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Thermoelastic materials with heat flux evolution equation

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THE RESULTS obtained in this paper refer to the class of materials for which *specific free energy* ψ, the *specific entropy* η, and *the first Piola–Kirchhoff stress tensor* **S** are, respectively, determined through the *constitutive functionals* $\overline{\psi}$, $\overline{\eta}$, and $\overline{S}$ which are defined in their common domain consisting of quadruples $(\mathbf{F}, \theta, \mathbf{G}, \mathbf{Q})$, called *states* of the material, and where **F** is the *deformation gradient*, θ is the *absolute temperature*, **G** is the *material gradient* of the temperature, and **Q** is the *referential heat flux*. The heat flux **Q** behaves as a "*hidden variable*" or an "*internal variable*" [1] and its evolution in time is described by a differential equation $\dot{\mathbf{Q}} = \mathbf{H}(\mathbf{F}, \theta, \mathbf{G}; \mathbf{Q})$, where **H** is a constitutive functional of the material. Such materials will be called *thermoelastic materials with heat flux evolution equation*. To a certain extent, this class of materials may be considered as a limit case of thermomechanical materials with internal state variables examined by COLEMAN and GURTIN [1]. It is for this reason that this fundamental work of modern continuum thermodynamics inspired much of the results in this paper. On the other hand, the above heat flux evolution equation is generalizing Cattaneo's heat conduction equation [2] for isotropic materials. So this theory is convenient for predicting thermal waves propagating at finite speed.

Introduction

THE BASIC FUNCTIONAL and conceptual underpinnings of the classical continuum thermodynamics are briefly presented in Sec. 1.

The axiomatic definition of thermoelastic materials with heat flux evolution equation and their constitutive equations are given in Sec. 2.

The general form of constitutive functionals $\overline{\psi}$, $\overline{\eta}$, and $\overline{S}$ in the assumption that the *heat evolution functional* **H** is linear in **G** and **Q**, i.e. in the Cattaneo's case, is presented in Sec. 3.

The notions of *equilibrium state* (E.S.), *isothermal E.S.* and its *domain of attraction* for a given material point are introduced in Sec. 4. We point out that our definition of E.S. includes the usual one as a special case, but it is not confined to it. A state $(\boldsymbol{\lambda}_0, \mathbf{G}_0; \mathbf{Q}_0)$, $\boldsymbol{\lambda}_0 = (\mathbf{F}_0, \theta_0)$ is an E.S. if

$$\mathbf{G}_0 \cdot \mathbf{Q}_0 = 0, \qquad \mathbf{H}(\boldsymbol{\lambda}_0, \mathbf{G}_0; \mathbf{Q}_0) = \mathbf{0}.$$

The strictly E.S., i.e. a state $(\boldsymbol{\lambda}_0, \mathbf{0}; \mathbf{Q}_0)$ which satisfies the condition

$$\mathbf{H}(\boldsymbol{\lambda}_0, \mathbf{0}; \mathbf{Q}_0) = \mathbf{0},$$

coincides with what is usually understood by an E.S. It is showed that the free energy function has a local minimum at an asymptotically stable isothermal E.S. and that if a strictly isothermal E.S. is a strict local minimum for the free energy function then this E.S. is Lyapunov stable. Results regarding asymptotic and Lyapunov stability of a strict isothermal E.S. for *strictly dissipative materials* are also obtained.

A theorem of *consistency with thermostatics* [3] on the set of asymptotically isothermal E.S. is proved in Sec. 5.

The specific entropy is taken as an independent variable in Sec. 6. In this case the implications of the Cattaneo's equation on constitutive functionals are derived and conditions of asymptotic and Lyapunov stability of an *isentropic* E.S. at constant strain for a material point are obtained.

In Sec. 7 the specific internal energy is taken as an independent variable and results regarding the asymptotic and Lyapunov stability of an *isoenergetic* E.S., similar to the results in Sec. 4 and Sec. 6, are established. Some links between asymptotic and Lyapunov stability of isothermal, isentropic, and isoenergetic E.S. are rendered evident, and the restrictions the Cattaneo heat flux evolution equation imposes upon constitutive functionals are pointed out.

Finally, we mention that some of the problems here discussed have been approached by the author in [10].

1. General formulae

1.1. The basic functional framework

Let $\mathbf{E}$ be the three-dimensional Euclidean point space, $\mathbf{V}$ the translation space of $\mathbf{E}$, and Lin the space of linear transformations of $\mathbf{V}$. We denote by $\mathcal{V}$ the set of triplets

$$\mathbf{\Lambda} = (\mathbf{A}, a, \mathbf{a}) \in \mathrm{Lin} \times \mathbb{R} \times \mathbf{V}. \tag{1.1}$$

$\mathcal{V}$ is a 13-dimensional Euclidean space with respect to the linear operation

$$\alpha(\mathbf{A}, a, \mathbf{a}) + \beta(\mathbf{B}, b, \mathbf{b}) = (\alpha\mathbf{A} + \beta\mathbf{B},\ \alpha a + \beta b,\ \alpha\mathbf{a} + \beta\mathbf{b}) \tag{1.2}$$

for every $(\mathbf{A}, a, \mathbf{a}), (\mathbf{B}, b, \mathbf{b}) \in \mathcal{V}$, $\alpha, \beta \in \mathbb{R}$, and the inner product

$$(\mathbf{A}, a, \mathbf{a}) \cdot (\mathbf{B}, b, \mathbf{b}) = \mathbf{A} \cdot \mathbf{B} + ab + \mathbf{a} \cdot \mathbf{b}, \tag{1.3}$$

where $\mathbf{A} \cdot \mathbf{B} = \mathrm{tr}\,(AB^T)$ and $\mathbf{a} \cdot \mathbf{b}$ are the inner product in Lin and $\mathbf{V}$, respectively. The corresponding Euclidean norm in $\mathcal{V}$ is given by

$$(\mathbf{A}, a, \mathbf{a}) \mapsto |(\mathbf{A}, a, \mathbf{a})| = (\mathbf{A} \cdot \mathbf{A} + a^2 + \mathbf{a} \cdot \mathbf{a})^{1/2} \geq 0, \qquad (\mathbf{A}, a, \mathbf{a} \in \mathcal{V}). \tag{1.4}$$

Also, the notation $\boldsymbol{\lambda} = (\mathbf{A}, a) \in \mathrm{Lin} \times \mathbb{R}$ will be used, so that

$$\mathbf{\Lambda} = (\mathbf{A}, a, \mathbf{a}) = (\boldsymbol{\lambda}, \mathbf{a}). \tag{1.5}$$

We denote by $\mathcal{V}^+$ the subset of $\mathcal{V}$ defined by

$$\mathcal{V}^+ = \mathrm{Lin}^+ \times \mathbb{R}^+ \times \mathbf{V}, \tag{1.6}$$

where $\mathrm{Lin}^+ = \{\mathbf{A} \in \mathrm{Lin} \,/\, \det A > 0\}$ and $\mathbb{R}^+ = (0, \infty)$.

Of course, $\mathcal{V}$ is a Banach space with respect to the Euclidean norm (1.4) and $\mathcal{V}^+$ is an open set in $\mathcal{V}$.

1.2. Classical continuum thermodynamics

A body [3], or a *continuous medium*, $\mathcal{B}$ is identified with the region [5] $\mathbf{B} \subset \mathbf{E}$ it occupies in a fixed reference configuration κ, and the material element, or particle $X \in \mathcal{B}$ is identified with its position $\mathbf{X} \in \mathbf{B}$. It is assumed that a *referential mass density* $\varrho_\kappa : \mathbf{B} \to (0, \infty)$ of $\mathcal{B}$ in the reference configuration is given such that the mass of the subpart $\mathbf{P}$ of $\mathcal{B}$ is

$$m(\mathbf{P}) = \int_P \varrho_\kappa \, dm.$$

Along with $\mathcal{B}$ and its referential mass distribution, the *process class* $\mathbb{P}(\mathcal{B})$ ([4, 5]) is given characterizing the material comprising $\mathcal{B}$. The elements $\pi \in \mathbb{P}(\mathcal{B})$ are called *processes* and they are ordered 8-tuples of mappings on $\mathbf{B} \times \mathbb{R}$.

$$\pi = (\boldsymbol{\chi}, \theta, \varepsilon, \eta, \mathbf{S}, \mathbf{Q}, \mathbf{b}, r), \tag{1.7}$$

where, during the process π, at particle $\mathbf{X}$, and time t, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) \in \mathbf{E}$ is the *motion*, $\theta = \theta(\mathbf{X}, t) \in \mathbb{R}^+$ is the *absolute temperature*, $\varepsilon = \varepsilon(\mathbf{X}, t) \in \mathbb{R}$ is the *specific internal energy* per unit mass, $\eta = \eta(\mathbf{X}, t) \in \mathbb{R}$ is the *specific entropy per unit mass*, $\mathbf{S} = \mathbf{S}(\mathbf{X}, t) \in$ Lin is the *first Piola–Kirchhoff stress tensor*, $\mathbf{Q} = \mathbf{Q}(\mathbf{X}, t) \in \mathbf{V}$ is the *referential heat flux*, $\mathbf{b} = \mathbf{b}(\mathbf{X}, t) \in \mathbf{V}$ is the *specific body force* per unit mass, and $r = r(\mathbf{X}, t) \in \mathbb{R}$ is the *radiant heating* per unit mass.

DEFINITION 1.1. *A process $\pi \in \mathbb{P}(\mathcal{B})$ is said to be admissible if its components mappings are satisfying sufficiently smooth conditions and the laws of balance of linear momentum, balance of moment of momentum, balance of energy, and imbalance of entropy* [3].

The *deformation gradient*

$$\mathbf{F} = \mathbf{F}(\mathbf{X}, t) = \operatorname{Grad} \boldsymbol{\chi}(\mathbf{X}, t), \tag{1.8}$$

where Grad denotes the gradient with respect to $\mathbf{X}$, is assumed to be in Lin^+, i.e. $J = \det F > 0$. The velocity $\mathbf{v}$ of particle $\mathbf{X}$ at time t is determined by the material time derivate of motion

$$\mathbf{v} = \mathbf{v}(\mathbf{X}, t) = \dot{\boldsymbol{\chi}}(\mathbf{X}, t). \tag{1.9}$$

The *mass conservation law* requires

$$\varrho = J \varrho_\kappa, \tag{1.10}$$

where $\varrho = \varrho(\mathbf{X}, t)$ is the mass density at particle $\mathbf{X}$ at time t.

For any admissible process $\pi \in \mathbb{P}(\mathcal{B})$ the laws of balance of linear momentum, balance of moment of momentum, balance of energy, and imbalance of entropy are equivalent to the local referential equations [3]:

$$\varrho_\kappa \dot{v} = \operatorname{Div} \mathbf{S} + \varrho_\kappa \mathbf{b}, \tag{1.11}$$

$$\mathbf{F}\mathbf{S}^T = \mathbf{S}\mathbf{F}^T, \tag{1.12}$$

$$\varrho_\kappa \dot{\boldsymbol{\varepsilon}} = \mathbf{S} \cdot \dot{\mathbf{F}} - \operatorname{Div} \mathbf{Q} + \varrho_\kappa r, \tag{1.13}$$

$$\varrho_\kappa \dot{\boldsymbol{\eta}} \geq \varrho_\kappa (r/\theta) - \operatorname{Div}(\mathbf{Q}/\theta), \tag{1.14}$$

where Div denotes the divergence operator with respect to $\mathbf{X}$.

By using the *specific free energy* $\psi = \psi(\mathbf{X}, t)$ per unit mass defined by

$$\psi = \varepsilon - \theta\eta \tag{1.15}$$

and taking into account the energy balance equation (1.13), it results that the *Clausius–Duhem inequality* (1.14) takes the form

$$\varrho_\kappa \left(\dot{\psi} + \eta \dot{\theta} \right) - \mathbf{S} \cdot \dot{\mathbf{F}} + \mathbf{Q} \cdot (\mathbf{G}/\theta) \leq 0, \tag{1.16}$$

where $\mathbf{G} = \mathbf{G}(\mathbf{X}, t) = \operatorname{Grad} \theta(\mathbf{X}, t)$ is the *temperature gradient* with respect to the reference configuration κ. The inequality (1.16) is called the *Reduced Dissipation Inequality* [3]. If $\gamma = \gamma(\mathbf{X}, t)$ denotes the *specific rate of entropy production* [1] of particle $\mathbf{X}$ at time t

$$\varrho_\kappa \gamma = \varrho_\kappa \dot{\eta} - [\varrho_\kappa (r/\theta) - \operatorname{Div}(\mathbf{Q}/\theta)], \tag{1.17}$$

then the Clausius-Duhem inequality (1.14) asserts that

$$\gamma \geq 0. \tag{1.18}$$

From the energy balance equation (1.13) it follows that for any admissible process $\pi \in \mathbb{P}(\mathcal{B})$ we may write (1.17) in the form

$$\gamma = \dot{\eta} - \dot{\varepsilon}/\theta + (1/\varrho_\kappa \theta)\mathbf{S} \cdot \dot{\mathbf{F}} - (1/\varrho_\kappa \theta^2)\mathbf{Q} \cdot \mathbf{G} \tag{1.19}$$

and, since $\psi = \dot{\varepsilon} - \theta \dot{\eta} - \dot{\theta}\eta$, from where we get

$$\theta\gamma = -\dot{\psi} - \eta\dot{\theta} + (1/\varrho_\kappa)\mathbf{S} \cdot \dot{\mathbf{F}} - (1/\varrho_\kappa \theta)\mathbf{Q} \cdot \mathbf{G}. \tag{1.20}$$

The following implications hold:

$$\dot{\theta} = 0, \quad \dot{\mathbf{F}} = 0, \quad \text{and} \quad \mathbf{Q} \cdot \mathbf{G} = 0 \ \Rightarrow \ \dot{\psi} \leq 0, \tag{1.21}$$

$$\dot{\eta} = 0, \quad \dot{\mathbf{F}} = 0, \quad \text{and} \quad \mathbf{Q} \cdot \mathbf{G} = 0 \ \Rightarrow \ \dot{\varepsilon} \leq 0, \tag{1.22}$$

$$\dot{\varepsilon} = 0, \quad \dot{\mathbf{F}} = 0, \quad \text{and} \quad \mathbf{Q} \cdot \mathbf{G} = 0 \ \Rightarrow \ \dot{\eta} \leq 0. \tag{1.23}$$

2. Thermoelastic materials with heat flux evolution

The theory studied in this paper assumes that the material comprising the body undergoes only admissible processes, in the sense of Definition 1.1, and that the specific free energy $\psi(\mathbf{X},t)$, the specific entropy $\eta(\mathbf{X},t)$, the first Piola-Kirchhoff stress tensor $\mathbf{S}(\mathbf{X},t)$, and the specific time rate of the heat flux $\dot{\mathbf{Q}}$ of a particle $\mathbf{X}$ and at time t are determined by the *state functions corresponding to the admissible processes* $\pi \in \mathbb{P}(\mathcal{B})$

$$(2.1)\qquad (\mathbf{\Lambda};\mathbf{Q}) \equiv (\mathbf{F},\theta,\mathbf{G};\mathbf{Q}) : \mathbf{B}\times\mathbb{R} \to \mathrm{Lin}^+ \times \mathbb{R}^+ \times \mathbf{V}\times\mathbf{V} = \mathcal{V}^+ \times \mathbf{V}$$

through the *constitutive functionals* of the material

$$(2.2)\qquad \psi(t) = \overline{\psi}(\mathbf{\Lambda}(t);\mathbf{Q}(t)),$$

$$(2.3)\qquad \eta(t) = \overline{\eta}(\mathbf{\Lambda}(t);\mathbf{Q}(t)),$$

$$(2.4)\qquad \mathbf{S}(t) = \overline{\mathbf{S}}(\mathbf{\Lambda}(t);\mathbf{Q}(t)),$$

$$(2.5)\qquad \dot{\mathbf{Q}}(t) = \mathbf{H}(\mathbf{\Lambda}(t);\mathbf{Q}(t)).$$

The variable $\mathbf{X}\in\mathbf{B}$ is understood to enter both sides of (2.2)–(2.5), but it is not written there because all the subsequent considerations refer to one particular material point $X \in \mathcal{B}$.

We now make the following *constitutive assumptions* defining the material under consideration. These assumptions refer to the common domain of the constitutive functionals $\overline{\psi}$, $\overline{\eta}$, $\overline{\mathbf{S}}$, $\mathbf{H}$ and their smoothness properties.

A 1. The constitutive functionals $\overline{\psi}$, $\overline{\eta}$, $\overline{\mathbf{S}}$, and $\mathbf{H}$ have for their domain of definition the set $\mathcal{D}\times\mathbf{V}$, where $\mathcal{D}\subset\mathcal{V}^+$ is an open and connected set satisfying the condition

$$(2.6)\qquad (\mathbf{A},a,\mathbf{a})\in\mathcal{D} \Rightarrow (\mathbf{A},a,\mathbf{0})\in\mathcal{D}.$$

A 2. The free energy functional $\overline{\psi}$ is continuous differentiable on $\mathcal{D}\times\mathbf{V}$, i.e. for every $\mathbf{\Lambda} = (\boldsymbol{\lambda},\mathbf{G})\in\mathcal{D}$, $\boldsymbol{\lambda} = (\mathbf{F},\theta)$, and $\mathbf{Q}\in\mathbf{V}$ we have

$$(2.7)\qquad \overline{\psi}(\mathbf{\Lambda}+\mathbf{\Gamma};\mathbf{Q}+\mathbf{u}) = \overline{\psi}(\mathbf{\Lambda};\mathbf{Q}) + \partial_{\mathbf{\Lambda}}\overline{\psi}(\mathbf{\Lambda};\mathbf{Q})\cdot\mathbf{\Gamma} + \partial_{\mathbf{Q}}\overline{\psi}\mathbf{\Lambda};\mathbf{Q})\cdot\mathbf{u} + o(|\mathbf{\Gamma}|+|\mathbf{u}|),$$

for any $\mathbf{\Gamma} = (\mathbf{A},a,\mathbf{a})\in\mathcal{V}$, and $\mathbf{u}\in\mathbf{V}$, with $(\mathbf{\Lambda}+\mathbf{\Gamma};\mathbf{Q}+\mathbf{u})\in\mathcal{D}\times\mathbf{V}$.

Moreover, the *partial derivative of* $\overline{\psi}$ with respect to $\mathbf{\Lambda}$

$$(2.8)\qquad \partial_{\mathbf{\Lambda}}\overline{\psi} = (\partial_{\boldsymbol{\lambda}}\overline{\psi},\partial_{\mathbf{G}}\overline{\psi}) : \mathcal{D}\times\mathbf{V}\to\mathcal{V}, \qquad \partial_{\boldsymbol{\lambda}}\overline{\psi} = (\partial_{\mathbf{F}}\overline{\psi},\partial_{\theta}\overline{\psi}),$$

and the *partial derivative* of $\overline{\psi}$ *with respect to* $\mathbf{Q}$

$$(2.9)\qquad \partial_{\mathbf{Q}}\overline{\psi} : \mathcal{D}\times\mathbf{V}\to\mathbf{V},$$

are continuous applications on $\mathcal{D}\times\mathbf{V}$.

A 3. The mappings $\overline{\eta}$, $\overline{\mathbf{S}}$ and $\mathbf{H}$ are continuous on $\mathcal{D} \times \mathbf{V}$.

A 4. The *heat evolution function* $\mathbf{H}$ is locally Lipschitzian, with respect to $\mathbf{Q}$, on $\mathcal{D} \times \mathbf{V}$ for *any fixed mapping*

$$\mathbf{\Lambda} : \mathbf{B} \times \mathbb{R} \to \mathcal{D}.$$

REMARK 2.1. From the assumption **A 1** it results that if $(\mathbf{A}, a, \mathbf{a}) \in \mathcal{D}$, then for every $\mathbf{a} \in \mathbf{V}\backslash\{\mathbf{0}\}$ there exists $\delta > 0$ such that $(\mathbf{A}, a, \alpha\mathbf{a}) \in \mathcal{D}$ as soon as $|\alpha| < \delta$.

REMARK 2.2. Suppose we are giving an initial time t_0, an *initial heat flux distribution* on $\mathcal{B}$,

$$\mathbf{X} \mapsto \mathbf{Q}_0 = \mathbf{Q}_0(\mathbf{X}) \in \mathbf{V}, \qquad \mathbf{X} \in \mathbf{B},$$

a smooth motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, and a smooth temperature field $\theta = \theta(\mathbf{X}, t)$ such that

$$\mathbf{\Lambda}(t) = (\mathbf{F}(X, t),\ \theta(\mathbf{X}, t),\ \mathbf{G}(\mathbf{X}, t)) \in \mathcal{D}, \qquad t \in \mathbf{I},$$

where $I \subset \mathbb{R}$ is an interval containing t_0. Assumption **A 4** guarantees the existence and the uniqueness [6, 7] of the solution

$$\dot{\mathbf{Q}} = \mathbf{H}(\mathbf{\Lambda}(t); \mathbf{Q}). \tag{2.10}$$

With $(\mathbf{\Lambda}(t); \mathbf{Q}(t)) \in \mathcal{D} \times \mathbf{V}$, $t \in (t_0 - \delta, t_0 + \delta)$, determined in this manner, from (1.15), (2.2)–(2.4), we obtain $\psi(t) = \psi(\mathbf{X}, t)$, $\eta(t) = \eta(\mathbf{X}, t)$, $\varepsilon(t) = \varepsilon(\mathbf{X}, t) = \psi(t) + \theta(t)\eta(t)$, $\mathbf{S}(t) = \mathbf{S}(\mathbf{X}, t)$ and, from (1.11), (1.13), we get the specific body force $\mathbf{b}(\mathbf{X}, t)$ and the radiant heating $r(\mathbf{X}, t)$.

Thus to each sufficiently smooth choice of $\mathbf{Q}_0$, χ, and θ there corresponds a unique process

$$\pi^* = (\chi, \theta, \varepsilon, \eta, \mathbf{S}, \mathbf{Q}, \mathbf{b}, r) \in \mathbb{P}(\mathcal{B}), \quad \text{on} \quad (t_0 - \delta, t_0 + \delta). \tag{2.11}$$

REMARK 2.3. For every state $(\boldsymbol{\lambda}_0; \mathbf{Q}_0) = (\mathbf{F}_0, \theta_0, \mathbf{G}_0, \mathbf{Q}_0) \in \mathcal{D} \times \mathbf{V}$, given at the material point $X \in \mathcal{B}$ occupying the place $\mathbf{X} \in \mathbf{B}$, for every $t_0 \in \mathbb{R}$, and for arbitrarily chosen $\Gamma = (\mathbf{A}, a, \mathbf{a}) \in \mathcal{V}$ there exists an admissible process $\pi \in \mathbb{P}(\mathcal{B})$ such that the states

$$(\mathbf{\Lambda}(\mathbf{X}, t);\ \mathbf{Q}(\mathbf{X}, t)) = (\mathbf{F}(\mathbf{X}, t), \theta(\mathbf{X}, t), \mathbf{G}(\mathbf{X}, t); \mathbf{Q}(\mathbf{X}, t))$$

corresponding to the process π satisfy the conditions

$$(\mathbf{\Lambda}(\mathbf{X}, t_0); \mathbf{Q}(\mathbf{X}, t_)) = (\mathbf{\Lambda}_0; \mathbf{Q}_0), \qquad \dot{\mathbf{\Lambda}}\,(\mathbf{X}, t_0) = \Gamma. \tag{2.12}$$

The proof of the statements in this Remark may be found in [1, 4, 5].

DEFINITION 2.1. *The constitutive equations* (2.2)–(2.5) *are said to be* ***compatible with the Second Law of Thermodynamics*** *if for every choice of sufficiently smooth initial heat flux distribution* $\mathbf{Q}_0$, *motion* $\boldsymbol{\chi}$, *and temperature field* θ, *the process*

$\pi^* \in \mathbb{P}(\mathcal{B})$ *is an admissible process i.e. the constitutive functionals* $\overline{\psi}$, $\overline{\eta}$, $\overline{\mathbf{S}}$, *and* $\mathbf{H}$ *satisfy the dissipation inequality* (1.16) *at each time* t *and for all material points* $X \in \mathcal{B}$.

The content of this definition is referred to as the *Principle of Thermomechanically Compatible Determinism* [3].

Using the line of arguments in [1, 4, 5] and the results of the Remark 2.3 the following theorem can be proved (cf. [8]).

THEOREM 2.1. *If the functions* $\overline{\psi}$, $\overline{\eta}$, $\overline{\mathbf{S}}$, *and* $\mathbf{H}$ *obey the assumptions* **A** 1 – **A** 4 *then the constitutive equations* (2.2) – (2.4) *are compatible with the second law of thermodynamics if and only if for any smooth motion, temperature field, and initial heat flux distribution, the following conditions hold:*

1) *the free energy function* $\overline{\psi}$ *is independent of* $\mathbf{G}$, *i.e.*

$$\psi(t) = \widehat{\psi}(\boldsymbol{\lambda}(t); \mathbf{Q}(t)), \qquad \boldsymbol{\lambda}(t) = (\mathbf{F}(t), \theta(t)); \tag{2.13}$$

2) *the functions* $\overline{\eta}$ *and* $\overline{\mathbf{S}}$ *are independent of* $\mathbf{G}$, *i.e.*

$$\eta(t) = \widehat{\eta}(\boldsymbol{\lambda}(t); \mathbf{Q}(t)), \qquad \mathbf{S}(t) = \widehat{\mathbf{S}}(\boldsymbol{\lambda}(t); \mathbf{Q}(t)), \tag{2.14}$$

and the functionals $\widehat{\eta}$ *and* $\widehat{\mathbf{S}}$ *are determined by the function* $\widehat{\psi}$ *through the relations*

$$\widehat{\eta} = -\partial_\theta \widehat{\psi} \qquad \widehat{\mathbf{S}} = \varrho_\kappa \partial_{\mathbf{F}} \widehat{\psi}; \tag{2.15}$$

3) *the* ***Dissipation Inequality*** *is satisfied*

$$\varrho_\kappa \theta(t) \partial_{\mathbf{Q}} \widehat{\psi}(\boldsymbol{\lambda}(t); \mathbf{Q}(t)) \cdot \mathbf{H}(\boldsymbol{\Lambda}(t); \mathbf{Q}(t)) + \mathbf{Q}(t) \cdot \mathbf{G}(t) \leq 0. \tag{2.16}$$

REMARK 2.4. Following [1], the quantity

$$\sigma = \widehat{\sigma}(\boldsymbol{\Lambda}; \mathbf{Q}) = -(1/\theta) \partial_{\mathbf{Q}} \widehat{\psi}(\boldsymbol{\lambda}; \mathbf{Q}) \cdot \mathbf{H}(\boldsymbol{\Lambda}; \mathbf{Q}) \tag{2.17}$$

is referred to as the *internal dissipation*.

If we denote by $\widehat{\sigma}_0$ the restriction of $\widehat{\sigma}$ to the set

$$\Delta = \{(\boldsymbol{\Lambda}; \mathbf{Q}) \equiv (\mathbf{F}, \theta, \mathbf{G}; \mathbf{Q}) \in \mathcal{D} \times \mathbf{V} / \mathbf{Q} \cdot \mathbf{G} = 0\}, \tag{2.18}$$

then from (2.16) we get the inequality

$$\sigma_0 = \widehat{\sigma}_0(\boldsymbol{\lambda}, \mathbf{G}; \mathbf{Q}) \geq 0, \tag{2.19}$$

which is called *internal dissipation inequality*. In virtue of (2.6) we remark that $\delta \neq 0$ and that

$$\widehat{\sigma}_0(\boldsymbol{\lambda}, \mathbf{0}; \mathbf{Q}) = -(1/\theta) \partial_{\mathbf{Q}} \widehat{\psi}(\boldsymbol{\lambda}; \mathbf{Q}) \cdot \mathbf{H}(\boldsymbol{\Lambda}, \mathbf{0}; \mathbf{Q}) \geq 0. \tag{2.20}$$

Because of (2.15) we have

$$\dot{\psi} = \partial_{\mathbf{F}}\widehat{\psi}(\boldsymbol{\lambda};\mathbf{Q})\cdot\dot{\mathbf{F}} + \partial_{\boldsymbol{\theta}}\widehat{\psi}(\boldsymbol{\lambda};\mathbf{Q})\dot{\boldsymbol{\theta}} + \partial_{\mathbf{Q}}\widehat{\psi}(\boldsymbol{\lambda};\mathbf{Q})\cdot\mathbf{H}(\boldsymbol{\Lambda};\mathbf{Q}) = \mathbf{S}\cdot\mathbf{F} - \eta\dot{\boldsymbol{\theta}} - \theta\sigma \tag{2.21}$$

so that

$$\dot{\boldsymbol{\lambda}} = (\dot{\mathbf{F}}, \dot{\boldsymbol{\theta}}) = \mathbf{0} \Rightarrow \sigma = -(1/\theta)\,\dot{\psi}. \tag{2.22}$$

Since, as it results from (1.15) and $(2.21)_2$,

$$\dot{\varepsilon} = \mathbf{S}\cdot\dot{\mathbf{F}} - \theta\,\dot{\eta} - \theta\sigma, \tag{2.23}$$

we obtain the following implications

$$\dot{\mathbf{F}} = 0, \qquad \dot{\eta} = 0 \quad\Rightarrow\quad \sigma = -(\dot{\varepsilon}/\theta), \tag{2.24}$$

$$\dot{\mathbf{F}} = 0, \qquad \dot{\eta} = 0 \quad\Rightarrow\quad \sigma = \dot{\eta}. \tag{2.25}$$

In the present theory σ plays the part it did in [1].

REMARK 2.5. The Dissipation Inequality (2.16) imposes a severe limitation on the free energy functional $\widehat{\psi}$ and on the heat evolution functional **H**. The restriction of this limitation to the set Δ (see (2.18)) takes the form

$$\partial_{\mathbf{Q}}\widehat{\psi}(\boldsymbol{\lambda};\mathbf{Q})\cdot\mathbf{H}(\boldsymbol{\Lambda};\mathbf{Q}) \le 0, \qquad (\boldsymbol{\lambda},\mathbf{G};\mathbf{Q}) \in \mathcal{D}\times\mathbf{V}. \tag{2.26}$$

In particular, we have

$$\partial_{\mathbf{Q}}\widehat{\psi}(\boldsymbol{\lambda};\mathbf{Q})\cdot\mathbf{H}(\boldsymbol{\Lambda},\mathbf{0};\mathbf{Q}) \le 0, \qquad (\boldsymbol{\lambda},\mathbf{0};\mathbf{Q}) \in \mathcal{D}\times\mathbf{V}. \tag{2.27}$$

3. Materials with Cattaneo heat flux equation

In this section we suppose that the heat flux evolution functional is linear in **Q** and **G**, i.e.

$$\begin{gathered}\mathbf{H}(\boldsymbol{\Lambda};\mathbf{Q}) = \mathbf{M}(\boldsymbol{\lambda})\mathbf{Q} + \mathbf{N}(\boldsymbol{\lambda})\mathbf{G},\\ (\boldsymbol{\Lambda};\mathbf{Q}) = (\boldsymbol{\lambda},\mathbf{G};\mathbf{Q}) \in \mathcal{D}\times\mathbf{V}, \qquad \boldsymbol{\lambda} = (\mathbf{F},\boldsymbol{\theta}),\end{gathered} \tag{3.1}$$

where the second order tensor functions

$$\boldsymbol{\lambda} \to \mathbf{M}(\boldsymbol{\lambda}), \qquad \mathbf{N}(\boldsymbol{\lambda}) \in \mathrm{Lin}, \qquad (\boldsymbol{\lambda},\mathbf{G};Q) \in \mathcal{D}\times\mathbf{V} \tag{3.2}$$

are nonsingular, and we derive the implication of this assumption on the constitutive functionals $\widehat{\psi}$, $\widehat{\eta}$, $\widehat{\mathbf{S}}$, and

$$\widehat{\varepsilon} = \widehat{\psi} + \theta\widehat{\eta}. \tag{3.3}$$

Inserting (3.1) into (2.16) we conclude that the tensor functions (3.2) must satisfy the inequality

$$\varrho_\kappa \left[\mathbf{M}^T(\boldsymbol{\lambda})\partial_{\mathbf{Q}}\widehat{\psi}(\boldsymbol{\lambda};\mathbf{Q})\right] \cdot \mathbf{Q} + \left[\varrho_\kappa \mathbf{N}^T(\boldsymbol{\lambda})\partial_{\mathbf{Q}}\widehat{\psi}(\boldsymbol{\lambda};\mathbf{Q}) + (1/\theta)\mathbf{Q}\right] \cdot \mathbf{G} \leq 0 \tag{3.4}$$

for every $(\boldsymbol{\lambda}, \mathbf{G}; \mathbf{Q}) \in \mathcal{D} \times \mathbf{V}$.

THEOREM 3.1. *The inequality* (3.4) *holds on* $\mathcal{D} \times \mathbf{V}$ *if and only if the relations*

$$\left[\mathbf{M}^T(\boldsymbol{\lambda})\partial_{\mathbf{Q}}\widehat{\psi}(\boldsymbol{\lambda};\mathbf{Q})\right] \cdot \mathbf{Q} \leq 0, \tag{3.5}$$

$$\varrho_\kappa \mathbf{N}^T(\boldsymbol{\lambda})\partial_{\mathbf{Q}}\widehat{\psi}(\boldsymbol{\lambda};\mathbf{Q}) = -(1/\theta)\mathbf{Q} \tag{3.6}$$

are satisfied on $\mathcal{D} \times \mathbf{V}$.

P r o o f. If $\mathbf{Q}, \mathbf{G} \in \mathbf{V}$ are arbitrary, as they are supposed to be in [8], the theorem is rather evident. But this is not the case because $(\boldsymbol{\lambda}, \mathbf{G}; \mathbf{Q}) \in \mathcal{D} \times \mathbf{V}$, and the domain $\mathcal{D}$ is *a priori* given. It is obvious that (3.5) and (3.6) are sufficient for (3.4). On the other hand, (3.4) and (3.6) imply (3.5). So it remains to prove that (3.4) implies (3.6). To prove this implication we will prove its contrapositive assertion. The relation (3.6) does not hold on $\mathcal{D} \times \mathbf{V}$ if there exists $\boldsymbol{\lambda}_0 = (\mathbf{F}_0, \theta_0) \in \mathrm{Lin}^+ \times \mathbb{R}$ and $\mathbf{G} \in \mathbf{V}$ with $(\boldsymbol{\lambda}_0, \mathbf{G}) \in \mathcal{D}$ such that $\varrho_\kappa \mathbf{N}^T(\boldsymbol{\lambda}_0)\partial_{\mathbf{Q}}\widehat{\psi}(\boldsymbol{\lambda}_0; \mathbf{0}) = \mathbf{u} \neq \mathbf{0}$.

From the assumption **A** 1 and from Remark 2.1 it results that there exists $\alpha_0 > 0$ such that $(\boldsymbol{\lambda}_0, \mathbf{G}_0) \in \mathcal{D}$ where $\mathbf{G}_0 = \alpha_0 \mathbf{u}$. For the point $(\boldsymbol{\lambda}_0, \mathbf{G}_0; \mathbf{0}) \in \mathcal{D} \times \mathbf{V}$ the left-hand side of (3.4) becomes $\alpha_0 \mathbf{u} \cdot \mathbf{u} > 0$ and this contradicts (3.4). The theorem is proved.

REMARK 3.1. For any $(\boldsymbol{\lambda}, G) = (\mathbf{F}, \theta, \mathbf{G}) \in \mathcal{D}$ the mapping

$$\theta \mathbf{N}^T(\boldsymbol{\lambda})\partial_{\mathbf{Q}}\psi(\lambda; \cdot) \; : \; \mathbf{V} \to \mathbf{V} \tag{3.7}$$

is an invertible linear transformation on $\mathbf{V}$, namely a similarity transformation of coefficient $k = 1/\varrho_\kappa$.

REMARK 3.2. Let us introduce the notations

$$\mathbf{T} = -\mathbf{M}^{-1}, \qquad \mathbf{Z} = -\mathbf{N}^{-1}, \qquad \mathbf{K} = \mathbf{T}\,\mathbf{Z}^{-1}. \tag{3.8}$$

With these notations, from (3.1) and (2.5), we obtain

$$\mathbf{T}(\boldsymbol{\lambda}\;\dot{\mathbf{Q}} + \mathbf{Q} = -\mathbf{K}(\boldsymbol{\lambda})\mathbf{G}, \qquad (\boldsymbol{\lambda}, \mathbf{G}; \mathbf{Q}) \in \mathcal{D} \times \mathbf{V}, \tag{3.9}$$

where the tensor functions $\boldsymbol{\lambda} \to \mathbf{T}(\lambda)$, $\mathbf{K}(\lambda) \in \mathrm{Lin}$ are nonsingular.

Equation (3.9) is the *Cattaneo heat flux evolution equation*.

Supposing that $\widehat{\psi}$ is twice continuously differentiable on $\mathcal{D} \times \mathbf{V}$ it results that **Z**, and therefore **N**, is symmetric and it is given by

$$\mathbf{Z} = -\mathbf{N}^{-1} = \varrho_\kappa \theta \partial_{\mathbf{Q}^2}\widehat{\psi}(\boldsymbol{\lambda}; \mathbf{Q}). \tag{3.10}$$

On the other hand, in view of (3.8) and (3.6), from (3.5) it follows

$$\mathbf{Q}\cdot\mathbf{K}^{-1}(\boldsymbol{\lambda})\mathbf{Q} \geq 0, \tag{3.11}$$

which shows that $\mathbf{K}$ is positive definite because $\mathbf{Q} \in \mathbf{V}$ is arbitrary and $\mathbf{K}$ is invertible.

The internal dissipation (2.17) is now given by

$$\sigma = \widehat{\sigma}(\Lambda;\mathbf{Q}) = (1/\varrho_\kappa\theta^2)\left[\mathbf{Q}\cdot\mathbf{K}^{-1}\mathbf{Q} + \mathbf{Q}\cdot\mathbf{G}\right], \tag{3.12}$$

and (3.11) is a consequence of the internal dissipation inequality which now becomes

$$\sigma_0 = \widehat{\sigma}_0(\Lambda;\mathbf{Q}) = (1/\varrho_\kappa\theta^2)\mathbf{Q}\cdot\mathbf{K}^{-1}(\boldsymbol{\lambda})\mathbf{Q} \geq 0. \tag{3.13}$$

Taking into account that $\mathbf{N}^T = \mathbf{N} = -\mathbf{Z}^{-1}$, from (3.6) we obtain (see [9] and [8])

$$\varrho_\kappa\widehat{\psi}(\boldsymbol{\lambda};\mathbf{Q}) = \varrho_\kappa\widehat{\psi}_0(\boldsymbol{\lambda}) + (1/2\theta)\mathbf{Q}\cdot\mathbf{Z}(\boldsymbol{\lambda})\mathbf{Q}. \tag{3.14}$$

From this relation, (2.15), and (1.15) we get

$$\varrho_\kappa\widehat{\varepsilon}(\boldsymbol{\lambda};\mathbf{Q}) = \varrho_\kappa\widehat{\varepsilon}_0(\boldsymbol{\lambda}) + \mathbf{Q}\cdot\mathbf{A}(\boldsymbol{\lambda})\mathbf{Q}, \tag{3.15}$$

$$\varrho_\kappa\widehat{\eta}(\boldsymbol{\lambda};\mathbf{Q}) = \varrho_\kappa\widehat{\eta}_0(\boldsymbol{\lambda}) + \mathbf{Q}\cdot\mathbf{B}(\boldsymbol{\lambda})\mathbf{Q}, \tag{3.16}$$

$$\widehat{\mathbf{S}}(\Lambda;\mathbf{Q}) = \widehat{\mathbf{S}}_0(\boldsymbol{\lambda}) + \mathbf{Q}\cdot\mathbf{P}(\boldsymbol{\lambda})\mathbf{Q}, \tag{3.17}$$

where

$$\widehat{\eta}_0 = -\partial_\theta\widehat{\psi}_0\,, \qquad \widehat{\varepsilon}_0 = \widehat{\psi}_0 - \theta\partial_\theta\widehat{\psi}_0 = \widehat{\psi}_0 + \theta\widehat{\eta}_0\,, \qquad \widehat{\mathbf{S}}_0 = \varrho_x\partial_{\mathbf{F}}\widehat{\psi}_0 \tag{3.18}$$

and

$$\begin{aligned}\mathbf{A} &= -(1/2\theta^2)\partial_\theta\left[(1/\theta^2)\mathbf{Z}\right], \qquad \mathbf{B} = -(1/2\theta)\partial_\theta\left[(1/\theta)\mathbf{Z}\right],\\ \mathbf{P} &= (1/2\theta)\partial_{\mathbf{F}}\mathbf{Z}.\end{aligned} \tag{3.19}$$

REMARK 3.3. When the heat flux evolution functional $\mathbf{H}$ is of the form (3.1), the observations in Remark 2.5 are more specific. From (3.10) it results that for every $(\boldsymbol{\lambda},\mathbf{G}) \in \mathcal{D}$ the mapping

$$\widehat{\psi}(\boldsymbol{\lambda};\,\cdot\,) \;:\; \mathbf{V} \to \mathbb{R} \tag{3.20}$$

is a nonsingular quadratic form having the matrix $-(1/\varrho_\kappa\theta)\mathbf{N}^{-1}(\boldsymbol{\lambda}) = (1/\varrho_\kappa\theta)\mathbf{Z}$.

So the nonsingular tensor $\mathbf{N}(\boldsymbol{\lambda})$ in (3.10) is completely determined by the free energy functional $\widehat{\psi}$. The invertible tensor $\mathbf{M}(\boldsymbol{\lambda})$ in (3.10) depends on $\widehat{\psi}$ through the relation $\mathbf{M}(\boldsymbol{\lambda}) = \mathbf{N}(\boldsymbol{\lambda})\mathbf{K}^{-1}(\boldsymbol{\lambda})$ where $\mathbf{K}(\boldsymbol{\lambda})$ is a arbitrary positive definite second order tensor.

4. Stability of isothermal equilibrium states (E.S.)

Throughout this and the following section we suppose that the heat flux evolution functional $\mathbf{H}$ is continuously differentiable on $\mathcal{D} \times \mathbf{V}$ and that the second order tensors $\partial_{\mathbf{G}}\mathbf{H}(\boldsymbol{\lambda}, \mathbf{G}; \mathbf{Q})$ and $\partial_{\mathbf{Q}}\mathbf{H}(\boldsymbol{\lambda}, \mathbf{G}; \mathbf{Q})$ are nonsingular on $\mathcal{D} \times \mathbf{V}$.

With these conditions, the equation

$$\mathbf{H}(\boldsymbol{\lambda}, \mathbf{G}; \mathbf{Q}) = 0 \tag{4.1}$$

defines the implicit functions

$$\mathbf{Q} = \widehat{\mathbf{Q}}(\boldsymbol{\lambda}, \mathbf{G}), \qquad \mathbf{H}(\boldsymbol{\lambda}, \mathbf{G}; \widehat{\mathbf{Q}}(\boldsymbol{\lambda}, \mathbf{G})) = \mathbf{0}; \qquad \mathbf{Q}_0 = \widehat{\mathbf{Q}}(\boldsymbol{\lambda}_0, \mathbf{G}_0) \tag{4.2}$$

and

$$\mathbf{G} = \widehat{\mathbf{G}}(\boldsymbol{\lambda}; \mathbf{Q}), \qquad \mathbf{H}(\boldsymbol{\lambda}, \widehat{\mathbf{G}}(\boldsymbol{\lambda}; \mathbf{Q}); \mathbf{Q}) = \mathbf{0}; \qquad \mathbf{G}_0 = \widehat{\mathbf{G}}(\boldsymbol{\lambda}_0; \mathbf{Q}_0) \tag{4.3}$$

in a neighbourhood $U \subset \mathcal{D} \times \mathbf{V}$ of a solution $(\boldsymbol{\lambda}_0, \mathbf{G}_0; \mathbf{Q}_0)$ of the equation (4.1).

The functions $(4.2)_1$ and $(4.3)_1$ are satisfying the indentities

$$\widehat{\mathbf{Q}}(\boldsymbol{\lambda}, \widehat{\mathbf{G}}(\boldsymbol{\lambda}; \mathbf{Q})) = \mathbf{Q}, \qquad \widehat{\mathbf{G}}(\boldsymbol{\lambda}; \widehat{\mathbf{Q}}(\boldsymbol{\lambda}, \mathbf{G})) = \mathbf{G} \tag{4.4}$$

on U and are differentiable in certain neighbourhoods of $(\boldsymbol{\lambda}_0, \mathbf{G}_0)$ and $(\boldsymbol{\lambda}_0, \mathbf{Q}_0)$, respectively.

All the following considerations refer to an arbitrary fixed material point $X \in \mathcal{B}$ having the position $\mathbf{X} \in \mathbf{B}$ in the configuration κ.

DEFINITION 4.1. *The state* $(\boldsymbol{\lambda}_0, \mathbf{G}_0; \mathbf{Q}_0) \in \mathcal{D} \times \mathbf{V}$, $\boldsymbol{\lambda}_0 = (\mathbf{F}_0, \theta_0)$, *is called an* ***isothermal E.S. at constant strain*** $\mathbf{F}_0$ *for the material point* $X \in \mathcal{B}$ *if it is a solution of* Eq. (4.1) *and if* $\mathbf{G}_0 \cdot \mathbf{Q}_0 = 0$. *The state* $(\boldsymbol{\lambda}_0, \mathbf{0}; \mathbf{Q}_0) \in \mathcal{D} \times \mathbf{V}$, $\boldsymbol{\lambda}_0 = (\mathbf{F}_0, \theta_0)$ *is called a* ***strictly isothermal E.S. at constant strain*** $\mathbf{F}_0$ *for the material point* $X \in \mathcal{B}$ *if it verifies* Eq. (4.1).

With these definitions, the following theorem can be proved (see [8]).

THEOREM 4.1. *If the functional* $\mathbf{H}$ *satisfies the above conditions, then*:

1) *every state* $(\boldsymbol{\lambda}, \mathbf{0}; \mathbf{0}) \in \mathcal{D} \times \mathbf{V}$ *is a strictly E.S.*;

2) *the second order tensor*

$$[\partial_{\mathbf{Q}}\mathbf{H}(\boldsymbol{\lambda}, \mathbf{0}; \mathbf{0})]^{-1}\, \partial_{\mathbf{G}}\mathbf{H}(\boldsymbol{\lambda}, \mathbf{0}; \mathbf{0}) \tag{4.5}$$

is positive definite.

We denote by $\mathcal{E} \subset \mathcal{D} \times \mathbf{V}$ the set of isothermal E.S. and by $\mathcal{E}_0 \subset \mathcal{E}$ the subset of strictly isothermal E.S. at constant strain for a material point $X \in \mathcal{B}$.

REMARK 4.1. The preceding theorem shows that $\mathcal{E}_0$ and therefore $\mathcal{E}$ are nonvoid sets. Moreover, for every *a priori* given $\boldsymbol{\lambda}_0 = (\mathbf{F}_0, \theta_0) \in \mathrm{Lin}^+ \times \mathbb{R}^+$ at $X \in \mathcal{B}$ the nonvoid set

$$\Sigma(\boldsymbol{\lambda}_0) = \{(\boldsymbol{\lambda}_0, \mathbf{G}; \mathbf{Q}) \in \mathcal{D} \times \mathbf{V} \mid \mathbf{G} \cdot \mathbf{Q} = 0,\ \mathbf{H}(\boldsymbol{\lambda}_0, \mathbf{G}; \mathbf{Q}) = \mathbf{0}\} \subset \mathcal{E} \tag{4.6}$$

is a 2-dimensional manifold in the 6-dimensional space of tuples $(\mathbf{G};\mathbf{Q})$ and $(\mathbf{0};\mathbf{0}) \in \Sigma(\boldsymbol{\lambda}_0)$. From (4.1) – (4.4) it results that for every $(\mathbf{G}_0;\mathbf{Q}_0) \in \Sigma(\boldsymbol{\lambda}_0)$ there exists a neighbourhood $U(\mathbf{G}_0;\mathbf{Q}_0)$ such that

$$
\begin{aligned}
(4.7)\qquad U(\mathbf{G},\mathbf{Q}_0)\cap\Sigma(\boldsymbol{\lambda}_0) &= \left\{(\boldsymbol{\lambda}_0,\mathbf{G};\mathbf{Q})\in\mathcal{D}\times\mathbf{V}\mid \mathbf{G}=\widehat{\mathbf{G}}(\boldsymbol{\lambda}_0;\mathbf{Q}),\, \mathbf{Q}\cdot\widehat{\mathbf{G}}(\boldsymbol{\lambda}_0;\mathbf{Q})=0\right\}\\
&= \left\{(\boldsymbol{\lambda}_0,\mathbf{G};\mathbf{Q})\in\mathcal{D}\times\mathbf{V}\mid \mathbf{Q}=\widehat{\mathbf{Q}}(\boldsymbol{\lambda}_0;\mathbf{G}),\ \mathbf{G}\cdot\widehat{\mathbf{Q}}(\boldsymbol{\lambda}_0;\mathbf{G})=0\right\}\\
= \Big\{(\boldsymbol{\lambda}_0,\mathbf{G};\mathbf{Q})\in\mathcal{D}\times\mathbf{V}\mid \mathbf{G}&=\widehat{\mathbf{G}}(\boldsymbol{\lambda}_0;\mathbf{Q}),\ \mathbf{Q}=\widehat{\mathbf{Q}}(\boldsymbol{\lambda}_0;\mathbf{G}),\ \widehat{\mathbf{G}}(\boldsymbol{\lambda}_0;\mathbf{Q})\cdot\widehat{\mathbf{Q}}(\boldsymbol{\lambda}_0;\mathbf{G})=0\Big\}.
\end{aligned}
$$

REMARK 4.2. The only isothermal E.S. at constant strain for a material point $X \in \mathcal{B}$ of the thermoelastic materials with Cattaneo's heat flux evolution equation (3.9) is the strictly E.S. $(\boldsymbol{\lambda}_0,\mathbf{0};\mathbf{0})$.

DEFINITION 4.2 *If* $(\boldsymbol{\lambda}_0,\mathbf{G}_0;\mathbf{Q}_0) \in \mathcal{E}$, *then the set* $D(\boldsymbol{\lambda}_0,\mathbf{G}_0;\mathbf{Q}_0) \subset \mathbf{V}$ *of vectors* $\mathbf{Q}^* \in \mathbf{V}$ *for which the solution* $\mathbf{Q}=\mathbf{Q}(t)$ *of the Cauchy problem*

$$
(4.8)\qquad \dot{\mathbf{Q}} = \mathbf{H}(\boldsymbol{\lambda}_0,\mathbf{G}_0;\mathbf{Q}),\qquad \mathbf{Q}(0)=\mathbf{Q}^*,
$$

exists on $[0,\infty)$ *and satisfies the condition*

$$
(4.9)\qquad \lim_{t\to\infty}\mathbf{Q}(t)=\mathbf{Q}_0,
$$

is called the ***domain of attraction*** *of the E.S.* $(\boldsymbol{\lambda}_0,\mathbf{G}_0;\mathbf{Q}_0)$ *at constant strain and temperature.*

If $\mathbf{Q}_0 \in D(\boldsymbol{\lambda}_0,\mathbf{G}_0;\mathbf{Q}_0)$ *is an interior point, then* $(\boldsymbol{\lambda}_0,\mathbf{G}_0;\mathbf{Q}_0) \in \mathcal{E}$ *is said to be an* ***asymptotically stable*** E.S.

The E.S. $(\boldsymbol{\lambda}_0,\mathbf{G}_0;\mathbf{Q}_0) \in \mathcal{E}$ *is called* ***Lyapunov stable*** *if for each* $\varepsilon > 0$ *there exists* $\delta=\delta(\varepsilon)>0$ *such that every solution* $\mathbf{Q}=\mathbf{Q}(t)$ *of* Eq. $(4.8)_1$ *satisfies*

$$
(4.10)\qquad |\mathbf{Q}(t)-\mathbf{Q}_0| < \varepsilon,\qquad t\ge 0,
$$

whenever

$$
(4.11)\qquad |\mathbf{Q}(0)-\mathbf{Q}| < \delta.
$$

REMARK 4.3. For every $\mathbf{Q}^* \in D(\boldsymbol{\lambda}_0,\mathbf{G}_0;\mathbf{Q}_0)$ and every $\mathbf{X} \in \mathbf{B}$ there exists at least one process $\pi^* \in \mathbb{P}(\mathcal{B})$ such that

$$
(4.12)\qquad
\begin{aligned}
&\mathbf{Q}(\mathbf{X},0)=\mathbf{Q}^*,\qquad \mathbf{F}(\mathbf{X},t)=\mathbf{F}_0,\qquad \theta(\mathbf{X},t)=\theta_0,\\
&\mathbf{G}(\mathbf{X},t)\cdot\mathbf{Q}(\mathbf{X},t)=0,\qquad t\ge 0.
\end{aligned}
$$

Indeed, using the Remark 2.1 it results that the process $\pi^* \in \mathbb{P}(\mathcal{B})$ defined by the motion

$$
(4.13)\qquad \mathbf{x}=\mathbf{X}(\mathbf{Y},t)=\mathbf{X}+\mathbf{F}_0[\mathbf{Y}-\mathbf{X}],\qquad (\mathbf{Y},t)\in\mathbf{B}\times[0,\infty),
$$

and by the temperature field

$$\theta = \theta(\mathbf{Y}, t) = \theta_0 + \mathbf{g}(t) \cdot [\mathbf{Y} - \mathbf{X}], \qquad (\mathbf{Y}, t) \in \mathbf{B} \times [0, \infty), \tag{4.14}$$

where $t \to \mathbf{g}(t) \in \mathbf{V}$, $t \in [0, \infty)$, is a differentiable application satisfying the condition $\mathbf{g}(t) \cdot \mathbf{Q}(t) = 0$, $t \geq 0$, and $\mathbf{Q}(t) = \mathbf{Q}(\mathbf{X}, t)$, the solution of the Cauchy problem (4.8), satisfies (4.12). For the process here defined we have $\dot{\mathbf{F}}(\mathbf{x}, t) = 0$ and $\dot{\theta}(\mathbf{x}, t) = 0$.

THEOREM 4.2.

1) *If* $(\boldsymbol{\lambda}_0, \mathbf{G}_0; \mathbf{Q}_0) \in \mathcal{E}$, $\boldsymbol{\lambda}_0 = (\mathbf{F}_0, \theta_0)$, *then*

$$\widehat{\psi}(\boldsymbol{\lambda}_0; \mathbf{Q}^*) \geq \widehat{\psi}(\boldsymbol{\lambda}_0; \mathbf{Q}_0), \qquad \mathbf{Q} \in D(\boldsymbol{\lambda}_0, \mathbf{G}_0; \mathbf{Q}_0); \tag{4.15}$$

2) *if* $(\boldsymbol{\lambda}_0, \mathbf{G}_0; \mathbf{Q}_0) \in \mathcal{E}$ *is asymptotically stable then the preceding inequality holds in a neighbourhood* $U(\mathbf{Q}_0)$ *of* $\mathbf{Q}_0$, $U(\mathbf{Q}_0) \subset D(\boldsymbol{\lambda}_0, \mathbf{G}_0; \mathbf{Q}_0)$ *and, consequently, there exists* $\nu_0 \in \mathbb{R}$ *such that*

$$\partial_{\mathbf{Q}} \widehat{\psi}(\boldsymbol{\lambda}_0; \mathbf{Q}_0) = \nu_0 \mathbf{G}_0; \tag{4.16}$$

3) *if* $(\boldsymbol{\lambda}_0, 0; \mathbf{Q}_0) \in \mathcal{E}_0$ *and there exists a neighbourhood* $U(\mathbf{Q}_0)$ *of* $\mathbf{Q}_0$ *such that*

$$\widehat{\psi}(\boldsymbol{\lambda}_0; \mathbf{Q}) > \widehat{\psi}(\boldsymbol{\lambda}_0; \mathbf{Q}_0), \qquad \mathbf{Q} \neq \mathbf{Q}_0 \in U(\mathbf{Q}_0) \cap D(\boldsymbol{\lambda}_0, \mathbf{G}_0; \mathbf{Q}_0), \tag{4.17}$$

then $(\boldsymbol{\lambda}_0, \mathbf{0}; \mathbf{Q}_0)$ *is Lyapunov stable.*

P r o o f.

1. From (1.21) it results that for processes π^* constructed as in Remark 4.3 we have $\dot{\psi}(t) \leq 0$ on $[0, \infty)$, and consequently

$$\widehat{\psi}(\boldsymbol{\lambda}_0; \mathbf{Q}(t)) = \widehat{\psi}(t) \leq \widehat{\psi}(0) = \widehat{\psi}(\boldsymbol{\lambda}_0; \mathbf{Q}(0)) = \widehat{\psi}(\boldsymbol{\lambda}_0; \mathbf{Q}^*), \qquad t \geq 0.$$

If we make here $t \to \infty$, and take into account that $\lim_{t \to \infty} \mathbf{Q}(t) = \mathbf{Q}_0$ because $\mathbf{Q}^* \in D(\boldsymbol{\lambda}_0, \mathbf{G}_0; \mathbf{Q}_0)$, we obtain (4.15).

2. In our hypotheses the differentiable function $\widehat{\psi}(\boldsymbol{\lambda}_0, \cdot)$ attains its minimum at $\mathbf{Q}_0$ on the set $U(\mathbf{G}_0, \mathbf{Q}_0) \cap \Sigma(\boldsymbol{\lambda}_0)$, as described in $(4.7)_1$. This means that $\mathbf{Q}_0$ is a point of local conditional minimum under the side condition $\mathbf{Q} \cdot \mathbf{G}(\boldsymbol{\lambda}_0; \mathbf{Q}) = 0$. Therefore there exists $\nu_0 \in \mathbb{R}$ such that

$$\text{(a)} \qquad \partial_{\mathbf{Q}} \widehat{\psi}(\boldsymbol{\lambda}_0; \mathbf{Q}_0) = \nu_0 \left[\widehat{\mathbf{G}}(\boldsymbol{\lambda}_0; \mathbf{Q}_0) + \mathbf{Q}_0 \partial_{\mathbf{Q}} \widehat{\mathbf{G}}(\boldsymbol{\lambda}_0; \mathbf{Q}_0) \right].$$

On the other hand, differentiating the relation $\widehat{\mathbf{Q}}(\boldsymbol{\lambda}_0; \mathbf{G}) \cdot \widehat{\mathbf{G}}(\boldsymbol{\lambda}_0; \mathbf{Q}) = 0$ (see $(4.7)_3$) with respect to $\mathbf{Q}$ at the point $\mathbf{Q}_0$ and taking into account $(4.2)_3$ we obtain

$$\text{(b)} \qquad \mathbf{Q}_0 \partial_{\mathbf{Q}} \widehat{\mathbf{G}}(\boldsymbol{\lambda}_0; \mathbf{Q}_0) = 0.$$

From (a) and (b) we get (4.16).

3. From (2.20) we have

$$\partial_{\mathbf{Q}}\widehat{\psi}(\boldsymbol{\lambda}_0;\mathbf{Q})\cdot\mathbf{H}(\boldsymbol{\lambda}_0,\mathbf{0};\mathbf{Q}_0)\le 0,\qquad \mathbf{Q}\in\mathbf{V}.$$

This condition together with (4.17) shows that the function

$$\widehat{\psi}(\boldsymbol{\lambda};\,\cdot\,):\mathbf{V}\to\mathbb{R}$$

can serve as a Lyapunov function [6, 7] for the autonomous differential system

$$\dot{\mathbf{Q}} = \mathbf{H}(\boldsymbol{\lambda}_0,\mathbf{0};\mathbf{Q}_0) \tag{4.18}$$

and therefore $(\boldsymbol{\lambda}_0,\mathbf{0};\mathbf{Q}_0)\in\mathcal{E}_0$ is asymptotically stable.

Concluding this theorem we note that if $(\boldsymbol{\lambda}_0,\mathbf{0};\mathbf{Q}_0)\in\mathcal{E}_0$ is asymptotically stable then

$$\partial_{\mathbf{Q}}\widehat{\psi}(\boldsymbol{\lambda}_0;\mathbf{Q}_0)=\mathbf{0}. \tag{4.16'}$$

DEFINITION 4.3. *Let* $\boldsymbol{\lambda}^0=(\mathbf{F}^0,\theta^0)\in \mathrm{Lin}^+\times\mathbb{R}^+$ *be given at the material point* $X\in\mathcal{B}$. *The vectorial equation*

$$\partial_{\mathbf{Q}}\widehat{\psi}(\boldsymbol{\lambda}^0;\mathbf{Q})=\nu\mathbf{G} \tag{4.19}$$

is referred to as the ***equation of isothermal internal equilibrium*** *at constant temperature* θ^0 *and constant strain* $\mathbf{F}^0$ *for the material point* $X\in\mathcal{B}$.

REMARK 4.4. The unknowns in (4.19) are the triplets $(\mathbf{G},\mathbf{Q},\nu)\in\mathbf{V}\times\mathbf{V}\times\mathbb{R}$. The part 1 of the preceding theorem shows that if $(\boldsymbol{\lambda}^0,\mathbf{G}_0;\mathbf{Q}_0)$ is an asymptotically stable E.S. then there exists $\nu_0\in\mathbb{R}$ such that $(\mathbf{G}_0,\mathbf{Q}_0,\nu_0)$ satisfies (4.19), i.e. is a solution of the system

$$\mathbf{G}\cdot\mathbf{Q}=0,\qquad \mathbf{H}(\boldsymbol{\lambda}^0,G;\mathbf{Q})=\mathbf{0},\qquad \partial_{\mathbf{Q}}\widehat{\psi}(\boldsymbol{\lambda}^0;Q)=\nu\mathbf{G}. \tag{4.20}$$

DEFINITION 4.4. *The thermoelastic material under consideration is called* ***strictly dissipative*** [1] *if*

$$\dot{\boldsymbol{\lambda}}=(\dot{\mathbf{F}},\dot{\boldsymbol{\theta}})=\mathbf{0},\qquad \mathbf{G}\cdot\mathbf{Q}=0,\qquad \dot{\mathbf{Q}}\neq\mathbf{0}\ \Rightarrow\ \gamma>0, \tag{4.21}$$

where γ *is the specific rate of production of entropy defined by* (1.20).

REMARK 4.5. From (1.20) and (2.16) it follows that the considered material is strictly dissipative if and only if

$$\mathbf{G}_0\cdot\mathbf{Q}=0\quad\text{and}\quad(\boldsymbol{\lambda}_0,\mathbf{G}_0;\mathbf{Q})\notin\mathcal{E}\ \Rightarrow\ \partial_{\mathbf{Q}}\widehat{\psi}(\boldsymbol{\lambda}_0;\mathbf{Q})\cdot\mathbf{H}(\boldsymbol{\lambda}_0,\mathbf{G}_0;\mathbf{Q})<0. \tag{4.22}$$

Using the same line of arguments which leads us to the part 1 of the preceding theorem we can prove the

THEOREM 4.3. *If* $(\boldsymbol{\lambda}_0, \mathbf{G}_0; \mathbf{Q}_0) \in \mathcal{E}$ *is asymptotically stable and if there exists* $U(\mathbf{Q}_0) \subset D(\boldsymbol{\lambda}_0, \mathbf{G}_0; \mathbf{Q}_0)$, *a neighbourhood* $\mathbf{Q}_0$, *such that the inequality in* (4.22) *holds on* $U(\mathbf{Q}_0)\setminus\{\mathbf{Q}_0\}$, *then*

$$\widehat{\psi}(\boldsymbol{\lambda}_0, \mathbf{Q}) > \widehat{\psi}(\boldsymbol{\lambda}_0, \mathbf{Q}_0), \qquad \mathbf{Q}_0 \neq \mathbf{Q} \in U(\mathbf{Q}_0). \tag{4.23}$$

THEOREM 4.4. *If* $(\boldsymbol{\lambda}_0, \mathbf{0}; \mathbf{Q}_0) \in \mathcal{E}_0$ *and*

$$\partial_{\mathbf{Q}}\widehat{\psi}(\boldsymbol{\lambda}_0; \mathbf{Q}) \cdot \mathbf{H}(\boldsymbol{\lambda}_0, \mathbf{0}; \mathbf{Q}) < 0, \qquad \mathbf{Q}_0 \neq \mathbf{Q} \in U(\mathbf{Q}_0), \tag{4.24}$$

where $U(\mathbf{Q}_0)$ *is a neighbourhood of* $\mathbf{Q}_0$, *then:*

1) $(\boldsymbol{\lambda}_0, \mathbf{0}; \mathbf{Q}_0)$ *is asymptotically stable if and only if* (4.23) *holds;*

2) *if* $(\boldsymbol{\lambda}_0, \mathbf{0}; \mathbf{Q}_0)$ *is asymptotically stable then it is Lyapunov stable.*

P r o o f. The necessary part of 1 is a result of the preceding theorem. The sufficiency of 1 follows from Lyapunov's theorem on asymptotic stability since in this case the function $\widehat{\psi}(\boldsymbol{\lambda}_0; \cdot)$ is a Lyapunov function [6, 7] for the autonomous differential system (4.18). The part 2 of the theorem is a consequence of the preceding theorem and of the Lyapunov's stability theorem [6, 7].

REMARK 4.6. If the material is strictly dissipative and $(\boldsymbol{\lambda}_0, \mathbf{0}; \mathbf{Q}_0) \in \mathcal{E}_0$ is asymptotically stable, the inequality (4.23) holds and $(\boldsymbol{\lambda}_0, \mathbf{0}; \mathbf{Q}_0) \in \mathcal{E}_0$ is Lyapunov stable.

REMARK 4.7. The only E.S. $(\boldsymbol{\lambda}_0, \mathbf{0}; \mathbf{0}) \in \mathcal{E}_0$ (see Remark 4.2) of a thermoelastic material obeying the Cattaneo's heat flux evolution equation (3.9) is asymptotically stable if and only if the characteristic roots of $\mathbf{T}^{-1}(\boldsymbol{\lambda}_0)$ have positive real parts [6, 7].

5. Consistency with thermostatics

In this section we assume that for each $\boldsymbol{\lambda}_0 = (\mathbf{F}_0, \theta_0) \in \mathrm{Lin}^+ \times \mathbb{R}^+$ there exists a unique pair $(\mathbf{G}_0, \mathbf{Q}_0) \in \mathbf{V} \times \mathbf{V}$ such that $(\boldsymbol{\lambda}_0, \mathbf{G}_0; \mathbf{Q}_0) \in \mathcal{E}$. Using $(4.2)_3$ we denote

$$\mathcal{D}_0 = \{(\boldsymbol{\lambda}_0, \mathbf{G}_0) \in \mathcal{D}\ (\boldsymbol{\lambda}_0, \mathbf{G}_0; \mathbf{Q}_0) = (\boldsymbol{\lambda}_0, \mathbf{G}_0; \widehat{\mathbf{Q}}(\boldsymbol{\lambda}_0; \mathbf{G}_0)) \in \mathcal{E}\}. \tag{5.1}$$

The set $\mathcal{D}_0 \subset \mathcal{D}$ is referred to as the *equilibrium part* of $\mathcal{D}$ and is supposed to be a subdomain of $\mathcal{D}$.

On $\mathcal{D}_0$ we define the *equilibrium response functions* $\overline{\psi}_0$, $\overline{\eta}_0$, and $\overline{\mathbf{S}}_0$ giving the *equilibrium free energy* ψ_0, the *equilibrium entropy* η_0, and the *equilibrium first Piola–Kirchhoff stress tensor* $\mathbf{S}_0$ through *equilibrium constitutive equations*.

$$\psi_0 = \overline{\psi}_0(\boldsymbol{\lambda}_0; \mathbf{G}_0) \equiv \widehat{\psi}(\boldsymbol{\lambda}_0; \widehat{Q}(\boldsymbol{\lambda}_0; \mathbf{G}_0)), \tag{5.2}$$

$$\eta_0 = \overline{\eta}_0(\boldsymbol{\lambda}_0; \mathbf{G}_0) \equiv \widehat{\eta}(\boldsymbol{\lambda}_0; \widehat{Q}(\boldsymbol{\lambda}_0; \mathbf{G}_0)) = -\partial_\theta\widehat{\psi}(\boldsymbol{\lambda}_0; \widehat{\mathbf{Q}}(\boldsymbol{\lambda}_0, \mathbf{G}_0)), \tag{5.3}$$

$$\mathbf{S}_0 = \overline{\mathbf{S}}_0(\boldsymbol{\lambda}_0; \mathbf{G}_0) \equiv \widehat{\mathbf{S}}(\boldsymbol{\lambda}_0; \widehat{\mathbf{Q}}(\boldsymbol{\lambda}_0; \mathbf{G}_0)) = \partial_{\mathbf{F}}\widehat{\psi}(\boldsymbol{\lambda}_0; \widehat{\mathbf{Q}}(\boldsymbol{\lambda}_0, \mathbf{G}_0)). \tag{5.4}$$

REMARK 5.1. If $(\boldsymbol{\lambda}_0, \mathbf{G}_0) \in \mathcal{D}_0$ is asymptotically stable, i.e. $(\boldsymbol{\lambda}_0, \mathbf{G}_0; \widehat{\mathbf{Q}}(\boldsymbol{\lambda}_0; \mathbf{G}_0)) \in \mathcal{E}$ is asymptotically stable, then

$$\partial_{\mathbf{G}} \overline{\psi}_0(\boldsymbol{\lambda}_0; \mathbf{G}_0) = \mathbf{0}, \tag{5.5}$$

$$\partial_{\theta} \overline{\psi}_0(\boldsymbol{\lambda}_0; \mathbf{G}_0) = \partial_{\theta} \widehat{\psi}(\boldsymbol{\lambda}_0; \mathbf{G}_0), \tag{5.6}$$

$$\partial_{\mathbf{F}} \overline{\psi}_0(\boldsymbol{\lambda}_0; \mathbf{G}_0) = \partial_{\mathbf{F}} \widehat{\psi}(\boldsymbol{\lambda}_0; \mathbf{G}_0). \tag{5.7}$$

Indeed from $(4.7)_2$ and (5.2) it results that in a neighbourhood of $(\boldsymbol{\lambda}_0, \mathbf{G}_0; \mathbf{Q}_0)$ we have

$$\overline{\psi}_0(\boldsymbol{\lambda}; \mathbf{G}) \equiv \widehat{\psi}(\boldsymbol{\lambda}; \mathbf{Q}(\boldsymbol{\lambda}, \mathbf{G})), \qquad \mathbf{Q} = \widehat{\mathbf{Q}}(\lambda, \mathbf{G}), \qquad \mathbf{G} \cdot \widehat{\mathbf{Q}}(\boldsymbol{\lambda}; \mathbf{G}) = 0. \tag{5.8}$$

Applying the chain rules with respect to $\mathbf{G}$, θ and $\mathbf{F}$ for $(5.8)_1$ we obtain

$$\partial_{\mathbf{G}} \overline{\psi}_0(\boldsymbol{\lambda}; \mathbf{G}) \equiv \partial_{Q} \widehat{\psi}(\boldsymbol{\lambda}; \widehat{\mathbf{Q}}(\boldsymbol{\lambda}; \mathbf{G})) \partial_{\mathbf{G}} \widehat{\mathbf{Q}}(\lambda; G), \tag{5.9}$$

$$\partial_{\theta} \overline{\psi}_0(\boldsymbol{\lambda}; \mathbf{G}) \equiv \partial_{\theta} \widehat{\psi}(\boldsymbol{\lambda}; \widehat{\mathbf{Q}}(\boldsymbol{\lambda}; \mathbf{G})) + \partial_{\mathbf{Q}} \widehat{\psi}(\boldsymbol{\lambda}; \widehat{\mathbf{Q}}(\boldsymbol{\lambda}; \mathbf{G})) \partial_{\theta} \widehat{\mathbf{Q}}(\boldsymbol{\lambda}; \mathbf{G}), \tag{5.10}$$

$$\partial_{\mathbf{F}} \overline{\psi}_0(\boldsymbol{\lambda}_0; \mathbf{G}) \equiv \partial_{\mathbf{F}} \widehat{\psi}(\boldsymbol{\lambda}; \widehat{\mathbf{Q}}(\boldsymbol{\lambda}; \mathbf{G})) + \partial_{\mathbf{Q}} \widehat{\psi}(\boldsymbol{\lambda}; \widehat{\mathbf{Q}}(\boldsymbol{\lambda}; \mathbf{G})) \partial_{\mathbf{F}} \widehat{\mathbf{Q}}(\boldsymbol{\lambda}; \mathbf{G}). \tag{5.11}$$

If $(\boldsymbol{\lambda}_0, \mathbf{G}_0) \in \mathcal{D}_0$, is asymptotically stable then in view of Theorem 4.2, there exists $\nu_0 \in \mathbb{R}$ such that (a) $\partial_{\mathbf{Q}} \widehat{\psi}(\boldsymbol{\lambda}_0; \mathbf{Q}_0) = \nu_0 \mathbf{G}_0$, and therefore we have (b) $\partial_{\mathbf{G}} \overline{\psi}_0(\boldsymbol{\lambda}_0; \mathbf{G}_0) = \nu_0 \mathbf{G}_0 \partial_{\mathbf{G}} \widehat{\mathbf{Q}}(\boldsymbol{\lambda}_0, \mathbf{G}_0)$. Differentiating $\widehat{\mathbf{Q}}(\boldsymbol{\lambda}; \mathbf{G}) \cdot \widehat{\mathbf{G}}(\boldsymbol{\lambda}; \mathbf{Q}) = 0$ with respect to $\mathbf{G}$ in the point $(\boldsymbol{\lambda}_0, \mathbf{G}_0) \in \mathcal{D}_0$ and taking into account $(4.2)_3$ we have $\mathbf{G}_0 \partial_{\mathbf{G}} \widehat{\mathbf{Q}}(\boldsymbol{\lambda}_0; \mathbf{G}_0) = \mathbf{0}$ which, together with (5.9) and (b), implies (5.5). Differentiating $(5.8)_3$ with respect to θ and $\mathbf{F}$ in the point $(\boldsymbol{\lambda}_0, \mathbf{G}_0) \in \mathcal{D}_0$ we obtain (c) $\mathbf{G}_0 \partial_{\theta} \widehat{\mathbf{Q}}(\boldsymbol{\lambda}_0; \mathbf{G}_0) = 0$ and (d) $\mathbf{G}_0 \partial_{\mathbf{F}} \widehat{\mathbf{Q}}(\boldsymbol{\lambda}_0; \mathbf{G}_0) = \mathbf{0}$. If we evaluate (5.10), (5.11) in $(\boldsymbol{\lambda}_0, \mathbf{G}_0) \in \mathcal{D}_0$ we obtain (5.6) and (5.7) in virtue of (b), (c) and (d).

Thus we obtain the following *theorem of consistency with thermostatics* ([1, 3]).

THEOREM 5.1. *If the set $\mathcal{D}_0^* \subset \mathcal{D}_0$ of asymptotically stable pairs $(\boldsymbol{\lambda}^*, \mathbf{G}^*)$, $\boldsymbol{\lambda}^* = (F^*, \theta^*)$ is an open and connected set then:*

1) *the equilibrium function of free energy is independent of* $\mathbf{G}^*$, *i.e.*

$$\psi_0 = \widehat{\psi}_0(\boldsymbol{\lambda}^*); \tag{5.12}$$

2) *the equilibrium functions of entropy and of the stress tensor are independent of* $\mathbf{G}^*$, *i.e.*

$$\eta_0 = \widehat{\eta}_0(\boldsymbol{\lambda}^*), \qquad \mathbf{S}_0 = \widehat{\mathbf{S}}_0(\boldsymbol{\lambda}^*), \tag{5.13}$$

and they are determined by the function $\widehat{\psi}_0$ through the relations

$$\widehat{\eta}_= - \partial_{\theta^*} \widehat{\psi}_0, \qquad \widehat{\mathbf{S}}_0 = \partial_{\mathbf{F}^*} \widehat{\psi}_0. \tag{5.14}$$

REMARK 5.2. Of course we have

$$\mathbf{G}^* \cdot \widehat{\mathbf{Q}}(\boldsymbol{\lambda}^*; \mathbf{G}^*) = 0, \qquad (\boldsymbol{\lambda}^*, \mathbf{G}^*) \in \mathcal{D}_0^* . \tag{5.15}$$

Writing the first order Taylor's formula of $\widehat{\mathbf{Q}}$ in the point $(\boldsymbol{\lambda}^*, \mathbf{0}) \in \mathcal{D}_0^*$ we obtain

$$\widehat{\mathbf{Q}}(\boldsymbol{\lambda}^*; \mathbf{G}^*) = \mathbf{K}_0(\boldsymbol{\lambda}^*)\mathbf{G}^* + o(|\mathbf{G}^*|), \tag{5.16}$$

where

$$\mathbf{K}_0(\boldsymbol{\lambda}^*) = \left[\partial_{\mathbf{G}^*} \widehat{\mathbf{Q}}(\boldsymbol{\lambda}^*; \mathbf{0})\right]^T , \tag{5.17}$$

because (5.15) implies $\widehat{\mathbf{Q}}(\boldsymbol{\lambda}^*; \mathbf{0}) = \mathbf{0}$ [4].

The relation (5.16) shows that at an asymptotically E.S. the Fourier law holds within an error of order $o(|\mathbf{G}^*|)$ ([4, 5, 1]).

6. Entropy as an independent variable

The quantity

$$c = \partial_\theta \widehat{\varepsilon}(\boldsymbol{\lambda}; \mathbf{Q}), \tag{6.1}$$

is called the *heat capacity* of the body. In virtue of (3.3) and $(2.15)_1$

$$c = \theta \partial_\theta \widehat{\eta}(\boldsymbol{\lambda}; \mathbf{Q}). \tag{6.2}$$

In what follows we suppose $c > 0$ [1] on $\mathcal{D} \times \mathbf{V}$. This hypothesis implies that the function

$$(\boldsymbol{\lambda}; \mathbf{Q}) \to \widehat{\eta}(\boldsymbol{\lambda}; \mathbf{Q}) \in \mathbb{R}, \qquad (\boldsymbol{\lambda}, \mathbf{G}; \mathbf{Q}) \in \mathcal{D} \times \mathbf{V}, \qquad \boldsymbol{\lambda} = (\mathbf{F}, \theta), \tag{6.3}$$

is smoothly invertible with respect to θ on $\mathcal{D} \times \mathbf{V}$. Consequently the constitutive functionals of the thermoelastic material may be written as follows

$$\varepsilon = \widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}; \mathbf{Q}), \tag{6.4}$$

$$\theta = \widetilde{\theta}(\widetilde{\boldsymbol{\lambda}}; \mathbf{Q}), \tag{6.5}$$

$$\mathbf{S} = \widetilde{\mathbf{S}}(\widetilde{\boldsymbol{\lambda}}; \mathbf{Q}), \tag{6.6}$$

$$\dot{\mathbf{Q}} = \widetilde{\mathbf{H}}(\widetilde{\boldsymbol{\lambda}}, \mathbf{G}; \mathbf{Q}), \qquad (\widetilde{\boldsymbol{\lambda}}, \mathbf{G}; \mathbf{Q}) \in \widetilde{\mathcal{D}} \times \mathbf{V}, \qquad \widetilde{\boldsymbol{\lambda}} = (\mathbf{F}, \eta), \tag{6.7}$$

where the function $\widetilde{\theta}(\mathbf{F}, \cdot; \mathbf{Q})$ is the inverse of the function $\widehat{\eta}(\mathbf{F}, \cdot; \mathbf{Q})$ defined in (6.3), $\widetilde{\mathcal{D}} \subset \mathrm{Lin}^+ \times \mathbb{R} \times \mathbf{V}$ is a domain completely determined by the domain $\mathcal{D}$, and

$$\widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}; \mathbf{Q}) = \widehat{\varepsilon}(\mathbf{F}, \widetilde{\theta}(\widetilde{\boldsymbol{\lambda}}; \mathbf{Q}); \mathbf{Q}) = \widehat{\psi}(\mathbf{F}, \widetilde{\theta}(\widetilde{\boldsymbol{\lambda}}; \mathbf{Q}); \mathbf{Q}) + \eta \widetilde{\theta}(\widetilde{\boldsymbol{\lambda}}; \mathbf{Q}), \tag{6.8}$$

$$\widetilde{\mathbf{S}}(\widetilde{\boldsymbol{\lambda}}; \mathbf{Q}) = \widehat{\mathbf{S}}(\mathbf{F}\widetilde{\theta}(\widetilde{\boldsymbol{\lambda}}; \mathbf{Q}); \mathbf{Q}), \tag{6.9}$$

$$\widetilde{\mathbf{H}}(\widetilde{\boldsymbol{\lambda}}, \mathbf{G}; \mathbf{Q}) = H(\mathbf{F}, \widetilde{\theta}(\widetilde{\boldsymbol{\lambda}}; \mathbf{Q}), \mathbf{G}; \mathbf{Q}). \tag{6.10}$$

Applying the chain rule to (6.8) with respect to η and $\mathbf{F}$ and taking into account the entropy relation $(2.15)_1$, we obtain

$$\widetilde{\theta} = \partial_\eta \widetilde{\varepsilon}, \qquad \widetilde{\mathbf{S}} = \varrho_\kappa \partial_{\mathbf{F}} \widetilde{\varepsilon} \tag{6.11}$$

which means that the temperature functional $\widetilde{\theta}$ and the stress tensor functional $\widetilde{\mathbf{S}}$ are determined by the internal energy functional $\widetilde{\varepsilon}$.

The chain rule with respect to $\mathbf{Q}$ applied to (6.8) and the entropy relation $(2.15)_1$ leads to

$$\partial_{\mathbf{Q}} \widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}; \mathbf{Q}) = \partial_{\mathbf{Q}} \widehat{\psi}(\mathbf{F}, \widetilde{\theta}(\widetilde{\boldsymbol{\lambda}}; \mathbf{Q}), \mathbf{G}; \mathbf{Q}), \tag{6.12}$$

so that the Dissipation Inequality (2.16) becomes

$$\varrho_\kappa \partial_\eta \widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}; \mathbf{Q}) \partial_{\mathbf{Q}} \widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}; \mathbf{Q}) \cdot \widetilde{\mathbf{H}}(\widetilde{\boldsymbol{\lambda}}, \mathbf{G}; \mathbf{Q}) + \mathbf{G}\mathbf{Q} \leq 0. \tag{6.13}$$

Therefore

$$\mathbf{Q} \cdot \mathbf{G} = 0 \;\Rightarrow\; \partial_{\mathbf{Q}} \widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}; \mathbf{Q}) \cdot \widetilde{\mathbf{H}}(\widetilde{\boldsymbol{\lambda}}, \mathbf{G}; \mathbf{Q}) \leq 0, \tag{6.14}$$

and, in particular, we have

$$\partial_{\mathbf{Q}} \widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}; \mathbf{Q}) \cdot \widetilde{\mathbf{H}}(\widetilde{\boldsymbol{\lambda}}, \mathbf{0}; \mathbf{Q}) \leq 0. \tag{6.15}$$

The counterpart of theorem 3.1 is the

THEOREM 6.1. *If $\widetilde{\varepsilon}$ is twice continuously differentiable and the heat flux evolution equation* (6.7) *has the Cattaneo's form*

$$\widetilde{\mathbf{T}}(\widetilde{\boldsymbol{\lambda}})\,\dot{\mathbf{Q}} + \mathbf{Q} = -\widetilde{\mathbf{K}}(\widetilde{\boldsymbol{\lambda}})\mathbf{G}, \qquad \widetilde{\boldsymbol{\lambda}} = (\mathbf{F}, \eta), \tag{6.16}$$

where the second order tensors $\widetilde{\mathbf{T}}$ and $\widetilde{\mathbf{K}}$ are nonsingular, then the dissipation inequality (6.13) *holds if and only if on $\mathcal{D} \times \mathbf{V}$:*

1) $\widetilde{\mathbf{K}}(\widetilde{\boldsymbol{\lambda}})$ *is positive definite and*

2) *the second order tensor function*

$$\widetilde{\boldsymbol{\lambda}} \to \widetilde{\mathbf{Z}}(\widetilde{\boldsymbol{\lambda}}) \equiv \left[\widetilde{\mathbf{K}}(\widetilde{\boldsymbol{\lambda}})\right]^{-1} \widetilde{\mathbf{T}}(\widetilde{\boldsymbol{\lambda}}) \in \mathrm{Lin}\,, \qquad \widetilde{\boldsymbol{\lambda}} = (\mathbf{F}, \eta), \tag{6.17}$$

is given by

$$\widetilde{\mathbf{Z}} = \varrho_\kappa \left[\partial^2_{\eta\mathbf{Q}} \widetilde{\varepsilon}) \otimes (\partial_{\mathbf{Q}} \widetilde{\varepsilon}) + \partial_\eta \widetilde{\varepsilon} \partial^2_Q \widetilde{\varepsilon}\right]. \tag{6.18}$$

P r o o f. As in the proof of Theorem 3.1, we conclude that the inequality obtained by inserting (6.16) into (6.13) holds if and only if we have

$$\left[(\widetilde{\mathbf{T}}^{-1}(\boldsymbol{\lambda}))^T \partial_{\mathbf{Q}} \widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}; \mathbf{Q})\right] \cdot \mathbf{Q} \geq 0, \tag{6.19}$$

$$\varrho_\kappa \partial_\eta \widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}; \mathbf{Q})(\widetilde{\mathbf{Z}}^{-1}(\boldsymbol{\lambda}))^T \partial_{\mathbf{Q}} \widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}; \mathbf{Q}) = \mathbf{Q}. \tag{6.20}$$

Using the *temperature relation* $(6.11)_1$ we write (6.20) in the form

$$(6.20') \qquad \varrho_\kappa \partial_\eta \tilde{\varepsilon}(\tilde{\boldsymbol{\lambda}}; \mathbf{Q}) \partial_{\mathbf{Q}} \tilde{\varepsilon}(\tilde{\boldsymbol{\lambda}}; \mathbf{Q}) = \tilde{\mathbf{Z}}^T(\tilde{\boldsymbol{\lambda}}) \mathbf{Q}.$$

Differentiating this relation with respect to $\mathbf{Q}$ we get (6.18). From (6.20′) and (6.19) it results

$$(6.21) \qquad \mathbf{Q} \cdot \tilde{\mathbf{K}}^{-1}(\tilde{\boldsymbol{\lambda}}) \mathbf{Q} \geq 0, \qquad \mathbf{Q} \in \mathbf{V},$$

which means that $\mathbf{K}^{-1}$, and therefore $\tilde{\mathbf{K}}$ is positive definite.

REMARK 6.1. We have to note that in this case it is difficult to derive relations similar to the relations (3.14) – (3.19). On the other hand, the $\tilde{\mathbf{Z}}$ is not symmetric.

DEFINITION 6.1. *The state* $(\tilde{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0) \in \tilde{\mathcal{D}} \times \mathbf{V}$, $\tilde{\boldsymbol{\lambda}}_0 = (\mathbf{F}_0, \eta_0)$ *is called an* ***isentropic E.S. at constant strain*** $\mathbf{F}_0$ *for the material point* $X \in \mathcal{B}$ *if*

$$(6.22) \qquad \mathbf{G}_0 \cdot \mathbf{Q}_0 = 0, \qquad \tilde{\mathbf{H}}(\tilde{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0) = \mathbf{0}.$$

The state $(\tilde{\boldsymbol{\lambda}}_0, \mathbf{0}; \mathbf{Q}_0) \in \tilde{\mathcal{D}} \times \mathbf{V}$, $\tilde{\boldsymbol{\lambda}}_0 = (\mathbf{F}_0, \eta_0)$, *is a* ***strictly isentropic E.S. at constant strain*** $\mathbf{F}_0$ *for the material point* $X \in \mathcal{B}$ *if*

$$(6.23) \qquad \tilde{\mathbf{H}}(\tilde{\boldsymbol{\lambda}}_0, \mathbf{0}; \mathbf{Q}_0) = \mathbf{0}.$$

We will denote by $\tilde{\mathcal{E}}$ *the set of isentropic E.S. and by* $\tilde{\mathcal{E}}_0 \subset \tilde{\mathcal{E}}$ *the* ***subset of strictly isentropic*** *E.S. for a given material point* $X \in \mathcal{B}$.

REMARK 6.2. From (6.10) it follows that if $(\boldsymbol{\lambda}_0, \mathbf{G}_0; \mathbf{Q}_0) \in \mathcal{D} \times \mathbf{V}$ and $(\tilde{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0) \in \tilde{\mathcal{D}} \times \mathbf{V}$ are two states related by $\theta_0 = \tilde{\theta}(\tilde{\boldsymbol{\lambda}}; \mathbf{Q}_0)$, then

$$(6.24) \qquad (\boldsymbol{\lambda}, \mathbf{G}_0; \mathbf{Q}_0) \in \mathcal{E} \;\Leftrightarrow\; (\tilde{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0) \in \tilde{\mathcal{E}}.$$

DEFINITION 6.2. *If* $(\tilde{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0) \in \tilde{\mathcal{E}}$, *then the set* $\tilde{D}(\boldsymbol{\lambda}_0, \mathbf{G}_0; \mathbf{Q}_0) \subset \mathbf{V}$ *of points* $\mathbf{Q}^* \in \mathbf{V}$ *for which the solution* $\mathbf{Q} = \mathbf{Q}(t)$ *of the Cauchy problem*

$$(6.25) \qquad \dot{\mathbf{Q}} = \tilde{\mathbf{H}}(\tilde{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}), \qquad \mathbf{Q}(0) = \mathbf{Q}^*,$$

exists in $[0, \infty)$ *and satisfies the condition*

$$(6.26) \qquad \lim_{t \to \infty} \mathbf{Q}(t) = Q_0,$$

will be referred to as the ***domain of attraction*** *of the E.S.* $(\tilde{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0)$ *at constant strain and entropy* $\tilde{\boldsymbol{\lambda}}_0 = (\mathbf{F}_0, \eta_0)$.

The isentropic E.S. $(\tilde{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0)$ is said to be *asymptotically stable* if $\mathbf{Q}_0 \in \tilde{D}$ $(\tilde{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0)$ is an interior point.

The isentropic E.S. $(\widetilde{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0)$ will be called *Lyapunov stable* if for every $\varepsilon > 0$ there exists a $\delta = (\varepsilon) > 0$ such that every solution $\mathbf{Q} = \mathbf{Q}(t)$ of the differential equation $(6.25)_1$ satisfies the condition $|\mathbf{Q}(t) - \mathbf{Q}_0| < \varepsilon$ on $[0, \infty)$ whenever $|\mathbf{Q}(0) - \mathbf{Q}_0| < \delta$.

Similar results to those of theorem 4.2 are given by the

THEOREM 6.2.

1) *If* $(\widetilde{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0) \in \widetilde{\mathcal{E}}$, $\widetilde{\boldsymbol{\lambda}}_0 = (\mathbf{F}_0, \eta_0)$, *then*

$$\widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}_0; \mathbf{Q}^*) \geq \widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}_0; \mathbf{Q}_0), \qquad \mathbf{Q}^* \in \widetilde{D}(\widetilde{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0); \tag{6.27}$$

2) *if* $(\widetilde{\boldsymbol{\lambda}}_0; \mathbf{G}_0; \mathbf{Q}_0) \in \widetilde{\mathcal{E}}$ *is asymptotically stable then the preceding inequality holds in a neighbourhood* $U(\mathbf{Q}_0) \subset \widetilde{D}(\widetilde{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0)$ *of* $\mathbf{Q}_0$ *and there exists* $\widetilde{\nu}_0 \in \mathbb{R}$ *such that*

$$\partial_{\mathbf{Q}} \widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}_0, \mathbf{Q}_0) = \widetilde{\nu}_0 \mathbf{G}_0 ; \tag{6.28}$$

3) *if* $(\widetilde{\boldsymbol{\lambda}}_0, \mathbf{0}; \mathbf{Q}_0) \in \widetilde{\mathcal{E}}_0$ *and if there exists a neighbourhood* $U(\mathbf{Q}_0)$ *of* $\mathbf{Q}_0$ *such that*

$$\widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}_0; \mathbf{Q}) > \widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}_0; \mathbf{Q}_0), \qquad \mathbf{Q}_0 \neq \mathbf{Q} \in U(\mathbf{Q}_0) \cap \widetilde{D}(\widetilde{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0), \tag{6.29}$$

then $(\widetilde{\boldsymbol{\lambda}}_0, \mathbf{0}; \mathbf{Q}_0)$ *is Lyapunov stable E.S.*

REMARK 6.3. From (6.10), (6.12) and Remark 4.5 it results that the material is strictly dissipative if and only if

$$\mathbf{G}_0 \cdot \mathbf{Q} = 0 \quad \text{and} \quad (\widetilde{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}) \notin \widetilde{\mathcal{E}} \;\Rightarrow\; \partial_{\mathbf{Q}} \widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}_0; \mathbf{Q}) \cdot \widetilde{\mathbf{H}}(\widetilde{\boldsymbol{\lambda}}_0; \mathbf{G}_0; \mathbf{Q}_0) < 0. \tag{6.30}$$

Thus we obtain the following two theorems which are counterparts of Theorems 4.3 and 4.4.

THEOREM 6.3. *If* $(\widetilde{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0) \in \widetilde{\mathcal{E}}$ *is asymptotically stable and if there exists a neighbourhood* $U(\mathbf{Q}_0) \subset \widetilde{D}(\widetilde{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0)$ *of* $\mathbf{Q}_0$ *such that the inequality in* (6.30) *holds on* $U(\mathbf{Q}_0) \backslash \{\mathbf{Q}_0\}$ *then*

$$\widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}_0; \mathbf{Q}) > \widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}_0; \mathbf{Q}_0), \qquad \mathbf{Q}_0 \neq \mathbf{Q} \in U(\mathbf{Q}_0). \tag{6.31}$$

THEOREM 6.4. *If* $(\widetilde{\boldsymbol{\lambda}}_0; \mathbf{0}; \mathbf{Q}_0) \in \widetilde{\mathcal{E}}_0$ *and*

$$\partial_{\mathbf{Q}} \widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}_0; \mathbf{Q}) \cdot \widetilde{\mathbf{H}}(\widetilde{\boldsymbol{\lambda}}_0; \mathbf{0}; \mathbf{Q}) < 0, \qquad \mathbf{Q}_0 \neq \mathbf{Q} \in U(\mathbf{Q}_0), \tag{6.32}$$

where $U(\mathbf{Q}_0)$ *is a neighbourhood of* $\mathbf{Q}_0$ *then:*

1) $(\widetilde{\boldsymbol{\lambda}}_0, \mathbf{0}; \mathbf{Q}_0)$ *is asymptotically stable if and only if* (6.31) *holds and*

2) *if* $(\widetilde{\boldsymbol{\lambda}}_0, \mathbf{0}; \mathbf{Q}_0)$ *is asymptotically stable then it is Lyapunov stable.*

REMARK 6.4. From theorems 6.3 and 6.4 we conclude that if the material is strictly dissipative and $(\widetilde{\boldsymbol{\lambda}}_0, \mathbf{0}; \mathbf{Q}_0) \in \widetilde{\mathcal{E}}_0$ is asymptotically stable, then (6.31) holds and $(\boldsymbol{\lambda}_0, \mathbf{0}; \mathbf{Q}_0)$ is Lyapunov stable.

7. Internal energy as an independent variable

Because $\theta > 0$, the temperature relation $(6.11)_1$

$$\theta = \partial_\eta \tilde{\varepsilon}(\tilde{\boldsymbol{\lambda}};\mathbf{Q}), \qquad \tilde{\boldsymbol{\lambda}} = (\mathbf{F},\eta), \tag{7.1}$$

implies that the function $\eta \to \varepsilon = \tilde{\varepsilon}\,(\mathbf{F},\eta;\mathbf{Q}) \in \mathbb{R}$, $\eta \in \mathbb{R}$, is smoothly invertible for any fixed $\mathbf{F}$ and $\mathbf{Q}$. Denoting by $\varepsilon \to \eta = \check{\eta}(\mathbf{F},\varepsilon;\mathbf{Q}) \in \mathbb{R}$, $\varepsilon \in \mathbb{R}$ the inverse of the function $\tilde{\varepsilon}(\mathbf{F},\,\cdot\,;\mathbf{Q})$ and substituting it into (6.5) – (6.7) we obtain the following constitutive equations of the thermoelastic material

$$\eta = \check{\eta}(\check{\boldsymbol{\lambda}};\mathbf{Q}), \tag{7.2}$$

$$\theta = \check{\theta}(\check{\boldsymbol{\lambda}};\mathbf{Q}), \tag{7.3}$$

$$\mathbf{S} = \check{\mathbf{S}}(\check{\boldsymbol{\lambda}};\mathbf{Q}), \tag{7.4}$$

$$\dot{\mathbf{Q}} = \check{\mathbf{H}}(\check{\boldsymbol{\lambda}},\mathbf{G};\mathbf{Q}), \qquad (\check{\boldsymbol{\lambda}},\mathbf{G};\mathbf{Q}) \in \check{\mathcal{D}} \times \mathbf{V}, \qquad \check{\boldsymbol{\lambda}} = (\mathbf{F},\eta), \tag{7.5}$$

where $\check{\mathcal{D}} \subset \mathrm{Lin}^+ \times \mathbb{R} \times \mathbf{V}$ is a domain completely determined by the domain $\tilde{\mathcal{D}}$ and therefore by the domain $\mathcal{D}$, and

$$\check{\theta}(\check{\boldsymbol{\lambda}};\mathbf{Q}) = \tilde{\theta}(\mathbf{F},\check{\eta}(\check{\boldsymbol{\lambda}};\mathbf{Q});\mathbf{Q}), \tag{7.6}$$

$$\check{\mathbf{S}}(\check{\boldsymbol{\lambda}};\mathbf{Q}) = \tilde{\mathbf{S}}(\mathbf{F},\check{\eta}(\check{\boldsymbol{\lambda}};\mathbf{Q});\mathbf{Q}), \tag{7.7}$$

$$\check{\mathbf{H}}(\check{\boldsymbol{\lambda}};\mathbf{Q}) = \tilde{\mathbf{H}}(\mathbf{F},\check{\eta}(\check{\boldsymbol{\lambda}};\mathbf{Q});\mathbf{Q}). \tag{7.8}$$

Applying the chain rules with respect to ε, $\mathbf{F}$, and $\mathbf{Q}$ to the identity

$$\varepsilon = \tilde{\varepsilon}(\mathbf{F},\check{\eta}(\check{\boldsymbol{\lambda}};\mathbf{Q});\mathbf{Q}), \qquad \check{\boldsymbol{\lambda}} = (\mathbf{F},\varepsilon), \tag{7.9}$$

and taking into account the temperature relation $(6.11)_1$, we obtain

$$\check{\theta} = (\partial_\varepsilon \check{\eta})^{-1}, \qquad \check{\mathbf{S}} = -\varrho_\kappa \check{\theta}\partial_{\mathbf{F}}\check{\eta} = -\varrho_\kappa(\partial_\varepsilon\check{\eta})^{-1}\partial_{\mathbf{F}}\check{\eta} \tag{7.10}$$

which means that the temperature functional $\check{\theta}$ and the stress tensor functional are determined by the entropy functional $\check{\eta}$.

Differentiating (7.9) with respect to $\mathbf{Q}$ and using (7.1) we get

$$\partial_{\mathbf{Q}}\tilde{\varepsilon}(\tilde{\boldsymbol{\lambda}};\mathbf{Q}) = -\theta\partial_{\mathbf{Q}}\check{\eta}(\check{\boldsymbol{\lambda}};\mathbf{Q}). \tag{7.11}$$

Thus the Dissipation Inequality (6.13) becomes

$$\varrho_\kappa\left[\partial_\varepsilon\check{\eta}(\check{\boldsymbol{\lambda}};\mathbf{Q})\right]^{-2}\partial_{\mathbf{Q}}\check{\eta}(\check{\boldsymbol{\lambda}};\mathbf{Q})\cdot\check{\mathbf{H}}(\check{\boldsymbol{\lambda}},\mathbf{G};\mathbf{Q}) - \mathbf{G}\cdot\mathbf{Q} \geq 0. \tag{7.12}$$

From here we have the implication

$$\mathbf{Q}\cdot\mathbf{G}=0 \;\Rightarrow\; \partial_{\mathbf{Q}}\check{\eta}(\check{\boldsymbol{\lambda}};\mathbf{Q})\cdot\check{\mathbf{H}}(\check{\boldsymbol{\lambda}},\mathbf{G};\mathbf{Q})\geq 0, \tag{7.13}$$

and in particular we get

$$\partial_{\mathbf{Q}}(\check{\eta};\mathbf{Q})\cdot\check{\mathbf{H}}(\boldsymbol{\lambda},\mathbf{0};\mathbf{Q})\geq 0. \tag{7.14}$$

DEFINITION 7.1. *The state* $(\check{\boldsymbol{\lambda}}_0,\mathbf{G}_0;\mathbf{Q}_0)\in\check{\mathcal{D}}\times\mathbf{V}$, $\check{\boldsymbol{\lambda}}_0=(\mathbf{F}_0,\varepsilon_0)$, *is called an* ***isoenergetic*** *E.S. at constant strain* $\mathbf{F}_0$ *for the material point* $X\in\mathcal{B}$ *if*

$$\check{\mathbf{H}}(\check{\boldsymbol{\lambda}}_0,\mathbf{G}_0;\mathbf{Q}_0)=\mathbf{0},\qquad \mathbf{G}_0\cdot\mathbf{Q}_0=\mathbf{0}. \tag{7.15}$$

The state $(\check{\boldsymbol{\lambda}},\mathbf{0};\mathbf{Q}_0)\in\check{\mathcal{D}}\times\mathbf{V}$ is a *strictly isoenergetic* E.S. at *constant strain* $\mathbf{F}_0$ for the material point $X\in\mathcal{B}$ if

$$\check{\mathbf{H}}(\check{\boldsymbol{\lambda}},\mathbf{0};\mathbf{Q}_0)=\mathbf{0}. \tag{7.16}$$

We will note by $\check{\mathcal{E}}$ the *set of isoenergetic* E.S. and by $\check{\mathcal{E}}_0\subset\check{\mathcal{E}}$ the *subset of strictly isoenergetic* E. S. of the material point $X\in\mathcal{B}$.

REMARK 7.1. From (7.8) it follows that if $(\widetilde{\boldsymbol{\lambda}}_0,\mathbf{G}_0,\mathbf{Q}_0)\in\widetilde{\mathcal{D}}\times\mathbf{V}$, $\widetilde{\boldsymbol{\lambda}}_0=(\mathbf{F}_0,\eta_0)$, and $(\check{\boldsymbol{\lambda}},\mathbf{G}_0;\mathbf{Q}_0)\in\check{\mathcal{D}}\times\mathbf{V}$, $\check{\boldsymbol{\lambda}}_0=(\mathbf{F}_0,\varepsilon_0)$ are two states related by $\eta_0=\check{\eta}(\check{\boldsymbol{\lambda}};\mathbf{Q}_0)$ then

$$(\widetilde{\boldsymbol{\lambda}},\mathbf{G}_0;\mathbf{Q}_0)\in\widetilde{\mathcal{E}} \;\Leftrightarrow\; (\check{\boldsymbol{\lambda}},\mathbf{G}_0;\mathbf{Q}_0)\in\check{\mathcal{E}}. \tag{7.17}$$

DEFINITION 7.2. *If* $(\check{\boldsymbol{\lambda}}_0,\mathbf{G}_0;\mathbf{Q}_0)\in\check{\mathcal{E}}$ *then the set* $\check{D}(\check{\boldsymbol{\lambda}}_0,\mathbf{G}_0;\mathbf{Q}_0)\subset\mathbf{V}$ *of vectors* $\mathbf{Q}^*\in\mathbf{V}$ *for which the solution* $\mathbf{Q}=\mathbf{Q}(t)$ *of the Cauchy problem*

$$\dot{\mathbf{Q}}=\widehat{\mathbf{H}}(\check{\boldsymbol{\lambda}}_0,\mathbf{G}_0;\mathbf{Q}),\qquad \mathbf{Q}(0)=\mathbf{Q}^* \tag{7.18}$$

is defined on $[0,\infty)$ *and satisfies the condition*

$$\lim_{t\to\infty}\mathbf{Q}(t)=\mathbf{Q}_0, \tag{7.19}$$

is called the ***domain of attraction*** *of the E.S.* $(\check{\boldsymbol{\lambda}}_0,\mathbf{G}_0;\mathbf{Q}_0)$.

The isoenergetic E.S. $(\check{\boldsymbol{\lambda}}_0,\mathbf{G}_0;\mathbf{Q}_0)$ is said to be *asymptotically stable* if $\mathbf{Q}_0$ is an interior point of the set $\check{D}(\check{\boldsymbol{\lambda}},\mathbf{G}_0;\mathbf{Q}_0)$.

The isoenergetic E.S. $(\check{\boldsymbol{\lambda}},\mathbf{G}_0;\mathbf{Q}_0)$ will be referred to as Lyapunov stable if for each $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)>0$ such that every solution $\mathbf{Q}=\mathbf{Q}(t)$ of the differential system $(7.18)_1$ with $|\mathbf{Q}(0)-\mathbf{Q}_0|<\delta$ satisfies $|\mathbf{Q}(t)-\mathbf{Q}_0<\varepsilon$ for all $t\geq 0$.

The following three theorems are counterparts of theorems (4.2)–(4.4) and (6.2)–(6.4).

THEOREM 7.1.

1) *If* $(\check{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0) \in \check{\mathcal{E}}$, $\check{\boldsymbol{\lambda}}_0 = (\mathbf{F}_0, \varepsilon_0)$, *then*

$$\check{\eta}(\check{\boldsymbol{\lambda}}_0; \mathbf{Q}) \leq \check{\eta}(\check{\boldsymbol{\lambda}}_0; \mathbf{Q}_0), \qquad \mathbf{Q} \in \check{D}(\check{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0); \tag{7.20}$$

2) *if* $(\check{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0) \in \check{\mathcal{E}}$ *is asymptotically stable then the preceding inequality holds in a neighbourhood* $U(\mathbf{Q}_0) \subset \check{D}(\check{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0)$ *of the point* $\mathbf{Q}_0$ *and there exists* $\check{\nu}_0 \in \mathbb{R}$ *such that*

$$\partial_{\mathbf{Q}}\check{\eta}(\check{\boldsymbol{\lambda}}_0; \mathbf{Q}_0) = \check{\nu}_0 \mathbf{G}_0 ; \tag{7.21}$$

3) *if* $(\check{\boldsymbol{\lambda}}_0, \mathbf{0}; \mathbf{Q}_0) \in \check{\mathcal{E}}_0$ *and for a neighbourhood* $U(\mathbf{Q}_0)$ *of* $\mathbf{Q}_0$ *we have*

$$\check{\eta}(\check{\boldsymbol{\lambda}}_0; \mathbf{Q}) < \check{\eta}(\check{\boldsymbol{\lambda}}_0; \mathbf{Q}_0), \qquad \mathbf{Q}_0 \neq \mathbf{Q} \in U(\mathbf{Q}_0) \subset \check{D}(\check{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0), \tag{7.22}$$

then $(\check{\boldsymbol{\lambda}}_0, \mathbf{0}; \mathbf{Q}_0)$ *is Lyapunov stable.*

REMARK 7.2. From (7.8), (7.11), and (6.30) we come to the conclusion that the thermoelastic material is strictly dissipative if and only if

$$\mathbf{G}_0 \cdot \mathbf{Q} = 0 \quad \text{and} \quad (\check{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}) \notin \check{\mathcal{E}} \;\Rightarrow\; \partial_{\mathbf{Q}}\check{\eta}(\check{\boldsymbol{\lambda}}_0; \mathbf{Q}) \cdot \check{\mathbf{H}}(\check{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}) > 0. \tag{7.23}$$

THEOREM 7.2. *If* $(\check{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0) \in \check{\mathcal{E}}$ *is asymptotically stable and there exists a neighbourhood* $U(\mathbf{Q}_0)$ *of* $\mathbf{Q}_0$, $U(\mathbf{Q}_0) \subset \check{D}(\check{\boldsymbol{\lambda}}_0, \mathbf{G}_0; \mathbf{Q}_0)$, *such that the inequality in* (7.22) *holds on* $U(\mathbf{Q}_0)\backslash\{\mathbf{Q}_0\}$, *then*

$$\check{\eta}(\check{\boldsymbol{\lambda}}_0; \mathbf{Q}) < \check{\eta}(\check{\boldsymbol{\lambda}}_0; \mathbf{Q}_0), \qquad \mathbf{Q}_0 \neq \mathbf{Q} \in U(\mathbf{Q}_0). \tag{7.24}$$

THEOREM 7.3. *If* $(\check{\boldsymbol{\lambda}}_0, \mathbf{0}; \mathbf{Q}_0) \in \check{\mathcal{E}}_0$ *and*

$$\partial_{\mathbf{Q}}\check{\eta}(\check{\boldsymbol{\lambda}}_0; \mathbf{Q}) \cdot \check{\mathbf{H}}(\check{\boldsymbol{\lambda}}_0, \mathbf{0}; \mathbf{Q}) > 0, \qquad \mathbf{Q}_0 \neq \mathbf{Q} \in U(\mathbf{Q}_0), \tag{7.25}$$

$U(\mathbf{Q}_0)$ *being a neighbourhood of* $\mathbf{Q}_0$, *then*

1) $(\check{\boldsymbol{\lambda}}_0, \mathbf{0}; \mathbf{Q}_0)$ *is asymptotically stable if and only if* (7.24) *holds and*

2) *if* $(\check{\boldsymbol{\lambda}}_0, \mathbf{0}; \mathbf{Q}_0)$ *is asymptotically stable then it is Lyapunov stable.*

REMARK 7.3. In virtue of Theorems 7.1 and 7.2 it results that if the thermoelastic material is strictly dissipative and $(\check{\boldsymbol{\lambda}}_0, \mathbf{0}; \mathbf{Q}_0) \in \check{\mathcal{E}}_0$ is asymptotically stable, then the inequality (7.24) holds and $(\check{\boldsymbol{\lambda}}_0, \mathbf{0}; \mathbf{Q}_0)$ is a Lyapunov stable E.S.

Now, by using arguments similar to those in Sec. 9 of [1] we prove the following theorem giving some relations between isothermal, isentropic and isoenergetic asymptotic stability of an E.S.

THEOREM 7.4. *Let* $(\boldsymbol{\lambda}_0, \mathbf{0}; \mathbf{Q}_0) \in \mathcal{E}_0$, $\boldsymbol{\lambda}_0 = (\mathbf{F}_0, \theta_0)$, *be a strictly isothermal E.S. at constant strain* $\mathbf{F}_0$ *for the material point* $X \in \mathcal{B}$, *and let us suppose that*

$$\eta_0 = \widehat{\eta}_0(\boldsymbol{\lambda}_0; \mathbf{Q}_0); \qquad \varepsilon_0 = \widetilde{\varepsilon}(\widetilde{\lambda}_0; \mathbf{Q}_0), \qquad \widetilde{\lambda}_0 = (\mathbf{F}_0, \eta_0); \tag{7.26}$$

1) *if the inequalities* (4.24) *and* (6.32) *hold, then the asymptotic stability of* $(\boldsymbol{\lambda}_0, \mathbf{0}; \mathbf{Q}_0)$ *entails the asymptotic stability of* $(\widetilde{\lambda}_0, \mathbf{0}; \mathbf{Q}_0) \in \widetilde{\mathcal{E}}_0$;

2) *if the inequalities* (6.32) *and* (7.25) *hold, then* $(\widetilde{\boldsymbol{\lambda}}_0, \mathbf{0}; \mathbf{Q}_0) \in \widetilde{\mathcal{E}}_0$ *is asymptotically stable if and only if* $(\check{\boldsymbol{\lambda}}_0, \mathbf{0}; \mathbf{Q}_0) \in \check{\mathcal{E}}_0$, $\check{\boldsymbol{\lambda}}_0 = (\mathbf{F}_0, \varepsilon_0)$ *is asymptotically stable.*

P r o o f.

1. Making use of the assumption

$$c = \partial_\theta \widehat{\varepsilon}(\boldsymbol{\lambda}; \mathbf{Q}) = \theta \partial_\theta \widehat{\eta}(\widetilde{\lambda}; \mathbf{Q}) > 0$$

from $(2.15)_1$ we obtain

$$\partial_\theta^2 \widehat{\psi}(\boldsymbol{\lambda}; \mathbf{Q}) < 0, \tag{7.27}$$

due to the hypothesis that $\widehat{\psi}$ is twice continuous differentiable.

Writing the second order Taylor's formula with respect to the variable θ and using again $(2.15)_1$ we have

$$\widehat{\psi}(\mathbf{F}, \theta, \mathbf{Q}) - \widehat{\psi}(\mathbf{F}, \theta'; \mathbf{Q}) + (\theta - \theta')\widehat{\eta}(\mathbf{F}, \theta'; \mathbf{Q}) = 1/2(\theta - \theta')^2 \partial_\theta^2 \widehat{\psi}(\mathbf{F}, \theta_*, \mathbf{Q}),$$

where $\theta_* = \theta_*(\mathbf{F}, \theta, \theta', \mathbf{Q}) \in (\theta, \theta')$, and in view of (8.27) we get

$$\text{(a)} \qquad \widehat{\psi}(\mathbf{F}, \theta'; \mathbf{Q}) \geq \widehat{\psi}(\mathbf{F}, \theta; \mathbf{Q}) + (\theta - \theta')\widehat{\eta}(\mathbf{F}, \theta'; \mathbf{Q}).$$

From (6.5), (6.8), and $(7.26)_1$ we get

$$\begin{aligned}\text{(b)} \qquad \widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}_0; \mathbf{Q}) - \widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}_0; \mathbf{Q}_0) = \left[\widehat{\psi}(\mathbf{F}_0, \widetilde{\theta}(\widetilde{\boldsymbol{\lambda}}_0; \mathbf{Q}); \mathbf{Q}) - \widehat{\psi}(\boldsymbol{\lambda}_0; \mathbf{Q}_0)\right] \\ + \left[\widetilde{\theta}(\boldsymbol{\lambda}_0; \mathbf{Q}) - \theta_0\right] \eta_0 .\end{aligned}$$

Because $\widetilde{\theta}(\mathbf{F}, \cdot; \mathbf{Q})$ is the inverse of $\widehat{\eta}(\mathbf{F}, \cdot; \mathbf{Q})$ we have $\widehat{\eta}(\mathbf{F}, \widetilde{\theta}(\widetilde{\boldsymbol{\lambda}}_0; \mathbf{Q}); \mathbf{Q}) = \eta_0$ and, in view of (a) and (b), we obtain

$$\widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}_0; \mathbf{Q}) - \varepsilon_0 = \widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}_0; \mathbf{Q}) - \widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}_0; \mathbf{Q}_0) \geq \widehat{\psi}(\boldsymbol{\lambda}_0; \mathbf{Q}) - \widehat{\psi}(\boldsymbol{\lambda}_0; \mathbf{Q}_0). \tag{7.28}$$

Our hypotheses, Theorem 4.3, and (7.28) imply that if $(\boldsymbol{\lambda}_0, \mathbf{0}; \mathbf{Q}_0) \subset \mathcal{E}_0$ is asymptotically stable then it holds (6.27). Now by Theorem (6.4) we have that $(\widetilde{\boldsymbol{\lambda}}_0, \mathbf{0}; \mathbf{Q}_0) \in \widetilde{\mathcal{E}}_0$ is asymptotically stable. The conclusion 2 of the same Theorem 6.4 shows that $(\boldsymbol{\lambda}_0, \mathbf{0}; \mathbf{Q}_0) \in \mathcal{E}_0$ is even Lyapunov stable.

2. From $(7.10)_1$ it follows that the function $\check{\eta}(\mathbf{F}, \cdot; \mathbf{Q})$, which is inverse of $\widetilde{\varepsilon}(\mathbf{F}, \cdot; \mathbf{Q})$, is a strictly increasing function and therefore we have

$$\varepsilon_0 < \widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}_0; \mathbf{Q}) \;\Leftrightarrow\; \eta_0 = \check{\eta}(\check{\boldsymbol{\lambda}}_0; \mathbf{Q}) < \check{\eta}(\mathbf{F}_0, \widetilde{\varepsilon}(\widetilde{\boldsymbol{\lambda}}_0; \mathbf{Q}); \mathbf{Q}). \tag{7.29}$$

Now, the desired result is an immediate consequence of the Conclusion 1 of the Theorem (6.4), of the equivalence (7.29), and of the Conclusion 1 of the Theorem 7.3.

REMARK 7.4. Combining this result with the point 2 of Theorems 6.4 and 7.3 it follows that if $(\boldsymbol{\lambda}_0, \mathbf{0}; \mathbf{Q}_0)$ is an isentropic (resp. isoenergetic) asymptotically stable E.S., then it is an isoenergetic (resp. isentropic) Lyapunov stable E.S.

The counterpart of Theorems (3.1) and (6.1) is the following

THEOREM 7.5. *If the functional $\check{\eta}$ is twice continuously differentiable and the heat flux evolution equation* (7.5) *is of the Cattaneo kind*

$$\check{\mathbf{T}}(\check{\boldsymbol{\lambda}})\dot{\mathbf{Q}} + \mathbf{Q} = -\check{\mathbf{K}}(\check{\boldsymbol{\lambda}})\mathbf{G}, \tag{7.30}$$

where the second order tensor functions $\check{\mathbf{T}}$ *and* $\check{\mathbf{K}}$ *are invertible, then the Dissipation Inequality (7.12) is satisfied if and only if on* $\check{\mathcal{D}} \times \mathbf{V}$

1. $\mathbf{K}(\check{\boldsymbol{\lambda}})$ *is positive definite and*
2. *The second order tensor function*

$$\check{\boldsymbol{\lambda}} \to \check{\mathbf{Z}}(\check{\boldsymbol{\lambda}}) = \left[\check{\mathbf{K}}(\check{\boldsymbol{\lambda}})\right]^{-1} \check{\mathbf{T}}(\check{\boldsymbol{\lambda}}) \in \mathrm{Lin}, \qquad \check{\boldsymbol{\lambda}} = (\mathbf{F}, \varepsilon), \tag{7.31}$$

is given by

$$\check{\mathbf{Z}} = \varrho_\kappa \left[\partial_\varepsilon \check{\eta} \partial^2_{\mathbf{Q}} \check{\eta} - 2\partial_{\varepsilon \mathbf{Q}} \check{\eta} \otimes \partial_{\mathbf{Q}} \check{\eta}\right] (\partial_\varepsilon \check{\eta})^{-3}. \tag{7.32}$$

The proof of the theorem follows by using the same line of arguments as in the proof of Theorem (6.1).

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