}$$

In order to solve the above integral system, with the functions $g_j(x) = f_j^+(x) - f_j^-(x)$, $x \in S$, $j = \overline{1,3}$, as unknowns, we consider the velocity field $\tilde{\mathbf{v}}$ and pressure $\tilde{p}$, given by

$$\tilde{v}_j(x) = \int_S G_{ji}(x,y) g_j(y)\, d\sigma_y\,, \qquad x \in \Omega \cup \Omega^1, \quad j = \overline{1,3}, \tag{4.32}$$

$$\tilde{p}(x) = \int_S q_j(x,y) g_j(y)\, d\sigma_y\,, \qquad x \in \Omega \cup \Omega^1. \tag{4.33}$$

From (3.1) – (3.4) and (4.31), we deduce that $(\tilde{\mathbf{v}}, \tilde{p})$ represent a Stokes flow in Ω and Ω^1, respectively, with zero velocity on the planes $\mathcal{P}_0$ and $\mathcal{P}_1$, on the surface S, and at infinity. Using the uniqueness result of solution for the Stokes problem (2.5) – (2.9) in Ω (see the Remark 1), we conclude that $\tilde{\mathbf{v}}(x) = 0$, $\forall\, x \in \overline{\Omega}$. Hence, $\tilde{p}$ is a constant in Ω. But, $\tilde{p}(x) \to 0$, as $|x| \to \infty$. Hence, $\tilde{p}(x) = 0$, $\forall\, x \in \Omega$. Analogously, we obtain $\tilde{\mathbf{v}}(x) = 0$, $\forall\, x \in \overline{\Omega^1}$ and $\tilde{p}(x) = c \in \mathbf{R}$, $\forall\, x \in \Omega^1$. Let $\tilde{\mathbf{T}}$ be the stress field of the flow $(\tilde{\mathbf{v}}, \tilde{p})$. From the above reasons it follows that

$$\lim_{\substack{x' \to x \in S \\ x' \in \Omega}} \tilde{T}_i(x') - \lim_{\substack{x' \to x \in S \\ x' \in \Omega^1}} \tilde{T}_i(x') = -c n_i(x), \qquad \forall\, x \in S, \quad i = \overline{1,3}. \tag{4.34}$$

On the other hand, the jump properties (4.19), (4.20) give

$$\lim_{\substack{x' \to x \in S \\ x' \in \Omega}} \tilde{T}_i(x') - \lim_{\substack{x' \to x \in S \\ x' \in \Omega^1}} \tilde{T}_i(x') = -g_i(x), \qquad \forall\, x \in S, \quad i = \overline{1,3}. \tag{4.35}$$

Thus, we have

$$g_i(x) = c n_i(x), \qquad \forall\, n \in S, \quad i = \overline{1,3}. \tag{4.36}$$

These equalities show that the function $\mathbf{g} = \mathbf{f}^+ - \mathbf{f}^-$ not depends by the distribution $\boldsymbol{\varphi}$. If $\boldsymbol{\varphi}$ is the null function, then we must have $\mathbf{g} \equiv \mathbf{0}$. We conclude that

$$\mathbf{f}^+(x) = \lim_{\substack{x' \to x \in S \\ x' \in \Omega}} \mathbf{T}^2(\mathbf{w})(x') = \lim_{\substack{x' \to x \in S \\ x' \in \Omega^1}} \mathbf{T}^2(\mathbf{w})(x') = \mathbf{f}^-(x), \qquad \forall\, x \in S. \tag{4.37}$$

By using the Properties 1 – 4, we prove the existence and the uniqueness of solution of the Stokes problem (2.5) – (2.9).

From the boundary condition (2.7), the jump property (4.16) and the continuity property of the single-layer potentials (4.15) across S, we obtain the following Fredholm integral system of the second kind:

$$\frac{1}{2}\varphi_i(x) + \int_S T_{jik}(x,y)n_k(y)\varphi_j(y)\,d\sigma_y + \int_S G_{ij}(y,x)\varphi_j(y)\,d\sigma_y = -U_{\infty i}(x) + U_i(x), \qquad \forall\, x \in S, \quad i = \overline{1,3}. \tag{4.38}$$

The above double-layer integrals are understood as principal values in Cauchy's sense. For convenience, we have ommitted the symbol PV.

From Fredholm's result (see [12]), we deduce that the nonhomogeneous system (4.38) has a unique continuous solution if and only if the corresponding homogeneous system has only the null solution, in the space of continuous functions on S.

Let the following homogeneous integral system be satisfied:

$$\frac{1}{2}\varphi_i^0(x) + \int_S T_{jik}(x,y)n_k(y)\varphi_j^0(y)\,d\sigma_y + \int_S G_{ij}(y,x)\varphi_j^0(y)\,d\sigma_y = 0, \qquad \forall\, x \in S, \quad i = \overline{1,3}. \tag{4.39}$$

With the density $\boldsymbol{\varphi}^0$, supposed to be a continuous function on S, we define the following flow $(\mathbf{v}^0, p_0)$

$$v_i^0(x) = \int_S T_{jik}(x,y)n_k(y)\varphi_j^0(y)d\sigma_y + \int_S G_{ij}(y,x)\varphi_j^0(y)d\sigma_y, \quad x \in \Omega \cup \Omega^1, \tag{4.40}$$

$$p_0(x) = \int_S P_{jk}(y,x)n_k(y)\varphi_j^0(y)d\sigma_y + \int_S q_j(y,x)\varphi_j^0(y)d\sigma_y, \quad x \in \Omega \cup \Omega^1. \tag{4.41}$$

By using the Stokes equations and conditions (3.1) – (3.4) and the properties (4.2) – (4.5), (4.6) – (4.10), (4.39), we deduce that $(\mathbf{v}^0, p_0)$ represent a Stokes flow in Ω and Ω^1, respectively, with zero velocity on the planes $\mathcal{P}_0$ and $\mathcal{P}_1$, on the surface S and at infinity. By applying the uniqueness result of the Stokes flow in Ω, we conclude that

$$\mathbf{v}^0(x) = 0, \qquad p_0(x) = 0, \quad \forall\, x \in \Omega. \tag{4.42}$$

As a consequence of the above equalities (4.42), we deduce

$$\mathbf{f}^{0+}(x) = \lim_{\substack{x' \to x \in S \\ x' \in \Omega}} \mathbf{f}^0(x') = 0, \qquad \forall\, x \in S, \tag{4.43}$$

where $\mathbf{f}^0$ is the stress field of the flow $(\mathbf{v}^0, p_0)$, defined as in (4.18).

Now, by applying the Properties 1 and 2, we obtain

$$v_i^{0+}(x) - v_i^{0-}(x) = \varphi_i^0(x), \qquad \forall\, x \in S, \quad i = \overline{1,3}, \tag{4.44}$$

where $v_i^{0+}(x) = \lim\limits_{\substack{x' \to x \in S \\ x' \in \Omega}} v_i^0(x')$ and $v_i^{0-}(x) = \lim\limits_{\substack{x' \to x \in S \\ x' \in \Omega^1}} v_i^0(x)$, respectively. But (4.42) shows that the first term of the left-hand side of (4.44) is zero. Hence, we have the following equalities

$$v_i^{0-}(x) = -\varphi_i^0(x), \qquad \forall\, x \in S, \quad i = \overline{1,3}. \tag{4.45}$$

If we use the Properties 3 and 4, then we obtain

$$f_i^{0+}(x) - f_i^{0-}(x) = -\varphi_i^0(x), \qquad \forall\, x \in S, \quad i = \overline{1,3}. \tag{4.46}$$

By combining the result (4.43) to the above relations (4.46), we obtain

$$f_i^{0-}(x) = \varphi_i^0(x), \qquad \forall\, x \in S, \quad i = \overline{1,3}. \tag{4.47}$$

Using simple computations and the Green's formula, we obtain the following identity:

$$\int_S v_i^{0-}(x) f_i^{0-}(x)\, d\sigma_x = 2 \int_{\Omega^1} e_{ik}^0(x) e_{ik}^0(x)\, dx, \tag{4.48}$$

where

$$e_{ik}^0(x) = \frac{1}{2}\left(\frac{\partial v_i^0}{\partial x_j}(x) + \frac{\partial v_j^0}{\partial x_i}(x)\right), \qquad i,k = \overline{1,3}, \tag{4.49}$$

define the rate of the deformation tensor.

From (4.45) and (4.47), we deduce

$$-\int_C \varphi_i^0(x) \varphi_i^0(x)\, d\sigma_x = 2 \int_{\Omega^1} e_{ik}^0(x) e_{ik}^0(x)\, dx. \tag{4.50}$$

Because the left-hand side of (4.50) is non-positive and the right-hand side is non-negative, we conclude that both the sides of (4.50) are zero. By using the continuity of $\boldsymbol{\varphi}^0$ on S, it follows that $\boldsymbol{\varphi}^0(x) = 0$, $\forall\, x \in S$. This argument and the Fredholm's result (see [12]) imply that the nonhomogeneous integral system (4.38) has a unique, continuous solution $\boldsymbol{\varphi}$. With this function and by the formulas (4.11), (4.12), we determine the unique solution $(\mathbf{u}, p)$ of the Stokes problem (2.5) – (2.9). Hence, the existence and uniqueness of the solution of the Stokes problem (2.5) – (2.9) is completely proved.

REMARK 1. By using Green's formula in the domain Ω, the Stokes equations (2.5), (2.6), the boundary and asymptotic conditions (2.8), (2.9), we obtain the following identity:

$$-\frac{1}{2}\int_{\Omega}\left(\frac{\partial u_i}{\partial x_j}(x)+\frac{\partial u_j}{\partial x_i}(x)\right)^2 dx=\int_S T_i(x)u_i(x)\,d\sigma_x\,, \tag{4.51}$$

where $\mathbf{T}(T_1,T_2,T_3)$ is the surface force on S, of the flow $(\mathbf{u},p)$.

If we suppose that the Stokes problem (2.5)–(2.9) has two solutions $(\mathbf{u}^1,p^1)$ and $(\mathbf{u}^2,p^2)$, then the function $\mathbf{u}^0=\mathbf{u}^1-\mathbf{u}^2$ satisfies null boundary conditions on $\mathcal{P}_0,\mathcal{P}_1,S$, and at infinity. From (4.51) we deduce

$$\frac{\partial u_i^0}{\partial x_j}(x)+\frac{\partial u_j^0}{\partial x_i}(x)=0,\qquad \forall\, x\in\Omega,\quad i,j=\overline{1,3}. \tag{4.52}$$

The above system has the linear independent solutions, given below

$$\mathbf{U}^i(x)=(\delta_{1i},\delta_{2i},\delta_{3i}),\qquad i=\overline{1,3}, \tag{4.53}$$

$$\mathbf{U}^4(x)=(0,x_3,-x_2),\qquad \mathbf{U}^5(x)=(-x_3,0,x_1),\qquad \mathbf{U}^6(x)=(x_2,-x_1,0). \tag{4.54}$$

Another solution of (4.52) has the following form (see [1])

$$\mathbf{u}^0(x)=\mathbf{A}_0+\omega_0\times(\mathbf{x}-\mathbf{x}_0),\qquad \forall\, x\in\Omega, \tag{4.55}$$

where $\mathbf{A}_0$ and ω_0 are constant vectors, $\mathbf{x}$ is the position vector of the point $x=(x_1,x_2,x_3)$ and $\mathbf{x}_0$ is the position vector of a point x_0 of Ω^1.

By using the null condition on the walls $\mathcal{P}_0$, $\mathcal{P}_1$, on the surface S, and at infinity, satisfied by $\mathbf{u}^0$, we conclude that $\mathbf{A}_0=\omega_0=0$. Hence $\mathbf{u}^0=0$. This proves the uniqueness of solution for the problem (2.5)–(2.9).

5. Numerical results

From the Green formula, applied in the domain Ω^1, and the Stokes equations (3.1), (3.2), we obtain the following property

$$\int_S T_{ijk}(x,y)n_k(y)\,d\sigma_y=\begin{cases}-\delta_{ij}, & \text{for}\quad x\in\Omega^1,\\ 0, & \text{for}\quad x\in\Omega,\\ -\dfrac{1}{2}\delta_{ij}, & \text{for}\quad x\in S.\end{cases} \tag{5.1}$$

In the last case the integral is evaluated in the sense of the principal value.

With the above property, the system (4.38), can be written in the following form

$$\int_S T_{jik}(x,y)n_k(y)(\varphi_j(y)-\varphi_j(x))\,ds_y + \int_S G_{ij}(y,x)\varphi_j(y)\,d\sigma_y = -U_{\infty i}(x)+U_i(x), \qquad \forall\ x\in S,\quad i=\overline{1,3}. \tag{5.2}$$

Now, if we use the expression (4.17) of the kernel of the double-layer potential, then we can easily deduce that the double-layer potentials of (5.2) are proper integrals. Hence, the singularities of these integrals can be removed by considering their integrands to be equal to zero for $y = x$ (see [13]).

In order to reduce the system (5.2) to a linear system of algebraic equations, we use a boundary element method. Thus, we divide the surface S into N elements Δ_j, $j=\overline{1,N}$, and we suppose that the function $\boldsymbol{\varphi}$ is constant on each Δ_j and equal to its value at the center of this element. With these assumptions, the system (5.2) can be written approximatelly as folows,

$$\sum_{j=1}^{N}(\varphi_l^j-\varphi_l^m)\int_{\Delta_j} T_{lik}(x^m,y)n_k(y)\,d\sigma_y + \sum_{j=1}^{N}\varphi_l^j\int_{\Delta_j} G_{il}(y,x^m)\,d\sigma_y = -U_{\infty i}(x^m)+U_i(x^m), \qquad m=\overline{1,N},\quad i=\overline{1,3}, \tag{5.3}$$

where x^m is the center of Δ_m, φ_l^m is the constant value of φ_l on Δ_m and the terms $(\varphi_l^j-\varphi_l^m)\int_{\Delta_j} T_{lik}(x^m,y)n_k(y)d\sigma_y$ are equal to zero, when $l=m$, due to the removal of singularities.

Also the integral $\int_{\Delta_j} G_{il}(y,x^m)d\sigma_y$ becomes singular when $j=m$. Then we consider the following equality:

$$\int_{\Delta_j} G_{il}(y,x^m)\,d\sigma_y = \int_{\Delta_j}[G_{il}(y,x^m)-E_{il}(x-y)]\,d\sigma_y - \int_{\Delta_j} E_{il}(x-y)\,d\sigma_y\,.$$

The first integral of the right-hand side of the above equality is proper, hence, it can be computed by a Gauss quadrature formula. By using two variables (w_1,w_2) over the element Δ_j, we have $d\sigma_y = h_{w_1}dw_1dw_2$, where $h_{w_1} = \left|\dfrac{\partial \mathbf{y}}{\partial w_1}\times\dfrac{\partial \mathbf{y}}{\partial w_2}\right|$.

The second integral of the right-hand side of the above equality can be computed exactly in the (w_1,w_2) plane (see also [6]).

The algebraic system (5.3) can be numerically solved by using some integration and matrix inversion techniques.

By using the properties of the functions $\mathbf{G}$, $\mathbf{q}$ and $\mathbf{T}$, we obtain the total force $\boldsymbol{\mathcal{F}}'$ and torque $\mathbf{M}'$ on S, given by

$$\boldsymbol{\mathcal{F}}' = -\int_S \boldsymbol{\varphi}(y)\,d\sigma_y\,, \qquad \mathbf{M}' = -\int_S \mathbf{y}\times\boldsymbol{\varphi}(y)\,d\sigma_y\,. \tag{5.4}$$

After solving the system (5.3), the total force on the surface S has the following components

$$\mathcal{F}_i' = -\sum_{j=1}^{N} \varphi_i^j \int_{\Delta_j} d\sigma_y\,, \qquad i = \overline{1,3}. \tag{5.5}$$

The numerical integrations presented in this paper have been given by using the Gauss quadrature formulas, and the linear system (5.3) was solved by means of the Gaussian elimination.

The numerical results are presented in the case of the Poiseuille flow $\mathbf{U}_\infty(x) = (-U(d^2 - x_3^2), 0, 0)$, $U > 0$, past a fixed sphere Ω^1, with the radius $a < d$.

Here we use the following notation

$$s = \frac{d - |Z_0|}{2d} = \frac{1}{2} - \frac{|Z_0|}{2d}\,,$$

where (X_0, Y_0, Z_0) is the center of sphere. Hence, $|Z_0| \in (0, d]$ and $s \in (0, 1/2]$.

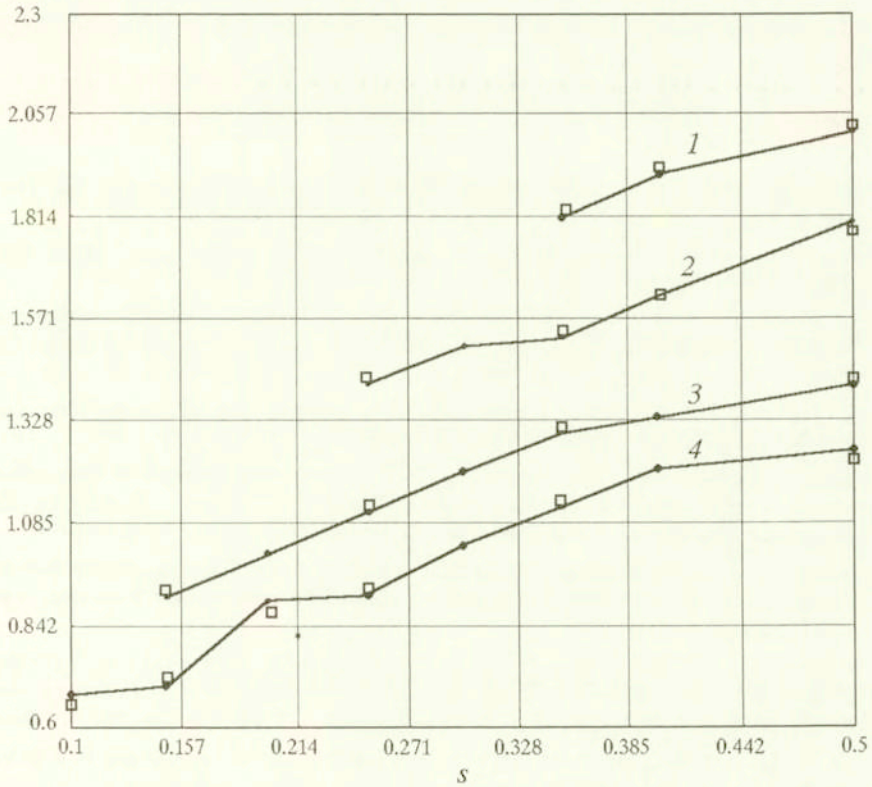

FIG. 2. —•— present method, □ Ganatos Pfeffer Weinbaum method, *1* — $d/a = 1.5$, *2* — $d/a = 2$, *3* — $d/a = 3$, *4* — $d/a = 5$.

Figure 2 gives the dependence between the modulus of the drag force $\mathcal{F} = \mathcal{F}'/(6\pi aU)$ and the parameter s, for various values of the ratio d/a. We deduce that the modulus of this force decreases when the ratio d/a increases. The maximum value is obtained when the center of the sphere is located on the plane Ox_1x_2.

Also, from Fig. 2, we conclude that the drag force $F = |\mathcal{F}|$ increases if the parameter s increases. Figure 2 shows that our results are in good agreement with similar results, obtained by P. GANATOS, R. PFEFFER and S.WEINBAUM in [7]. The partition of the sphere consists of 56 elements.

6. Conclusions

In this paper we have applied the direct boundary integral equations method to the Stokes flow past a smooth obstacle, between two plane parallel walls. Green's functions for the equations of the Stokes flow are obtained. These functions, together with the nonslip boundary condition on the surface obstacle, determine a Fredholm system of integral equations of the second kind, over the boundary of the obstacle. The integral formulation is simple and does not truncate the flow domain. This fact has the advantage of improving the accuracy of the numerical computations.

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FACULTY OF MATHEMATICS
BABEŞ-BOLYAI UNIVERSITY, CLUJ-NAPOCA, ROMANIA.
e-mail: mkohr@math.ubbcluj.ro

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