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Some existence result for a Stokes flow between two arbitrarily closed curves

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THE PROBLEM of determining the slow viscous flow of a fluid between two arbitrarily closed curves is formulated as a system of Fredholm integral equations of the second kind, addying a pair of singularities located outside of the flow region. We show that the integral equations proposed here have a unique continuous solution, when the two closed curves are Lyapunov curves and the fluid velocity is continuous on these curves.

1. Mathematical formulation

WE CONSIDER the creeping flow of an incompressible viscous fluid between two arbitrary closed Lyapunov curves (i.e. they have a continuously varying normal vector) denoted by C^1 and C^2, and supposed to be on the upper half plane $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\}$. Also, we suppose that the Reynolds number of the flow is very small. Under this condition, the governing equations for the velocity $\mathbf{u}(u_1, u_2)$ and pressure p can be reduced to the Stokes equations:

$$
\begin{aligned}
\Delta \mathbf{u}(x) &= \nabla p(x), \qquad && x \in \Omega, \\
\nabla \cdot \mathbf{u}(x) &= 0, && x \in \Omega,
\end{aligned}
\tag{1.1}
$$

where the symbols ∇ and Δ mean the gradient operator and the Laplace operator, respectively. Here $x(x_1, x_2) \in \Omega$ and Ω is the two-dimensional bounded domain with the boundaries C^1 and C^2, respectively, such that C^1 is located inside of the domain bounded by C^2.

The fluid velocity $\mathbf{u}$ must satisfy the following boundary conditions on the curves C^1 and C^2:

$$
\begin{aligned}
\mathbf{u}(x) &= \mathbf{f}_1(x), \qquad && \text{for} \quad x \in \Omega, \\
\mathbf{u}(x) &= \mathbf{f}_2(x), && \text{for} \quad x \in \Omega,
\end{aligned}
\tag{1.2}
$$

where the boundary velocities $\mathbf{f}_1$ and $\mathbf{f}_2$ are supposed to be smooth vector functions.

Using the continuity equation $(1.1)_2$, we deduce the following relation:

$$
\int\limits_{C^1 \cup C^2} u_j(x) n_j(x)\, ds_x = 0,
$$

hence, a necessary condition for our problem to have a solution in Ω is that

$$(1.3)\qquad \int_{C^1} f_{1j}(x)n_j(x)\,ds_x = \int_{C^2} f_{2j}(x)n_j(x)\,ds_x.$$

Here $\mathbf{n}(n_1, n_2)$ is the unit outward normal vector at points of C^1 and C^2.

By applying the Green identity for a smooth and solenoidal vector $\mathbf{v}(v_1, v_2)$ and a scalar function q, we obtain:

$$(1.4)\qquad \int_\Omega \left(\Delta v_j - \frac{\partial q}{\partial x_j}\right) u_j\,dx + \frac{1}{2}\int_\Omega \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right) dx = \int_{C^1} T_{ij}(\mathbf{v})u_i n_j\,ds - \int_{C^2} T_{ij}(\mathbf{v})u_i n_j\,ds,$$

where

$$(1.5)\qquad T_{ij}(\mathbf{v}) = -q\delta_{ij} + \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}, \qquad i,j \in \{1,2\},$$

are the components of the stress tensor, corresponding to the flow $(\mathbf{v}, q)$.

The formula (1.4) applied to $\mathbf{u} = \mathbf{v}$ and $p = q$, gives the following equality:

$$(1.6)\qquad \frac{1}{2}\int_\Omega \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 dx = \int_{C^1} T_{ij}(\mathbf{u})u_i n_j\,ds - \int_{C^2} T_{ij}(\mathbf{u})u_i n_j\,ds.$$

If we suppose that our problem has two solutions $\mathbf{u}_1$ and $\mathbf{u}_2$, then the vector $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$ satisfies homogeneous boundary conditions on C^1 and C^2, and the formula (1.6) gives:

$$(1.7)\qquad \frac{\partial u_i}{\partial x_j}(x) + \frac{\partial u_j}{\partial x_i}(x) = 0, \qquad x \in \Omega, \quad i,j \in \{1,2\}.$$

This system has three linearly independent solutions:

$$(1.8)\qquad \mathbf{u}^1(x) = (1,0), \qquad \mathbf{u}^2(x) = (0,1), \qquad \mathbf{u}^3(x) = (x_2, -x_1), \qquad x \in \Omega.$$

Hence, we conclude that the fluid motion compatible with homogeneous boundary conditions on C^1 and C^2 is given by the null solution $\mathbf{u} = \mathbf{0}$.

In the following we consider the components of stress tensor $\widetilde{\Pi}$ corresponding to the Stokes equations (see [1] and [8]):

$$(1.9)_1\qquad T_{ijk}(x,y) = -q_j(x,y)\delta_{ik} + \frac{\partial q_{ij}}{\partial x_k}(x,y) + \frac{\partial q_{kj}}{\partial x_i}(x,y),$$

where q_{ij} and q_j are components of Green tensor G and pressure vector $\mathbf{q}$, respectively. G and $\mathbf{q}$ satisfy the following equations and conditions:

$$(1.9)_{2-5} \qquad \begin{aligned} \Delta_x q_{ij}(x,y) - \frac{\partial q_j}{\partial x_i}(x,y) &= -4\pi\delta_{ij}\delta(x-y), \qquad \text{for} \quad x_2 > 0,\\ \frac{\partial}{\partial x_i} q_{ij}(x,y) &= 0, \qquad \text{for} \quad x_2 > 0,\\ q_{ij}(x,y) &= 0, \qquad \text{for} \quad x_2 = 0,\\ q_{ij}(x,y) \to 0, \qquad q_i(x,y) &\to 0, \qquad \text{as} \quad |x| \to \infty, \end{aligned}$$

where δ is Dirac's distribution.

From [8] it results that the Green tensor G can be written as:

$$(1.9)_6 \quad G(x,y) = G^{ST}(x-y) - G^{ST}(x-y^{im}) + 2y_2^2 G^D(x-y^{im}) - 2y_2 G^{SD}(x-y^{im}),$$

where $y^{im} = (y_1, -y_2)$ is the image of the pole y with respect to the boundary $y_2 = 0$, the Green tensor G^{ST} has the components (see [8]):

$$(1.9)_7 \qquad q_{ij}^{ST}(x) = -\ln|x|\delta_{ij} + \frac{x_i x_j}{|x|^2}.$$

The matrices which correspond to the tensors G^D and G^{SD} are given by

$$(1.9)_{8-9} \qquad \begin{aligned} q_{ij}^D(x) &= \pm\left(\frac{\delta_{ij}}{|x|^2} - 2\frac{x_i x_j}{|x|^4}\right),\\ q_{ij}^{SD}(x) &= x_2 q_{ij}^D(x) \pm \frac{\delta_{j2}x_i - \delta_{i2}x_j}{|x|^2}, \end{aligned}$$

where the plus sign applies for $j = 1$, in the $0x_1$ direction, and the minus sign for $j = 2$, and in the $0x_2$ direction.

The pressure tensor $\mathcal{P}$, with components Π_{ij}, is associated with the tensor $\widetilde{\Pi}$. Precisely, we have

$$(1.10)_1 \qquad \Pi_{ij}(x,y) = -P(x,y)\delta_{ij} + \frac{\partial q_i}{\partial y_j}(x,y) + \frac{\partial q_j}{\partial y_i}(x,y),$$

where

$$(1.10)_2 \qquad -\frac{\partial P}{\partial x_i}(x,y) + \Delta_x q_i(y,x) = 0, \qquad \text{for} \quad x \neq y, \quad x \in \mathbb{R}_+^2$$

and

$$(1.10)_3 \qquad \frac{\partial q_i}{\partial x_i}(y,x) = 0, \qquad x \in \mathbb{R}_+^2, \qquad x \neq y.$$

The pressure vector $\mathbf{q}$ can be written as (see [8]):

$$\mathbf{q}(x,y) = \mathbf{q}^{ST}(x-y) - \mathbf{q}^{ST}(x-y^{im}) - 2y_2\mathbf{q}^{SD}(x-y^{im}), \tag{1.10$_4$}$$

where

$$q_i^{ST}(x) = \frac{2x_i}{|x|^2}, \qquad \mathbf{q}^{SD}(x) = -\frac{2}{|x|^4}(2x_1x_2,\ x_1^2 - x_2^2). \tag{1.10$_5$}$$

With the above notations, we consider the following relations:

$$\begin{aligned} K_{ij}(x,y) &= T_{jik}(y,x)n_k(y), \\ K_i(x,y) &= \Pi_{ij}(x,y)n_j(y), \end{aligned} \tag{1.11}$$

where $y(y_1,y_2) \in C^1 \cup C^2$.

We determine the solution $(\mathbf{u}, p)$ of the Stokes problem (1.1), (1.2) in terms of the following double-layer potentials:

$$\begin{aligned} u_j(x) &= \int\limits_{C^1\cup C^2} K_{jl}(x,y)\phi_l(y)\,ds_y, \qquad x\in\Omega, \quad j\in\{1,2\}, \\ p(x) &= \int\limits_{C^1\cup C^2} K_j(x,y)\phi_j(y)\,ds_y, \qquad x\in\Omega. \end{aligned} \tag{1.12}$$

From the boundary conditions (1.2) we obtain a Fredholm integral system of the second kind for the unknown density $\boldsymbol{\phi}(\phi_1,\phi_2)$:

$$\begin{aligned} -2\pi\phi_j(x) + \int\limits_{C^1\cup C^2} K_{jl}(x,y)\phi_l(y)\,ds_y &= f_{1j}(x), \qquad x\in C^1, \\ 2\pi\phi_j(x) + \int\limits_{C^1\cup C^2} K_{jl}(x,y)\phi_l(y)\,ds_y &= f_{2j}(x), \qquad x\in C^2. \end{aligned} \tag{1.13}$$

We used here the following jump relations of the double layer potentials:

$$\lim_{x'\to x\in C} \int\limits_C \phi_j(y)K_{ij}(x',y)\,ds_y = \pm 2\pi\phi_i(x) + \int\limits_C \phi_j(y)K_{ij}(x,y)\,ds_y \tag{1.13$'$}$$

where C is a closed Lyapunov curve, the sign + corresponds to the internal side of C, and the sign − to the external side.

The above integrals, which appear in (1.13), are considered as the principal values in the Cauchy means.

The system (1.13) has a solution if and only if the non-homogeneous term $\mathbf{f} : C^1 \cup C^2 \to \mathbb{R}^2$, $\mathbf{f}(x) = \mathbf{f}_i(x)$, for $x \in C^i$, $i \in \{1,2\}$, is orthogonal to the solutions of the corresponding adjoint homogeneous system. We used here the second Fredholm alternative for Fredholm's type integral equations (see [3, 4]).

Let us consider the homogeneous system of (1.13):

$$
(1.14)\qquad \begin{aligned}
-2\pi\phi_j^0(x) + \int\limits_{C^1\cup C^2} K_{jl}(x,y)\phi_l^0(y)\,ds_y &= 0, \qquad x\in C^1,\\
2\pi\phi_j^0(x) + \int\limits_{C^1\cup C^2} K_{jl}(x,y)\phi_l^0(y)\,ds_y &= 0, \qquad x\in C^2.
\end{aligned}
$$

Also, the homogeneous adjoint system of (1.13) has the form:

$$
(1.15)\qquad \begin{aligned}
-2\pi\tau_j(x) + \int\limits_{C^1\cup C^2} K_{lj}(y,x)\tau_l(y)\,ds_y &= 0, \qquad x\in C^1,\\
2\pi\tau_j(x) + \int\limits_{C^1\cup C^2} K_{lj}(y,x)\tau_l(y)\,ds_y &= 0, \qquad x\in C^2.
\end{aligned}
$$

From the first Fredholm's alternative (see [3, 4]) it results that the vector solutions of the system (1.14) and (1.15), respectively, form two vector spaces of same finite dimension d.

If we use the following properties of the stress tensor:

$$
(1.16)\qquad \begin{aligned}
\frac{\partial T_{ijk}}{\partial x_i}(x,y) = \frac{\partial T_{kji}}{\partial x_i}(x,y) &= -4\pi\delta_{kj}\delta(x-y),\\
\frac{\partial}{\partial x_k}\left[\varepsilon_{ilm}x_l T_{mjk}(x,y)\right] &= -4\pi\varepsilon_{ilj}x_l\delta(x-y),
\end{aligned}
$$

where δ is the Dirac distribution, and using the divergence theorem in a bounded domain $D\subset\mathbb{R}^2$, having the boundary C, we obtain the next properties:

$$
(1.17)\qquad \begin{aligned}
\int\limits_C T_{ijk}(y,x)n_k(y)\,ds_y &= \begin{cases} 2\pi\delta_{ij}, & \text{for } x\in C,\\ 0, & \text{for } x\notin D\cup C,\end{cases}\\
\int\limits_C \varepsilon_{ijk}y_jT_{klm}(y,x)n_m(y)\,ds_y &= \begin{cases} 2\pi\varepsilon_{ijl}x_l, & \text{for } x\in C,\\ 0, & \text{for } x\notin D\cup C,\end{cases}
\end{aligned}
$$

where the components T_{ijk} are given by $(1.9)_1$, the unit normal vector **n** is directed inside of D, and the symbol ε_{ijk} means:

$$
\varepsilon_{ijk} = \begin{cases} 1, & \text{for an old permutation of numbers } 1,\ 2,\ 3,\\ -1, & \text{for an eden permutation of numbers } 1,\ 2,\ 3.\end{cases}
$$

By applying the properties (1.13), (1.17) we deduce that the functions $\mathbf{u}^i$, $i \in \{1,2,3\}$, given by (1.8), are solutions of the following equations:

$$(1.18) \qquad -2\pi u_j^i(x)+\int\limits_{C^1} K_{jl}(x,y)u_l^i(y)\,ds_y = 0, \quad x\in C^1, \quad i\in\{1,2,3\}, \quad j\in\{1,2\}$$

and

$$(1.19) \qquad \int\limits_{C^1} K_{jl}(x,y)u_l^i(y)\,ds_y = 0, \quad x \in C^2, \quad i \in \{1,2,3\}.$$

Let the vector functions $\boldsymbol{\phi}_i^0 : C^1 \cup C^2 \to \mathbb{R}^2$, $i \in \{1,2,3\}$, be given by

$$\boldsymbol{\phi}_i^0(x) = \begin{cases} \mathbf{u}^i(x), & x \in C^1, \\ 0, & x \in C^2. \end{cases}$$

From (1.18) and (1.19), we deduce that these functions are three linearly independent solutions of homogeneous system (1.14). Hence, we conclude that $d \geq 3$. In the next we shall prove that $d = 3$. For this aim we consider the single-layer potentials

$$(1.20) \qquad V_i^0(x) = \int\limits_{C^1\cup C^2} q_{ij}(x,y)\tau_j(y)\,ds_y, \qquad i \in \{1,2\}$$

with their corresponding pressure

$$(1.20') \qquad P^0(x) = \int\limits_{C^1\cup C^2} q_j(x,y)\tau_j(y)\,ds_y$$

where $\boldsymbol{\tau}$ is a possible solution of the adjoint system (1.15), q_{ij} and q_j are given by $(1.9)_{6-9}$ and $(1.10)_{4,5}$, respectively.

From $(1.9)_{2,3}$ it results that the potentials (1.20), (1.20′) determine a Stokes flow in Ω.

Since the potentials (1.20) and (1.20′) are continuous on C^1 and C^2, it follows that (1.20) can be considered as a continuous velocity field at every point $x \in \mathbb{R}_+^2$. On the other hand, the vector tension, of (1.20) and (1.20′), has a jump in points of C^1 and C^2. It is easily seen that the limiting value of the vector tension, when $x \in \Omega^2 = \mathbb{R}_+^2\setminus(\Omega^1 \cup \overline{\Omega})$ tends to a point $x \in C^2$, is given by the left-hand side of Eqs. $(1.15)_2$. The limiting value of the vector tension, when $x' \in \Omega^1$ (the domain bounded by the curve C^1) tends to a point $x \in C^1$, is given by the left-hand side of Eqs. $(1.15)_1$.

We can see that, for $x \in \Omega^1$, the potentials (1.20) and (1.20′) represent a Stokes flow with zero vector tension in points of C^1. As in (1.6), we deduce that:

$$(1.21) \qquad V_j^0(x) = u_j^i(x), \qquad \text{for} \quad x \in \Omega^1, \quad j \in \{1,2\}, \quad i \in \{1,2,3\},$$

where the functions $\mathbf{u}^i$, $i \in \{1,2,3\}$ are given in (1.8) (or a linear combination of these functions).

In the same way, the potentials (1.20) and (1.20′), for all $x \in \Omega^2$, represent a Stokes flow in Ω^2 with zero vector tension on C^2, with zero velocity on the boundary $x_2 = 0$, and the asymptotic form at infinity:

$$\mathbf{V}^0(x) = O(1), \qquad \text{as} \quad |x| \to \infty.$$

In the above statement we consider the boundary $x_2 = 0$ as a rigid wall, bounding a Stokes flow in Ω^2.

By using the Green's formula in Ω^2, it results that

$$\mathbf{V}^0(x) = \mathbf{0}, \qquad \text{for all} \quad x \in \Omega^2. \tag{1.22}$$

The previous arguments show that the potentials (1.20), (1.20′) represent a Stokes flow in Ω with the following boundary conditions on C^1 and C^2:

$$\begin{aligned} V_j^0(x) &= u_j^i(x), \qquad x \in C^1, \quad j \in \{1,2\}, \quad i \in \{1,2,3\}, \\ V_j^0(x) &= 0, \qquad x \in C^2, \quad j \in \{1,2\}. \end{aligned} \tag{1.23}$$

The above conditions determine the following Fredholm integral system of the first kind for the unknown function $\boldsymbol{\tau}$:

$$\begin{aligned} \int\limits_{C^1 \cup C^2} q_{ij}(x,y)\tau_j(y)\, ds_y &= u_i^k(x), \quad x \in C^1, \quad i \in \{1,2\}, \quad k \in \{1,2,3\}, \\ \int\limits_{C^1 \cup C^2} q_{ij}(x,y)\tau_j(y)\, ds_y &= 0, \qquad x \in C^2, \quad i \in \{1,2\}. \end{aligned} \tag{1.24}$$

Using the Fredholm's alternative (see [3, 4]), we prove that the system (1.24) has a unique solution, for each $k \in \{1,2,3\}$. In fact we show that the corresponding homogeneous system (1.24) has only a trivial solution.

For this aim, let us consider the following system:

$$\begin{aligned} \int\limits_{C^1 \cup C^2} q_{ij}(x,y)\tau_j^0(y)\, ds_y &= 0, \qquad x \in C^1, \quad i \in \{1,2\}, \\ \int\limits_{C^1 \cup C^2} q_{ij}(x,y)\tau_j^0(y)\, ds_y &= 0, \qquad x \in C^2, \quad i \in \{1,2\}. \end{aligned} \tag{1.25}$$

If we consider the single-layer potentials (1.20) and (1.20′) with density given by any possible continuous solution $\boldsymbol{\tau}^0$ of (1.25), then we conclude that the Stokes velocity $\mathbf{V}^0 = \mathbf{V}^0(\boldsymbol{\tau}^0)$ vanishes identically on C^1 and C^2. From the uniqueness result of the solution corresponding to the boundary-value problem (1.1), (1.2), we conclude that $\mathbf{V}^0 = \mathbf{V}^0(\boldsymbol{\tau}^0)$ must be equal to zero in Ω.

On the other hand, from the continuity property of single-layer potentials $V_j^0 = V_j^0(\boldsymbol{\tau}^0)$, $j \in \{1,2\}$, in each point of upper halfplane $\mathbb{R}_+^2$, it results that

$V_j^0 = V_j^0(\boldsymbol{\tau})(x) = 0$, for all $x \in \Omega^2$. Therefore, $T_{ij}(\mathbf{V}^0(\boldsymbol{\tau}^0)(x)) = 0$, for all $x \in \Omega^2$, and in particular we obtain

$$\lim_{\substack{x' \to x \in C^2 \\ x' \in \Omega^2}} T_{ij}(\mathbf{V}^0(\boldsymbol{\tau}^0)(x'))n_j(x') = -2\pi\tau_i^0(x) - \int_{C^1 \cup C^2} K_{ji}(y,x)\tau_j^0(y)\,ds_y = 0. \tag{1.26}$$

Also, we have

$$\lim_{\substack{x' \to x \in C^2 \\ x' \in \Omega}} T_{ij}(\mathbf{V}^0(\boldsymbol{\tau}^0)(x'))n_j(x') = 2\pi\tau_i^0(x) - \int_{C^1 \cup C^2} K_{ji}(y,x)\tau_j^0(y)\,ds_y = 0. \tag{1.27}$$

From (1.26) and (1.27) we obtain that $\boldsymbol{\tau}^0(x) = 0$, for $x \in C^2$. Analogously, we can prove that $\boldsymbol{\tau}^0(x) = 0$, for $x \in C^1$. Hence, the only solution of the homogeneous system (1.25) is the trivial solution, and also the system (1.24) (with k fixed) has a unique continuous solution. Because the system (1.24) has three linearly independent non-homogeneous terms $\mathbf{u}^1$, $\mathbf{u}^2$, $\mathbf{u}^3$, it is easily shown that the corresponding solutions, denoted by $\boldsymbol{\tau}^1$, $\boldsymbol{\tau}^2$, $\boldsymbol{\tau}^3$, are linearly independent. For this aim, let us consider the real numbers γ_1, γ_2, γ_3, such that

$$\sum_{i=1}^{3} \gamma_i \boldsymbol{\tau}^i(x) = 0, \qquad x \in C^1 \cup C^2.$$

Using (1.24) and the above equality, we obtain:

$$0 = \int_{C^1 \cup C^2} \left\{ q_{lj}(x,y) \sum_{i=1}^{3} \gamma_i \tau_j^i(y) \right\} ds_y = \sum_{i=1}^{3} \gamma_i u_l^i(x), \qquad x \in C^1, \quad l \in \{1,2\}.$$

By applying the linearly independent property of the functions $\mathbf{u}^1$, $\mathbf{u}^2$, $\mathbf{u}^3$, we deduce that $\gamma_1 = \gamma_2 = \gamma_3 = 0$, hence the functions $\boldsymbol{\tau}^1$, $\boldsymbol{\tau}^2$, $\boldsymbol{\tau}^3$, are linearly independent.

On the other hand, each solution $\boldsymbol{\tau}$ of the adjoint system (1.15) is also a solution of system (1.24). Hence, the system (1.15) has at most three linearly independent solutions, which shows that $d \leq 3$. Now we conclude that $d = 3$ and that the system (1.15) has the same solutions as the system (1.24).

By following the second Fredholm alternative (see [3, 4]), it results that a necessary and sufficient condition for the solvability of system (1.13), can be written as:

$$\int_{C^1} f_{1j}(x)\tau_j^i(x)\,ds_x + \int_{C^2} f_{2j}(x)\tau_j^i(x)\,ds_x = 0, \qquad i \in \{1,2,3\}, \tag{1.28}$$

where $\boldsymbol{\tau}^1$, $\boldsymbol{\tau}^2$, $\boldsymbol{\tau}^3$, are linearly independent solutions of system (1.24).

Finally, we can formulate the following result:

THEOREM. *The Stokes problem* (1.1), (1.2) *with the boundary condition* (1.3), *has a unique solution* $(\mathbf{u}, p)$ *on the bounded domain* Ω, *if and only if the functions* $\mathbf{f}_1$ *and* $\mathbf{f}_2$ *satisfy the conditions* (1.28).

The above condition (1.28) is restrictive. Then we consider a modified form for the flow $(\mathbf{u}, p)$.

2. Another form of solution

Using the singularity method, we determine the flow $(\mathbf{u}, p)$ as a sum of a double-layer potential plus some singularities located in a point x_c from the domain Ω^1:

$$
\begin{aligned}
u_j(x) &= \int\limits_{C^1 \cup C^2} K_{jl}(x,y)\phi_l(y)\,ds_y + \alpha_i q_{ji}(x,x_c) \\
&\qquad + w_l \varepsilon_{lmi} \frac{\partial q_{ji}}{\partial y_m}(x,x_c), \qquad j \in \{1,2\}, \\
p(x) &= \int\limits_{C^1 \cup C^2} K_j(x,y)\phi_j(y)\,ds_y + \alpha_i q_i(x,x_c) \\
&\qquad + \varepsilon_{lmj} \frac{\partial q_j}{\partial y_m}(x,x_c) w_l, \qquad x \in \Omega.
\end{aligned}
\tag{2.1}
$$

We choose the constants α_i, $w_3 \in \mathbb{R}$ in the following manner:

$$
\begin{aligned}
\alpha_j &= \int\limits_{C^1 \cup C^2} \phi_l(y) u_l^j(y)\,ds_y, \qquad j \in \{1,2\}, \\
w_3 = \alpha_3 &= \int\limits_{C^1 \cup C^2} \phi_l(y) u_l^3(y)\,ds_y,
\end{aligned}
\tag{2.2}
$$

where the functions $\mathbf{u}^1$, $\mathbf{u}^2$, $\mathbf{u}^3$ are given in (1.8).

By applying the boundary conditions (1.2), we obtain the following Fredholm integral system of second kind, with the unknown function $\boldsymbol{\phi}$:

$$
\begin{aligned}
&-2\pi\phi_j(x) + \int\limits_{C^1 \cup C^2} K_{jl}(x,y)\phi_l(y)\,ds_y + \alpha_i q_{ji}(x,x_c) \\
&\qquad + w_l \varepsilon_{lmi} \frac{\partial q_{ji}}{\partial y_m}(x) = f_{1j}(x), \qquad x \in C^1, \\
&2\pi\phi_j(x) + \int\limits_{C^1 \cup C^2} K_{jl}(x,y)\phi_l(y)\,ds_y + \alpha_i q_{ji}(x,x_c) \\
&\qquad + \varepsilon_{lmi} \frac{\partial q_{ji}}{\partial y_m}(x,x_c) w_l = f_{2j}(x), \qquad x \in C^2.
\end{aligned}
\tag{2.3}
$$

According to Fredholm's alternative (see [3, 4]), in order to prove the existence and uniqueness result of solution of system (2.3), it is sufficient to show that the following homogeneous system (2.47) has only the trivial solution:

$$(2.4)\quad \begin{aligned} &-2\pi\phi_j^0(x) + \int\limits_{C^1\cup C^2} K_{jl}(x,y)\phi_l^0(y)\,ds_y + \alpha_i^0 q_{ji}(x,x_c) \\ &\qquad + w_l^0\varepsilon_{lmi}\frac{\partial q_{ji}}{\partial y_m} = 0, \qquad x \in C^1, \\ &2\pi\phi_j^0(x) + \int\limits_{C^1\cup C^2} K_{jl}(x,y)\phi_l^0(y)\,ds_y + \alpha_i^0 q_{ji}(x,x_c) \\ &\qquad + w_l^0\varepsilon_{lmi}\frac{\partial q_{ji}}{\partial y_m}(x,x_c) = 0, \qquad x \in C^2, \end{aligned}$$

where

$$(2.5)\qquad \alpha_j^0 = \int\limits_{C^1\cup C^2} \phi_l^0(y)u_l^j(y)\,ds_y, \qquad j\in\{1,2,3\}$$

and $w_3^0 = \alpha_3^0$.

From (1.13′) and (2.4) it results that the vectors $\mathbf{v}^1$ and $\mathbf{v}^2$, given by:

$$(2.6)\quad \begin{aligned} v_j^1(x) &= \int\limits_{C^1\cup C^2} K_{jl}(x,y)\phi_l^0(y)\,ds_y\,, \\ v_j^2(x) &= \left\{\alpha_i^0 q_{ji}(x,x_c) + w_l^0\varepsilon_{lmi}\frac{\partial q_{ji}}{\partial y_m}(x,x_c)\right\}, \quad j\in\{1,2\} \end{aligned}$$

can be considered as Stokes velocity flows in Ω, which are equal on C^1 and C^2. From the uniqueness result of solution corresponding to the Stokes problem (1.1), (1.2) we deduce that $\mathbf{v}^1 = \mathbf{v}^2$ in Ω. It is easy to show that $\mathbf{v}^1$ gives zero total force on C^1 or C^2 (when the tension vector is considered in points of C^1 and C^2 as limiting values), and $\mathbf{v}^2$ gives a non-zero total force on C^1 or C^2, equal to $\pm 4\pi\boldsymbol{\alpha}^0$, where $\boldsymbol{\alpha}^0 = (\alpha_1^0, \alpha_2^0)$. Hence, we obtain

$$(2.7)\qquad \alpha_1^0 = \alpha_2^0 = 0.$$

On the other hand, $\mathbf{v}^1$ yields zero total torque on C^1 or C^2, and $\mathbf{v}^2$ yields a non-zero torque on C^1 or C^2. Precisely, this torque is equal to $\pm 8\pi\alpha_3^0\mathbf{k}$, where $\mathbf{k}$ is the unit vector of the $0x_3$ axis, orthogonal to the $0x_1x_2$ plane. We conclude that

$$(2.8)\qquad \alpha_3^0 = w_3^0 = 0.$$

From (2.7) and (2.8) it results that the system (2.4) is reduced to the system (1.14), which has three linearly independent solutions:

$$\mathbf{\Phi}_i^0(x) = \begin{cases} \mathbf{u}^i(x), & x \in C^1, \\ 0 & x \in C^2, \end{cases} \qquad i \in \{1,2,3\}.$$

Then, any solution of system (2.4) can be written as follows:

$$\mathbf{\Phi}^0(x) = \sum_{i=1}^{3} \beta_i \mathbf{\Phi}_i^0(x), \qquad x \in C^1 \cup C^2, \tag{2.9}$$

where β_1, β_2, β_3 are some real constants.

Using (2.7), (2.8) and (2.9) we obtain the following linear algebraic system with unknows β_i, $i \in \{1,2,3\}$:

$$\sum_{i=1}^{3} \beta_i \int_{C^1} u_l^i(y) u_l^j(y)\, ds_y = 0, \qquad j \in \{1,2,3\}. \tag{2.10}$$

Using the form of functions $\mathbf{u}^i$, $i \in \{1,2,3\}$ we infer that the corresponding determinant of system (2.10) is non-zero. Hence, $\beta_1 = \beta_2 = \beta_3 = 0$, which shows that the only solution of system (2.4) is the null solution. It results that the Fredholm integral system (2.3) has a unique continuous solution. With this argument we have proved the existence and uniqueness of solution corresponding to the Stokes problem (1.1)–(1.2).

REMARK. An analogous problem for the creeping flow of an incompressible viscous fluid between two arbitrary closed surfaces, was studied recently by H. POWER and G. MIRANDA (see [7]). Using the theory of single layer potentials, T.M. FISCHER, G.C. HSIAO, W.L. WENDLAND studied the slow viscous flows past obstacles in a half-plane (see [2]). Using the theory of double layer potentials, H. POWER and G. MIRANDA solved the problem of a three-dimensional Stokes flow past a rigid obstacle (see [5]).

The same method as that used in [5], was applied by H. Power to solve the problem of a Stokes flow past n bodies ($n \geq 1$) of arbitrary shapes (see [6]). A complete double-layer method was given by N.P. THIEN, D. TULLOCK and S. KIM in [9], to solve the problem of a Stokes flow past obstacles in a half-space.

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