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Scattering of oblique waves by a thin vertical wall with a submerged gap

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THIS PAPER is concerned with scattering of an obliquely incident train of surface water waves by a thin vertical wall with a submerged gap. Utilizing Havelock's expansion of water wave potential, two integral equations, one involving the horizontal component of velocity across the gap and the other involving the difference of velocity potential across the wall, are obtained. The quantities of physical interest, namely the reflection and transmission coefficients, are related to the solutions of these integral equations. For the case of normal incidence of the wave train these integral equations have exact solutions. These exact solutions provide one-term Galerkin approximations to the solutions of the corresponding oblique incidence integral equations. Identifying the reflection and transmission coefficients as some inner products involving the solutions of these integral equations and exploiting the properties of self-adjointness and positive semi-definiteness of the integral operators defining the integral equations, the one-term approximations result in some lower and upper bounds for the reflection and transmission coefficients. Numerical evaluation of these bounds for any angle of incidence and any wave number reveals that they are very close to each other, and as such they produce good approximations to the exact values of the quantities of physical interest. For the special case of normal incidence this method produces numerical results which are in good agreement with the results available in the literature obtained by other methods.

1. Introduction

WATER WAVE scattering problems involving fixed plane vertical barriers are being studied in the literature, assuming linear theory, over the last fifty years by employing various mathematical techniques. Since a thin barrier models a breakwater which shelters a port from the rough sea, study of its effect on surface water waves is of some physical importance. PORTER [1] considered the problem of water wave diffraction by a thin vertical wall with a submerged gap for the case of normal incidence of the wave train, and used a complex variable technique as well as an integral equation procedure based on Green's integral theorem to solve it in closed form. A number of researchers also studied the narrow gap problem assuming the gap width to be very small compared to the depth of submergence of its midpoint below the free surface. TUCK [2] used the method of matched asymptotic expansion to obtain the transmission coefficient approximately. PACKHAM and WILLIAMS [3] used an integral equation formulation based on a suitable use of Green's integral theorem for uniform finite depth of water, wherein the integral equation was solved approximately by exploiting the concept of narrowness of the gap, and then the transmission coefficient was obtained approximately. MANDAL [4] also considered the narrow gap problem for deep water by an integral equation formulation based on Havelock's expansion of water wave potential and used the idea of PACKHAM and WILLIAMS [3] to solve it approximately, and also obtained the transmission coefficient approximately.

For oblique incidence of the wave train, the narrow gap problem was considered by LIU and WU [5] who utilized TUCK'S [2] idea of matched asymptotic expansion to obtain the transmission coefficient apparently for low wave numbers, since approximation of Helmholtz's equation by Laplace equation for obtaining the near-field solution is not valid for large values of the wave number. MANDAL and KUNDU [6] used Havelock's expansion of water wave potential satisfying Helmholtz's equation to obtain an integral equation across the gap, which was then solved by assuming the gap to be narrow and the transmission coefficient was determined approximately.

MANDAL and DOLAI [7] recently used the idea of EVANS and MORRIS [8] to obtain very accurate lower and upper bounds for the reflection and transmission coefficients in oblique wave diffraction problems, involving four basic configurations of a thin vertical barrier present in water of uniform finite depth.

In the present paper the problem of oblique water wave diffraction by a thin vertical wall with a submerged gap (not necessarily narrow) is studied by utilizing the idea of EVANS and MORRIS [8]. The reflection and transmission coefficients are obtained in terms of two integrals involving the unknown horizontal component of velocity across the gap, and difference of velocity potential across the wall, respectively. These unknown functions satisfy some integral equations which have exact solutions for the case of normal incidence. Following EVANS and MORRIS [8], these known exact solutions for normally incident waves are utilized as one-term Galerkin approximations to the solutions of these two integral equations. The reflection and transmission coefficients are identified with some inner products involving the solutions of these integral equations. Exploiting the properties of self-adjointness and positive semi-definiteness of the integral operators defining the integral equations, the one-term Galerkin approximations produce upper and lower bounds for the reflection and transmission coefficients for any angle of incidence and any wave number. It is analytically verified that for the normal incidence case, the upper and lower bounds coincide. The bounds involve a number of integrals which are evaluated numerically by standard techniques. The numerical results reveal that the two bounds for any angle of incidence and any wave number are very close, and as such they produce very good approximations to the exact values of the reflection and transmission coefficients. In our numerical scheme, if the angle of incidence is taken to be zero (for the case of normal incidence of the wave train), the numerical values of the two bounds coincide by more than four decimal places. This verifies the correctness of the numerical scheme. Also, for the normal incidence case, the numerical results obtained by the present method are in good agreement with the graphical results obtained by PORTER [1] and TUCK [2].

2. Formulation of the problem

We choose a rectangular Cartesian coordinate system in which the y-axis is taken vertically downwards into the fluid, $y = 0$ is the undisturbed free surface. A train of progressive surface waves represented by the velocity potential

$$\psi_0(x, y, z, t) = \mathrm{Re}\,\{\exp(-Ky + i\mu x + i\nu z - i\sigma t)\},$$

where $\mu = K\cos\alpha$, $\nu = K\sin\alpha$, $K = \sigma^2/g$, and g is the gravity and σ is the circular freequency, is assumed to be obliquely incident (from negative infinity) on a fixed thin plane vertical wall at an angle α to the normal to the wall. The wall occupies the position $x = 0$ and has a gap which is represented by $x = 0$, $y \in S$, $S = (a, b)$. The geometry of the problem allows the z-dependence to be eliminated by assuming the velocity potential to be of the form

$$\psi(x, y, z, t) = \mathrm{Re}\,\{\phi(x, y)\exp(i\nu z - i\sigma t)\}$$

throughout. Then $\phi(x, y)$ satisfies the boundary value problem described by

$$(\nabla^2 - \nu^2)\phi = 0 \qquad \text{for} \quad y \geq 0, \tag{2.1}$$

$$K\phi + \phi_y = 0 \qquad \text{on} \quad y = 0, \tag{2.2}$$

$$\phi_x = 0, \qquad y = 0, \qquad y \in \overline{S} = (0, \infty) - S, \tag{2.3}$$

$$r^{1/2}\nabla\phi \qquad \text{is bounded as} \quad r \to 0, \tag{2.4}$$

where r is the distance from a submerged end of the wall,

$$\nabla\phi \to 0 \quad \text{as} \quad y \to \infty, \tag{2.5}$$

and

$$\phi(x, y) \sim \begin{cases} Te^{-Ky+i\mu x} & \text{as} \quad x \to \infty, \\ e^{-Ky+i\mu x} + Re^{-Ky-i\mu x} & \text{as} \quad x \to -\infty, \end{cases} \tag{2.6}$$

where R and T are the (complex) reflection and transmission coefficients, respectively, to be obtained.

3. Method of solution

By Havelock's expansion of water wave potential, a suitable representation for $\phi(x, y)$ satisfying (2.1), (2.2), (2.5) and (2.6) is given by

$$\phi(x, y) = \begin{cases} Te^{-Ky+i\mu x} + \int\limits_0^\infty A(k)L(k, y)e^{-(\nu^2+k^2)^{1/2}x}\,dk & \text{for} \quad x > 0, \\ e^{-Ky+i\mu x} + Re^{-Ky-i\mu x} + \int\limits_0^\infty B(k)L(k, y)e^{(\nu^2+k^2)^{1/2}x}\,dk & \\ & \text{for} \quad x < 0, \end{cases} \tag{3.1}$$

with $L(k, y) = k\cos ky - K\sin ky$.

Let us define

$$f(y) = \phi_x(0, y), \qquad 0 < y < \infty, \tag{3.2}$$

and

$$g(y) = \phi(+0, y) - \phi(-0, y), \qquad 0 < y < \infty, \tag{3.3}$$

then

$$f(y) = 0 \qquad \text{for} \quad y \in \overline{S}, \tag{3.4}$$

and

$$g(y) = 0 \qquad \text{for} \quad y \in S. \tag{3.5}$$

The constants T, R and the functions $A(k)$, $B(k)$ are related to $f(y)$ and $g(y)$ by

$$T = 1 - R = -\frac{2iK}{\mu} \int_S f(y) e^{-Ky}\, dy, \tag{3.6}$$

$$A(k) = -B(k) = -\frac{2}{\pi} \frac{1}{(\nu^2 + k^2)^{1/2}(k^2 + K^2)} \int_S f(y) L(k, y)\, dy, \tag{3.7}$$

$$R = -K \int_{\overline{S}} g(y) e^{-Ky}\, dy, \tag{3.8}$$

$$A(k) = \frac{1}{\pi} \frac{1}{k^2 + K^2} \int_{\overline{S}} g(y) L(k, y)\, dy. \tag{3.9}$$

Using (2.3) in (3.1) along with (3.9) we obtain an integral equation for $g(y)$ in the form

$$\int_{\overline{S}} g(u) M(y, u)\, du = \pi i \mu (1 - R) e^{-Ky} \qquad \text{for} \quad y \in \overline{S}, \tag{3.10}$$

where

$$M(y, u) = \operatorname*{Lt}_{\varepsilon \to 0} \int_0^\infty \frac{(k^2 + \nu^2)^{1/2}}{k^2 + K^2} L(k, y) L(k, u) e^{-\varepsilon k}\, dk, \tag{3.11}$$

so that $M(y, u) = M(u, y)$ and the exponential term is being introduced to ensure the convergence of the integral.

Again, use of (3.5) in (3.1) along with (3.7) produces an integral equation for $f(y)$

$$\int_S f(u)N(y,u)\,du = -\frac{\pi}{2}R\,e^{-Ky} \qquad \text{for} \quad y \in S, \tag{3.12}$$

where

$$N(y,u) = \int_0^\infty \frac{L(k,y)L(k,u)}{(\nu^2+k^2)^{1/2}(k^2+K^2)}\,dk, \tag{3.13}$$

so that $N(y,u) = N(u,y)$.

If we let

$$F(y) = -\frac{2}{\pi R}f(y) \qquad \text{for} \quad y \in S, \tag{3.14}$$

$$G(y) = \frac{1}{\pi i\mu(1-R)}g(y) \qquad \text{for} \quad y \in \overline{S}, \tag{3.15}$$

then $G(y)$ and $F(y)$ satisfy the integral equations

$$\int_{\overline{S}} G(u)M(y,u)\,du = e^{-Ky} \qquad \text{for} \quad y \in \overline{S}, \tag{3.16}$$

and

$$\int_S F(u)N(y,u)\,du = e^{-Ky} \qquad \text{for} \quad y \in S. \tag{3.17}$$

It may be noted that the functions $G(u)$ and $F(u)$ in (3.16) and (3.17), respectively, are real.

The relations (3.6) and (3.8) can be written as

$$\int_S F(y)e^{-Ky}\,dy = C, \tag{3.18}$$

and

$$\int_{\overline{S}} G(y)e^{-Ky}\,dy = \frac{1}{\pi^2K^2C}, \tag{3.19}$$

where

$$C = \frac{1-R}{i\pi R\sec\alpha}. \tag{3.20}$$

It is very important to note that C is real.

4. Upper and lower bounds for C

As in EVANS and MORRIS [8], we define an inner product

$$\langle f, g\rangle = \int\limits_{\overline{S}} f(y)g(y)\,dy. \tag{4.1}$$

Then obviously $\langle f, g\rangle = \langle g, f\rangle$ and $\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle$. Also, let us define the operator

$$(\mathcal{M}f)(y) = \langle M(y,u), f(u)\rangle. \tag{4.2}$$

Since

$$M(y,u) = M(u,y) \qquad \text{and} \qquad (\mathcal{M}(f_1+f_2))(y) = (\mathcal{M}f_1)(y) + (\mathcal{M}f_2)(y),$$

we find

$$\langle \mathcal{M}f, g\rangle = \langle f, \mathcal{M}g\rangle$$

and

$$\langle \mathcal{M}f, f\rangle \geq 0 \qquad \text{for all} \quad f(y).$$

Following EVANS and MORRIS [8], for the solution of (3.16) we choose a one-term approximation as

$$G(y) \approx a_1 g_1(y), \tag{4.3}$$

where a_1 is a constant and $g_1(y)$ is to be chosen suitably. Then

$$a_1 = \frac{\langle g_1(y), e^{-Ky}\rangle}{\langle g_1(y), (\mathcal{M}g_1)(y)\rangle}. \tag{4.4}$$

Hence from (3.19)

$$\begin{aligned}\frac{1}{\pi^2 K^2 C} &= \langle G(y), e^{-Ky}\rangle \\ &\geq \langle a_1 g_1(y), e^{-Ky}\rangle,\end{aligned}$$

by utilizing the properties of self-adjointness and positive semi-definiteness of the operator (cf. EVANS and MORRIS [8]).

Thus we get an upper bound for C as

$$C \leq A_0, \tag{4.5}$$

where

(4.6) $$A_0 = \frac{\underset{\varepsilon\to 0}{\mathrm{Lt}}\displaystyle\int_0^\infty \frac{(\nu^2+k^2)^{1/2}e^{-\varepsilon k}}{k^2+K^2}\left[\int_{\overline{S}} g_1(y)L(k,y)\,dy\right]^2 dk}{\pi^2K^2\left[\displaystyle\int_{\overline{S}} g_1(y)e^{-Ky}\,dy\right]^2}.$$

Again, let us define another inner product

(4.7) $$\{f,g\} = \int_S f(y)g(y)\,dy$$

and another operator

(4.8) $$(\mathcal{N}f)(y) = \{N(y,u),f(u)\}.$$

Then it is obvious that the inner product $\{f,g\}$ is symmetric, linear, and also the operator $\mathcal{N}$ is linear, self-adjoint and positive semi-definite.

Choosing a one-term approximation to $F(y)$ as

(4.9) $$F(y) \approx b_1 f_1(y),$$

where b_1 is a constant and $f_1(y)$ is to be chosen suitably, we find that

(4.10) $$b_1 = \frac{\{f_1(y),e^{-Ky}\}}{\{f_1(y),(\mathcal{N}f_1)(y)\}}.$$

Thus, by using (3.18) and the same argument as before, we find a lower bound for C as

(4.11) $$C \geq B_0,$$

where

(4.12) $$B_0 = \frac{\left[\displaystyle\int_S f_1(y)e^{-Ky}\,dy\right]^2}{\displaystyle\int_0^\infty \frac{1}{(\nu^2+k^2)^{1/2}(k^2+K^2)}\left[\int_{\overline{S}} f_1(y)L(k,y)\,dy\right]^2 dk}.$$

Hence for the unknown real constant C, which involves R, we find

(4.13) $$B_0 \leq C \leq A_0,$$

where A_0 and B_0 are given by (4.6) and (4.12), respectively. Thus the upper and lower bounds for $|R|$ and $|T|$ are obtained as

$$R_1 \le |R| \le R_2, \qquad T_1 \le |T| \le T_2, \tag{4.14}$$

where

$$R_1 = \frac{\cos\alpha}{(\cos^2\alpha + \pi^2 A_0^2)^{1/2}}, \qquad R_2 = \frac{\cos\alpha}{(\cos^2\alpha + \pi^2 B_0^2)^{1/2}}, \tag{4.15}$$

$$T_1 = \frac{\pi B_0}{(\cos^2\alpha + \pi^2 A_0^2)^{1/2}}, \qquad T_2 = \frac{\pi A_0}{(\cos^2\alpha + \pi^2 B_0^2)^{1/2}}. \tag{4.16}$$

5. Functions $g_1(y)$ and $f_1(y)$

The functions $g_1(y)$ and $f_1(y)$ are chosen as the explicit solutions of the appropriate integral equations associated with the problem of submerged gap in deep water for the case of normal incidence of the wave train. These are given by (cf. PORTER [1], MANDAL and DOLAI [7]).

$$g_1(y) = A_1\chi(y) \qquad (A_1 \neq 0) \tag{5.1}$$

and

$$f_1(y) = B_1\lambda'(y) \qquad (B_1 \neq 0), \tag{5.2}$$

where

$$\chi(y) = \begin{cases} -e^{-Ky}\displaystyle\int_a^y \frac{te^{Kt}}{S_1(t)}\left[\delta - \frac{2}{\pi}H(a,b,t)\right]dt & \text{for} \quad 0 < y < a, \\ e^{-Ky}\displaystyle\int_b^y \frac{te^{Kt}}{S_3(t)}\left[\delta - \frac{2}{\pi}H(a,b,t)\right]dt & \text{for} \quad y > b \end{cases} \tag{5.3}$$

and

$$\lambda(y) = e^{-Ky}\int_b^y \frac{te^{Kt}}{S_2(t)}\left[\delta - \frac{2}{\pi}H(a,b,t)\right]dt \qquad \text{for} \quad a < y < b \tag{5.4}$$

with

$$\delta = \frac{\dfrac{2}{\pi}\displaystyle\int_a^b \frac{te^{Kt}}{S_2(t)}H(a,b,t)\,dt + \frac{e^{Ka}}{K}}{\displaystyle\int_a^b \frac{te^{Kt}}{S_2(t)}\,dt},$$

$$H(a,b,t) = \int_0^a \frac{S_1(s)}{s^2 - t^2}\,ds$$

and

$$S_1(t) = \left\{(a^2 - t^2)(b^2 - t^2)\right\}^{1/2},$$
$$S_2(t) = \left\{(t^2 - a^2)(b^2 - t^2)\right\}^{1/2},$$
$$S_3(t) = \left\{(t^2 - a^2)(t^2 - b^2)\right\}^{1/2}.$$

Substituting these in the expressions (4.6) and (4.12), A_0 and B_0 are obtained as

$$A_0 = \frac{\int\limits_0^\infty \frac{(\nu^2+k^2)^{1/2}}{k^2+K^2}\left[-\frac{\sin ka}{k} + \int\limits_a^b \frac{y\cos ky}{S_2(y)}\left\{\delta - \frac{2}{\pi}H(a,b,y)\right\}dy\right]^2 dk}{\frac{1}{4}\pi^2 I^2} \tag{5.5}$$

and

$$B_0 = \frac{\frac{1}{4}J^2}{\int\limits_0^\infty \frac{k^2}{(\nu^2+k^2)^{1/2}(k^2+K^2)}\left[-\frac{\sin ka}{k} + \int\limits_a^b \frac{y\cos ky}{S_2(y)}\left\{\delta - \frac{2}{\pi}H(a,b,y)\right\}dy\right]^2 dk}, \tag{5.6}$$

where

$$I = \delta\{\alpha_1(K) - \alpha_3(K)\} - \frac{2}{\pi}\{\alpha_1(K,H) - \alpha_3(K,H)\},$$
$$J = \delta\alpha_2(K) - \frac{2}{\pi}\alpha_2(K,H) + \frac{e^{-Ka}}{K}$$

with

$$\alpha_i(K) = \int\limits_{\Gamma_i} \frac{ye^{-Ky}}{S_i(y)}dy, \qquad i = 1,2,3$$

and

$$\alpha_i(K,H) = \int\limits_{\Gamma_i} \frac{yH(a,b,y)e^{-Ky}}{S_i(y)}dy, \qquad i = 1,2,3,$$

where the curve Γ_1 is the interval $(-a,a)$, Γ_2 is (a,b) and Γ_3 is (b,∞).

For the case of normal incidence, the numerator of the expression (5.5) and the denominator of the expression (5.6) are identical and equal to

$$\int\limits_0^\infty \frac{k}{k^2+K^2}\left[-\frac{\sin ka}{k} + \int\limits_a^b \frac{y\cos ky}{S_2(y)}\left\{\delta - \frac{2}{\pi}H(a,b,y)\right\}dy\right]^2 dk. \tag{5.7}$$

Integrating

$$\frac{ze^{ikz}}{\{(z^2-a^2)(z^2-b^2)\}^{1/2}} \qquad (k>0)$$

in the complex z-plane along the contour consisting of the arc of the quarter circle of large radius and centre at the origin, the positive imaginary axis, and a line along the positive real axis cut from 0 to a and b to ∞, the line running slightly above the cuts, we get

$$-\int_0^a \frac{x\sin kx}{S_1(x)}dx + \int_b^\infty \frac{x\sin kx}{S_3(x)}dx = \int_a^b \frac{x\cos kx}{S_2(x)}dx \qquad (k>0).$$

Again, integrating

$$\frac{ze^{ikz}}{\{(z^2-a^2)(z^2-b^2)\}^{1/2}(z^2-v^2)} \qquad (k>0, \quad 0<v<a)$$

along the same contour together with an indentation above the pole at $z = v$ $(0 < v < a)$ on the positive real axis, we obtain

$$-\int_0^a \frac{x\sin kx}{(v^2-x^2)S_1(x)}dx + \int_b^\infty \frac{x\sin kx}{(v^2-x^2)S_3(x)}dx - \frac{\pi}{2}\frac{\cos kv}{S_1(v)}$$
$$= \int_a^b \frac{x\cos kx}{(v^2-x^2)S_2(x)}dx \qquad (k>0, \quad 0<v<a).$$

Using the above two identities suitably in the expression (5.7), interchanging the order of integration and utilizing the result (GRADSTEYN and RYZHIK [9]), pp. 415)

$$\int_0^\infty \frac{k\cos ky\sin ku}{k^2+K^2}dk = \begin{cases} \dfrac{\pi}{2}e^{-Ku}\cosh Ky, & 0<y<u, \\ -\dfrac{\pi}{2}e^{-Ky}\sinh Ku, & 0<u<y, \end{cases}$$

we obtain after some calculations that the expression (5.7) is equal to $-(\pi/4)JI$.

Hence for $\alpha = 0$, we have

$$A_0 = B_0 = -\frac{J}{\pi I}$$

thereby giving an exact value of C for $\alpha = 0$, so that in this case $R = iI/(J+iI)$, which was earlier obtained by PORTER [1].

6. Numerical results

The expressions (5.5) and (5.6) of A_0 and B_0, respectively, and hence the lower and upper bounds for $|R|$ and $|T|$ are evaluated numerically for a number of values of the non-dimensional parameters Kb, a/b and the angle of incidence α. The various single integrals appearing in these expressions are evaluated by using the Gauss quadrature formula appropriately. For the repeated integrals, the inner integrals are evaluated by using the Gauss quadrature formula while the outer integrals over $(0,\infty)$ are split into those over $(0,1)$ and $(1,\infty)$. The integrals over $(0,1)$ are computed by using the Gauss quadrature formula. The integrals over $(1,\infty)$ are evaluated by Simpson's rule over $(1,X)$ $(X \gg 1)$, where X increases till the values of the integrals correct to some desired decimal places are obtained. A representative set of values of the lower and upper bounds R_1 and R_2 of $|R|$ for various values of the parameters is displayed in Tables 1 to 3. Table 1 gives the bounds of $|R|$ for various values of the wave number Kb, the angle of incidence α and for $a/b = 0.05$. Tables 2 and 3 give the same for $a/b = 0.1$ and 0.5, respectively.

Table 1. Lower and upper bounds for the reflection coefficient $|R|$ for $a/b = 0.05$.

	$\alpha = 0°$	$\alpha = 30°$		$\alpha = 60°$		$\alpha = 85°$	
Kb	$R_1 = R_2$	R_1	R_2	R_1	R_2	R_1	R_2
0.05	0.7065	0.6256	0.6376	0.3899	0.4062	0.0712	0.0783
0.4	0.3250	0.2412	0.2580	0.1181	0.1376	0.0194	0.0233
1.2	0.0787	0.0492	0.0565	0.0215	0.0282	0.0034	0.0047
2.0	0.0316	0.0214	0.0245	0.0100	0.0132	0.0016	0.0023
3.0	0.0382	0.0320	0.0326	0.0178	0.0186	0.0030	0.0032
4.0	0.0657	0.0564	0.0565	0.0321	0.0323	0.0056	0.0056

Table 2. Lower and upper bounds for the reflection coefficient $|R|$ for $a/b = 0.1$.

	$\alpha = 0°$	$\alpha = 30°$		$\alpha = 60°$		$\alpha = 85°$	
Kb	$R_1 = R_2$	R_1	R_2	R_1	R_2	R_1	R_2
0.05	0.7072	0.6264	0.6384	0.3907	0.4070	0.0714	0.0785
0.4	0.3284	0.2444	0.2612	0.1200	0.1397	0.0197	0.0237
1.2	0.0963	0.0640	0.0722	0.0292	0.0374	0.0047	0.0063
2.0	0.0806	0.0636	0.0671	0.0335	0.0377	0.0056	0.0065
3.0	0.1545	0.1324	0.1327	0.0751	0.0756	0.0130	0.0131
4.0	0.2802	0.2411	0.2414	0.1378	0.1386	0.0239	0.0241

It is observed from the Tables 1 – 3 that in most cases the bounds are very close to each other so that their mean value provides a very good approximation to the actual value of $|R|$. It may be noticed that the difference between the bounds

Table 3. Lower and upper bounds for the reflection coefficient $|R|$ for $a/b = 0.5$.

	$\alpha = 0°$	$\alpha = 30°$		$\alpha = 60°$		$\alpha = 85°$	
Kb	$R_1 = R_2$	R_1	R_2	R_1	R_2	R_1	R_2
0.05	0.7251	0.6473	0.6586	0.4106	0.4266	0.0758	0.0831
0.4	0.4343	0.3456	0.3624	0.1824	0.2060	0.0308	0.0059
1.2	0.6502	0.5870	0.5891	0.3752	0.3793	0.0693	0.0705
2.0	0.9466	0.9230	0.9245	0.7861	0.7953	0.2094	0.2175
3.0	0.9960	0.9931	0.9936	0.9725	0.9771	0.5661	0.6106
4.0	0.9996	0.9993	0.9993	0.9967	0.9975	0.8949	0.9206

increases with the increase of the angle of incidence, but not significantly. For fixed Kb and a/b, from each table it is further observed that $|R|$ decreases with the increase of the angle of incidence. For fixed a/b and α, $|R|$ first decreases with the increase of Kb until a minimum is reached, and then it increases to unity asymptotically for further increase in Kb. This behaviour of $|R|$ is expected physically since for small Kb, the wavelength of the incident field is large compared to the width of the gap, so that there occurs a small energy transmission through the gap giving rise to large reflection coefficient. However as Kb increases, the wavelength of the incident field and the width of the gap become comparable, resulting in an increase of energy transmission through the gap. As Kb further increases, wavelength of the incident field further decreases and the waves are then confined within a thin layer below the free surface and as such, the wave energy is almost totally reflected by the part of the wall above the gap. The presence of the gap is hardly felt by these short waves and in the limit $|R| \to 1$ as $Kb \to \infty$. Thus $|R|$ has a minimum for some moderate value of Kb. For the normal incidence case, qualitatively similar behaviour of $|R|$ is noticed in the figure presented by PORTER [1]. It may be noted that for the complementary problem of submerged plate, the reflection coefficient exhibits the opposite behaviour (cf. EVANS [10]).

The results obtained from our numerical scheme for normal incidence have been compared with PORTER'S [1] results. PORTER [1] used the non-dimensional parameters μ and k which are given here by

$$\mu = \frac{2(b-a)}{b+a}, \qquad k = \frac{K(a+b)}{2}.$$

For $\mu = 0.1$ and 1.5 with $k = 0.5$, 2.0 we obtain from our results the following numerical values of $|R|$ (taken as the mean of the two bounds).

$k \backslash \mu$	0.1	1.5
0.5	0.8080	0.1609
2.0	0.9995	0.4368

These coincide with the results estimated from the graphical result of PORTER [1].

Again, for a narrow gap, the results obtained from the present numerical scheme for normal incidence are also compared with TUCK'S [2] numerical results obtained by utilizing the method of matched asymptotic expansion. The dimensionless parameters used in TUCK'S [2] analysis are $2c/h$ and h/λ (where $2c$ is the width of the gap, h is the depth of the mid-point of the gap and λ is the wavelength) which are given here by

$$\frac{2c}{h} = \frac{2(b-a)}{b+a} \qquad \text{and} \qquad \frac{h}{\lambda} = \frac{K(a+b)}{4\pi}.$$

For $2c/h = 0.05, 0.15, 0.4$ and $h/\lambda = 0.05$ we obtain here $|T|^2 = 1 - |R|^2$ as 0.3972, 0.5459 and 0.7202, respectively.

For $2c/h = 0.4$ and $h/\lambda = 0.1$, the corresponding value of $|T|^2$ as obtained here is 0.5982. As before, for $|R|$ we have taken the mean of its two bounds. These values of $|T|^2$ coincide with the results estimated from the graph of $|T|^2$ against h/λ given by TUCK [2].

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