Arch. Mech., 48, 5, pp. 869–892, Warszawa 1996

Travelling waves in laser sustained plasma Constant coefficient case

B. KAŹMIERCZAK (WARSZAWA)

WE USE the Conley index theory to prove existence of travelling waves to a system of partial differential equations describing a two-temperature model of plasma sustained by a laser beam. These waves connect two asymptotic state of gas: a cold one and a hot partially ionized one.

1. Introduction

THE AIM OF THIS PAPER is to prove the existence of travelling wave solution to the equations of a two-temperature model describing the laser-sustained plasma (see system (0)). The problem was positively solved by means of the implicit function theorem in [5] under the condition of sufficiently large values of the coupling parameter. This time we use the technique of Conley connection index theory (see [1, 2, 3, 4]). It seems interesting to compare these two methods. For simplicity, we consider the case of constant transport coefficients. The case of variable transport coefficients will be considered in the subsequent paper.

The evolution of temperatures T_1 and T_2 of electrons and heavy particles (i.e.atoms and ions) in plasma sustained by a laser beam under a constant pressure p are described by the following equations (see [5] and references therein):

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\mathbf{grad}\right) c_1 &= \operatorname{div}\,(k_1\,\mathbf{grad}\,T_1) + F_1 - (T_1 - T_2)\mathcal{W},\\
\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\mathbf{grad}\right) c_2 &= \operatorname{div}\,(k_2\,\mathbf{grad}\,T_2) + F_2 - (T_1 - T_2)\mathcal{W}.
\end{aligned}
\tag{0}
$$

Here k_i are effective heat conductivity coefficients, c_i their effective heat capacities per unit volume. F_1, F_2 are nonlinear source functions. The term $(T_1-T_2)\mathcal{W}$ describes collisional energy exchange betweeen electrons and heavy particles. $\mathcal{W}$ is proportional to the frequency of electron-heavy particle collisions. This frequency tends to infinity as $p \to \infty$. So, we can write $\mathcal{W}(p;T_1,T_2) = \lambda(p)W(T_1,T_2)$, where λ is a real parameter, $\lambda(p) \to \infty$ as $p \to \infty$. The functions $k_i > 0$, $c_i > 0$, F_i, $i \in \{1,2\}$, and $W > 0$ depend in general on T_1 and T_2. However, for simplicity of presentation we will assume that k_i, c_i, and W are constant. The dependence on T_1 and T_2 will be retained only in nonlinear source terms F_1 and F_2.

By looking for solutions in the form of a travelling wave, that is by making a substitution:

$$T_1(\mathbf{x},t) = u_1(\mathbf{x}\cdot\mathbf{n} + \chi t), \qquad T_2(\mathbf{x},t) = u_2(\mathbf{x}\cdot\mathbf{n} + \chi t),$$

where $n \in \mathbb{R}^3$ can be interpreted as the direction of propagation and χ as a speed of the wave, we are led to a system of ordinary differential equations of the form:

$$k_1 u_1'' - c_1\theta u_1' + F_1(u) + \lambda W(u_1 - u_2) = 0,$$
$$k_2 u_2'' - c_2\theta u_2' + F_2(u) + \lambda W(u_1 - u_2) = 0,$$

where $' := d/d\xi$, $\xi := \mathbf{x}\cdot\mathbf{n} + \chi t$, $\theta := (\chi + \mathbf{v}\cdot\mathbf{n})$ and $u := (u_1, u_2)$. It is obvious that by changing the scale of the independent variable and redefining the constants c_1, c_2 and λ we may obtain a simpler (but less symmetric) form of this system:

$$\begin{aligned} u_1'' - c_1\theta u_1' + F_1(u) - \lambda(u_1 - u_2) &= 0, \\ k u_2'' - c_2\theta u_2' + F_2(u) + \lambda(u_1 - u_2) &= 0. \end{aligned} \tag{1.1}$$

The roots of the corresponding algebraic system

$$F_1(u) - \lambda(u_1 - u_2) = 0,$$
$$F_2(u) + \lambda(u_1 - u_2) = 0,$$

are called constant states for (1.1). So, we are interested in *solutions defined for all $\xi \in \mathbb{R}^1$ whose derivatives vanish at $\pm\infty$ and such that $(u_1(\xi), u_2(\xi))$ tends to different constant states as $\xi \pm \infty$*. Such solutions are called *heteroclinics*. For a given λ such solutions can exist only for certain values of the parameter θ. (The problem considered is a sort of a nonlinear eigenvalue problem). Thus it makes sense to speak of heteroclinic triples (θ, u_1, u_2) satisfying Eqs. (1.1). Our aim is to prove existence of a heteroclinic connecting appropriate constant states of Eqs. (1.1). These constant states can be interpreted as the two states of gas: the cold incoming one (at $-\infty$) and the partially ionized hot one (at ∞). *The existence theorem is stated in Theorem at the end of Sec. 7.*

To analyze heteroclinic connections for Eqs. (1.1) we will consider the following family of systems:

$$\begin{aligned} u_1'' - c_1\theta u_1' + \mathcal{F}_{1\eta} - \lambda(u_1 - u_2) &= 0, \\ k u_2'' - c_{2\eta}\theta u_2' + \mathcal{F}_{2\eta} + \lambda(u_1 - u_2) &= 0, \end{aligned} \tag{1.η}$$

where $\eta \in [0, 1]$ and

$$c_{2\eta} = c_1 k(1 - \eta) + \eta c_2,$$
$$\mathcal{F}_{1\eta} := F_s + \eta(F_1 - F_s), \qquad \mathcal{F}_{2\eta} := F_s + \eta(F_2 - F_s),$$
$$F_s(u_1, u_2) := (F_1 + F_2)\left((1 + k)^{-1}(u_1 + k u_2),\ (1 + k)^{-1}(u_1 + k u_2)\right).$$

When we denote $w := (1 + k)^{-1}(u_1 + k u_2)$, $d := u_1 - u_2$, add and subtract the both sides of Eqs. (1.η), we obtain the system:

$$\begin{aligned} w'' - c_1\theta w' + 2(1 + k)^{-1}F_s(w, w) \qquad & \\ + \eta(1 + k)^{-1}\left(F_1 + F_2 - 2F_s - (c_2 - c_1 k)\theta u_2'\right) &= 0, \\ d'' - c_1\theta d' - (1 + k^{-1})\lambda d + \eta k^{-1}\left(k F_1 - F_2 + (c_2 - c_1 k)\theta u_2'\right) &= 0. \end{aligned} \tag{2}$$

Sections 2–5 have a preparatory character. Section 2 contains the assumptions imposed on the coefficients of system (1.1). In Sec. 3 we examine properties of constant states of system ($1.\eta$), and especially their behaviour for large λ. In Sec. 4 we prove *a priori* estimates for first derivatives of the solutions to ($1.\eta$) provided they are contained in a certain bounded region of (u_1, u_2)-space and prove that the set of θ, for which a heteroclinic orbit can exist is comprised in some bounded open interval (θ_0, θ_1), where $0 < \theta_0 < \theta_1$. Such estimations are necessary, because we want an isolating neighbourhood to be a compact subset of the phase space. In Sec. 5 we examine the eigenvalues and eigenvectors of the system linearized at its singular points. In Sec. 6 we construct an η-family of compact subsets of the phase space such that:

1) they are continuously varying with η,

2) each of them is an isolating neighbourhood with respect to the flow generated by Eqs. ($1.\eta$).

For $\eta = 0$ the system ($1.\eta$) has almost a "classical" structure and is relatively easy to analyze. Then, using the invariance of the connection index under continuation relation we can analyze existence of heteroclinics for the system (1.1). We did it in Sec. 7. For reader's convenience we have collected the necessary statements of the connection index theory taken from [1] in the Appendix A.

2. Assumptions

ASSUMPTION 1. All the considered functions are of C^2 class. □

ASSUMPTION 2. The constants k, c_1 and c_2 are positive. □

ASSUMPTION 3. In the interval $[-2\tau, 1+2\tau]$, $\tau > 0$, the equation

$$F(y,y) := F_1(y,y) + F_2(y,y) = 0 \tag{3}$$

has exactly three solutions 0, 1 and $y_0 \in (0,1)$ such that $F_{,y}(0,0) < 0$, $F_{,y}(1,1) < 0$ and $F_{,y}(y_0, y_0) > 0$. □

ASSUMPTION 4. $\displaystyle\int_0^1 F(y,y)\,dy := I > 0.$ □

3. Constant states during continuation

First of all, we will examine the behaviour of constant states for ($1.\eta$) i.e. solutions to the algebraic system:

$$\begin{aligned} \mathcal{F}_{1\eta}(u_1,u_2) - \lambda(u_1 - u_2) &= 0,\\ \mathcal{F}_{2\eta}(u_1,u_2) + \lambda(u_1 - u_2) &= 0, \end{aligned} \tag{4}$$

while the parameter η changes in the interval $[0,1]$.

REMARK. Let us note that the functions $\mathcal{F}_{1\eta}$, $\mathcal{F}_{2\eta}$ satisfy the relation

$$(\mathcal{F}_{1\eta} + \mathcal{F}_{2\eta})(y, y) = F(y, y)\gamma(\eta),$$

where $\gamma(\eta) = 2 - \eta$. □

To begin with, we will characterize the properties of the solutions to the system (4) with respect to the solutions of Eq. (3). First, it is easy to note, by means of the implicit function theorem, that for sufficiently large λ and $u_1 \in [-2\tau, 1 + 2\tau]$ the equations $\mathcal{F}_{1\eta}(u_1, u_2) - \lambda(u_1 - u_2) = 0$ and $\mathcal{F}_{2\eta}(u_1, u_2) + \lambda(u_1 - u_2) = 0$ are uniquely solvable with respect to u_2. The solutions to these equations will be denoted below respectively by $u_2 = \mu_\eta(u_1)$ and $u_2 = \vartheta_\eta(u_1)$.

Below, $\mathcal{F}_{i\eta,j}$ will denote the partial derivative of $\mathcal{F}_{i\eta}$ with respect to u_j.

LEMMA 1.

a. For all $\eta \in [0, 1]$ and sufficiently large $\lambda > 0$, the system (4) has exactly three solution pairs $(u_1, u_2)(\lambda, \eta)$ such that both u_1 and u_2 belong to the interval $[-\tau, 1 + \tau]$ and such that for $\lambda \to \infty$ they tend to appropriate solutions of Eq. (3).

b. Let V belong to the set $\{0, y_0, 1\}$ of solutions to Eq. (3) and let $F_{,y}(V) = [(F_1 + F_2)(y, y)]_{,y|y=V} > 0$ (< 0) in some open neighbourhood of V in R^1.

Let $(u_1, u_2)(\lambda, \eta)$ be this branch of solutions to (4) which tends to (V, V) as $\lambda \to \infty$. Then, in some open (in R^2) neighbourhood of this solution we have

$$[(\mathcal{F}_{1\eta} + \mathcal{F}_{2\eta})(u_1, u_2)]_{,1} + [(\mathcal{F}_{1\eta} + \mathcal{F}_{2\eta})(u_1, u_2)]_{,2} > 0 \quad (< 0).$$

Below $U_\nu(\lambda, \eta) := (U_{\nu 1}(\lambda, \eta), U_{\nu 2}(\lambda, \eta))$, $\nu \in \{, 0, +\}$, will denote the solution branch such that $U_-(\lambda, \eta) \to (0, 0)$, $U_0(\lambda, \eta)(y_0, y_0)$ and $U_+(\lambda, \eta) \to (1, 1)$ for $\lambda \to \infty$.

c. For all sufficiently large λ and all $u_1 \in [U_{-1}(\lambda, \eta), U_{+1}(\lambda, \eta)]$, we have $\mu'_\eta(u_1) > 0$, $\vartheta'_\eta(u_1) > 0$. Moreover in all sufficiently small neighbourhoods of the points $U_-(\lambda, \eta)$ and $U_+(\lambda, \eta)$ we have the inequalities, $\mu'_\eta > \vartheta'_\eta$. □

P r o o f. Adding and subtracting both sides of Eqs. (4) we obtain:

$$(4') \qquad \begin{aligned} \mathcal{F}_{1\eta} + \mathcal{F}_{2\eta} &= 0, \\ \ell(\mathcal{F}_{1\eta} - \mathcal{F}_{2\eta}) - 2(u_1 - u_2) &= 0, \end{aligned}$$

where $\ell = \lambda^{-1}$. If $\ell = 0$ and u_1 and u_2 satisfying the second equation are bounded, then they must satisfy the equality $u_2 = u_1$. Putting it into the first equation we infer, according to the definition of $\mathcal{F}_{i\eta}$, that it is equivalent to the equation $F(u_1, u_1) = 0$ i.e. to Eq. (3). Thus, for $\ell = 0$, in the rectangle $[-2\tau, 1 + 2\tau] \times [-2\tau, 1 + 2\tau]$ there are exactly three solutions equal to (V, V), $V \in \{0, y_0, 1\}$. The determinant of the Jacobian of the mapping $(R^2 \to R^2)$ determined by the left-hand sides of (4′) for $\ell = 0$ is equal to $2\gamma(\eta)F_{,y}(y, y)|_{y=V}$, $V \in \{0, y_0, 1\}$. So, according to Assumption 3 it is nonzero. Hence point a follows from the implicit function theorem. The proof of point b follows immediately from the continuity of the partial derivatives of $\mathcal{F}_{i\eta}$.

Now, differentiating both sides of Eqs. (4) with respect to u_1 we obtain the equalities:

$$\mu_\eta' = (-\mathcal{F}_{1\eta,1} + \lambda)(\mathcal{F}_{1\eta,2} + \lambda)^{-1}, \qquad \vartheta_\eta' = (\mathcal{F}_{2\eta,1} + \lambda)(-\mathcal{F}_{2\eta,2} + \lambda)^{-1}.$$

All the terms in the expression for μ_η' are taken at a point $(u_1, \mu_\eta(u_1))$, and in the expression for ϑ_η' at a point $(u_1, \vartheta_\eta(u_1))$. Suppose that $\vartheta_\eta' > \mu_\eta'$ in some neighbourhood of $U_-(\lambda, \eta)$ or $U_+(\lambda, \eta)$. Then, for λ sufficiently large, all brackets in the expressions above are positive and we would have

$$[-\mathcal{F}_{1\eta,1} + \lambda](u_1, \mu_\eta(u_1))[\mathcal{F}_{2\eta,2} + \lambda](u_1, \vartheta_\eta(u_1)) \\ -[\mathcal{F}_{2\eta,1} + \lambda](u_1, \vartheta_\eta(u_1))[\mathcal{F}_{1\eta,2} + \lambda](u_1, \mu_\eta(u_1)) < 0.$$

Sufficiently close to $U_\nu(\lambda, \eta)$ ($\nu = -$ or $\nu = +$) the difference $(\vartheta_\eta(u_1) - \mu_\eta(u_1))$ can be made arbitrarily small. Consequently, for sufficiently large λ, this would imply the inequality

$$-\lambda[(\mathcal{F}_{1\eta} + \mathcal{F}_{2\eta})_{,1} + (\mathcal{F}_{1\eta} + \mathcal{F}_{2\eta})_{,2}](U_\nu(\lambda, \eta)) < 0.$$

However, according to Assumption 3 and point b of this lemma, this would imply, that the left-hand side of the last inequality would be positive, which could not be true. This proves point c. □

LEMMA 2. Let the assumptions of Lemma 1 be fulfilled. Then for λ sufficiently large determinant of the matrix

$$\mathcal{M}(\lambda, \eta, u) := \begin{bmatrix} (\mathcal{F}_{1\eta,1}(u) - \lambda)k & (\mathcal{F}_{1\eta,2}(u) + \lambda)k \\ (\mathcal{F}_{2\eta,1}(u) + \lambda) & (\mathcal{F}_{2\eta,2}(u) - \lambda) \end{bmatrix} k^{-1} \tag{5}$$

has the sign opposite to the sign of the expression

$$[(\mathcal{F}_{1\eta} + \mathcal{F}_{2\eta})(u_1, u_2)]_{,1} + [(\mathcal{F}_{1\eta} + \mathcal{F}_{2\eta})(u_1, u_2)]_{,2}.$$

Proof. The determinant of $\mathcal{M}$ is equal to

$$(\mathcal{F}_{1\eta,1}F_{2\eta,2} - \mathcal{F}_{1\eta,2}F_{2\eta,1}) - \lambda(\mathcal{F}_{1\eta,1} + \mathcal{F}_{2\eta,2} + \mathcal{F}_{1\eta,2} + \mathcal{F}_{2\eta,1}).$$

Thus, for λ sufficiently large, we obtain the claim of this lemma. □

4. *A priori* estimates

Global properties of heteroclinic solutions.

According to Lemma 1, especially to the proof of point a of that lemma, for sufficiently large λ, the solutions $U_\nu(\lambda,\eta)$, $\nu \in \{-,0,+\}$ are isolated. To be more precise, there exists a number $\tau^* > 0$ such that in the rectangle

$$R_\eta := \{(u_1,u_2): \ u_i \in [U_{-i}(\lambda,\eta) - \tau^*, \ U_{+i}(\lambda,\eta) + \tau^*], \quad i = 1,2\}$$

there are no other solutions to the system (4).

LEMMA 3. For suffciently large λ there exists a constant $L^* < \infty$ independent of the values of θ, $\eta \in [0,1]$ and λ, such that for all bounded solutions to $(1.\eta)$, for which u_1 and u_2 stay in the rectangle R_η for all ξ we have the estimate:

$$\|u_1 - u_2\|_{C^0} < (\lambda)^{-1} L^*. \quad \square$$

P r o o f. First, suppose that the function $d(\xi) = u_1(\xi) - u_2(\xi)$ achieves a positive maximum (negative minimum) for some $\xi = \zeta \in (-\infty,\infty)$ and that $(u_1(\zeta), u_2(\zeta))$ lies in R_η. Then at this point $d' = 0$ and $d'' \le 0$ (≥ 0). Hence, due to (2),

$$|d| \le \eta k^{-1}(|kF_1 - F_2| + |(c_2 - c_1 k)\theta u_2'|)(1 + k^{-1})^{-1}\lambda^{-1}. \tag{6}$$

As we assume that the solution is bounded (for all times) and $u_2'(\xi) = 0$ for $|\xi| = \infty$, then $\theta u_2'$ must attain the global maximum somewhere. As $d' = 0$ implies $u_1' = u_2' = w'$ at the point of extremum, then by means of the first equation in (2), we can find an upper bound for $|(c_2 - c_1 k)\theta u_2'|$. It is not greater than

$$|(c_2 - c_1 k)((2 - 2\eta)F_s + \eta F_1 + \eta F_2)(c_1 + c_2\eta + c_1 k - c_1\eta k)^{-1}|.$$

After some computations one can prove that the right-hand side of (6) is not greater than $\max\{\eta, (1 + c_1 c_2^{-1}\eta^{-1})^{-1}\} \max\limits_u (2|F_1(u)| + 2|F_2(u)| + 2|F_s(u)|)\lambda^{-1}$, where the maximum is taken over R_η. This expression has a common bound independent of η. $\square$

By means of Lemma 3 we can prove:

LEMMA 4. For sufficiently large λ there exists a number $L < \infty$ independent of $\eta \in [0,1]$, $\theta \in (-\infty,\infty)$ and λ, such that for all solutions to $(1.\eta)$ which (for all ξ) stay in the rectangle R_η the estimates $|u_1'| < L$, $|u_2'| < L$ hold. $\square$

P r o o f. Let us consider an arbitrary solution satisfying the above conditions. Then there is M such that $|\mathcal{F}_{i\eta} + (-1)^i\lambda(u_1 - u_2)| < M$ for $i = 1,2$ and $(u_1,u_2) \in R_\eta$. Let us note, that due to Lemma 3, $|\lambda(u_1 - u_2)| < L^*$, so M *can also be treated as independent of* λ. First, let us examine the case: $(c_1\theta) \ge 1$. Suppose that for some solution $|u_1'|$ attains a value larger than M. Then u_1' and u_1'' have the same sign and this property is retained for all positive times. Consequently, this solution will grow exponentially, contrary to the boundedness of the solution. Hence, $|u_1'| \le M$. If $c_1\theta \le -1$, then changing the direction of "time" we arrive at the equation of the form: $u_1'' = c_1\theta u_1' - [\mathcal{F}_{1\eta} - \lambda(u_1 - u_2)]$, thus for all positive times u_1' and u_1'' have the same sign (as before) and the solution will grow exponentially. So, $|u_1'| \le M$. In the same way we can prove that $|u_2'| \le M$, if $|c_{2\eta}\theta| \ge 1$. Now, let us analyze the case $|c_1\theta| \le 1$. Then for sufficiently large $|u_1'|$, $|u_1''| < |u_1'| + M$ (Remember that $|\mathcal{F}_{i\eta} + (-1)^i\lambda(u_1 - u_2)| < M$). Thus, if $u_1'(0) = L_1 > 0$, then for $\xi > 0$, we would have $u_1'(\xi) > \exp(-\xi)[L_1 - M(\exp(\xi) - 1)]$ independently of the sign of $(c_1\theta)$. Integrating this inequality with respect to ξ over the interval $(0, 1)$ we obtain that $|u_1(1) - u_1(0)| > L_1(1 - e^{-1}) - M$. For L_1 sufficiently large, the right-hand side of this inequality is strictly larger than $(U_{+1}(\lambda, \eta) - U_{-1}(\lambda, \eta) + 2\tau^*)$, which is impossible due to the fact that the solution must lie in R_η. If $L_1 < 0$, the proof is carried out in the same way. Likewise the inequality $|c_{2\eta}\theta| \le 1$ implies the inequality $|u_2'| < L_2$. Consequently $|u_1'| < L$, $|u_2'| < L$, where $L = \max\{M, L_1, L_2\}$. □

Let $\Gamma = (c_2k^{-1} - c_1)$, $\Gamma_\eta = \eta|\Gamma|$. If $\Gamma \ge 0$, let $m = 2$, $\chi(\eta) = c_1$ and $s = k(1 + k)^{-1}$. If $\Gamma < 0$, then let $m = 1$, $\chi(\eta) = c_{2\eta}k^{-1}$ and $s = (1 + k)^{-1}$. Now, the first equation in (2) can be written as:

$$w'' - \chi_\eta\theta w' - \theta\Gamma_\eta s u_m' + (1 + k)^{-1}\{\mathcal{F}_{1\eta} + \mathcal{F}_{2\eta}\} = 0. \tag{7}$$

The next lemma estimates the "possible" values of θ.

LEMMA 5. There exists $\lambda_0 \in (0, \infty)$ such that for all $\lambda > \lambda_0$, $\eta \in [0, 1]$ the value of θ, for which a heteroclinic solution (with nonnegative derivatives) to system $(1.\eta)$ connecting the points $U_-(\lambda, \eta)$ and $U_+(\lambda, \eta)$ can exist, is positive and bounded uniformly from above and below i.e $\theta \in (\theta_0, \theta_1)$ with $0 < \theta_0 < \theta_1$. □

P r o o f. Suppose that, for some $\theta = \theta(\eta)$, $(u_1(\xi), u_2(\xi))$ satisfies $(1.\eta)$. Then there is no open interval (comprised in $(-\infty, \infty)$) such for $i = 1$ or $i = 2$, $u_i'(\xi) \equiv 0$ for ξ from this interval. For, then $u_i'' \equiv 0$, $u_i = \text{const}$ and $\mathcal{F}_{i\eta} - \lambda(u_1 - u_2)(-1)^{i-1}$ would be equal to 0 in this interval. Due to Lemma 1.c (for λ sufficiently large) the slope of the curve $(\mathcal{F}_{i\eta} - \lambda(u_1 - u_2)(-1)^{i-1}) = 0$ is positive and finite, so this would imply that $u_j = \text{const}$ also, where j is the index complementary to i. Consequently this would be a singular point. But this cannot happen for $|\xi| < \infty$.

Multiplying Eq. (7) by w' and integrating with respect to ξ from $(-\infty)$ to (∞) we obtain:

$$(\chi_\eta + \Gamma_\eta)\theta(\eta) \int_{-\infty}^{\infty} w'^2(\xi)\, d\xi - (1 + k)^{-1} \int_{-\infty}^{\infty} (\mathcal{F}_{1\eta} + \mathcal{F}_{2\eta})(u_1' d\xi + k u_2' d\xi) \ge 0.$$

We claim that for λ sufficiently large, the second term at the left-hand side is, independently of $\eta \in [0, 1]$, positive, say, larger than $8^{-1}I$ (Assumption 4). To prove this, let us consider for example the integral $\int_{-\infty}^{\infty} (\mathcal{F}_{1\eta} + \mathcal{F}_{2\eta})u_1' d\xi$. It is equal to $\int\{\mathcal{F}_{1\eta}(u_1, u_2(u_1)) + \mathcal{F}_{2\eta}(u_1, u_2(u_1))\}du_1$, where $u_2(u_1) = u_2(\xi(u_1))$, $\xi(u_1)$ is the inverse of the function $u_1(\xi)$ and the integration is made over the interval $[U_{-1}(\lambda, \eta), U_{+1}(\lambda, \eta)]$. Now, using point a of Lemma 1 (and its proof) and Lemma 3 we infer, by means of Taylor expansion with respect to $(u_2(u_1) - u_1)$, that this integral is larger than $(\gamma(\eta)\int_0^1 F(u_1)\,du_1 - C\lambda^{-1})$, where C is a constant independent of λ and η. The second integral can be estimated in the same way, so, we infer that the claim is true. The integral $\int_{-\infty}^{\infty} w'^2(\xi)\,d\xi$ can be written as the integral $\int w'(w)\,dw$ over the interval, which is bounded for every η. As, due to Lemma 4, $|w'(\xi)| < L$ independently of $\xi \in (-\infty, \infty)$ and $\eta \in [0, 1]$, and $(\chi_\eta + \Gamma_\eta) > 0$, then (for all $\eta \in [0, 1]$) we infer that $\theta(\eta) > 0$. Hence $\inf_\eta(\theta(\eta)) > \theta_0 > 0$.

To find an upper bound for $\theta(\eta)$ let us integrate Eq. (7) from $(-\infty)$ to ξ using the fact that $w' \geq 0$. We obtain:

$$2^{-1}w'^2(\xi) \geq \chi\theta(\eta)\int_{-\infty}^{\xi} w'^2(\xi)\,d\xi - (1+k)^{-1}\int_{-\infty}^{\xi}(\mathcal{F}_{1\eta} + \mathcal{F}_{2\eta})(u_1'\,d\xi + ku_2'\,d\xi).$$

Now, as before, one can easily prove that for λ sufficiently large and all $\eta \in [0, 1]$ there exists $\zeta \in (-\infty, \infty)$ such that for $\xi = \zeta$ the last term of the above inequality (respecting the sign) is positive, say, larger than $(-8^{-1}J)$, where $J = \min_y \int_0^y F(s)\,ds$, where minimum is taken over the interval $[0, 1]$. J is negative due to Assumption 3. Consequently, there is a point on the trajectory, where $2^{-1}w'^2 \geq -8^{-1}J$. Thus, at the point of maximum of w' it follows from (7) that $\theta(\eta) \leq \sup(\mathcal{F}_{1\eta}(u_1, u_2) + \mathcal{F}_{2\eta}(u_1, u_2))2[\chi_\eta(k+1)\sqrt{J}]^{-1}$, where the supremum is taken over $u \in R_\eta$. The right-hand side of this inequality is bounded from above by a number independent of η, let us denote it by θ_1. Thus, we obtain the claim of the lemma. □

5. Eigenvalues of the linearized system

Below z_1 *and* z_2 *will be variables standing for* u_1' *and* u_2' *and* $z := (z_1, z_2)$. Equations $(1.\eta)$ may be written as the first order system:

$$(u_1', u_2', z_1', z_2') = \Big(z_1, z_2, c_1\theta z_1 - \mathcal{F}_{1\eta} + \lambda(u_1 - u_2),$$
$$k^{-1}\{c_{2\eta}\theta z_2 - \mathcal{F}_{2\eta} - \lambda(u_1 - u_2)\}\Big).$$

It is seen that the zeros of the right-hand side have the following form:

$$(u_1, u_2, z_1, z_2) = (U(\lambda, \eta), 0, 0),$$

where $U(\lambda, \eta)$ is a solution to system (4). Thus, for (u_1, u_2) in the rectangle $[-\tau, 1+\tau] \times [-\tau, 1+\tau]$ and sufficiently large λ we have exactly three zeros: $(U_\nu(\lambda, \eta), 0, 0)$, $\nu \in \{-, 0, +\}$. Linearizing $(1.\eta)$ around $(U_\nu(\lambda, \eta), 0, 0)$, $\nu \in \{-, 0, +\}$, we obtain the system:

$$(8) \qquad \begin{bmatrix} u_1' \\ u_2' \\ z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \multicolumn{2}{c}{\multirow{2}{*}{$-\mathcal{M}_\nu(\lambda, \eta)$}} & c_1\theta & 0 \\ & & 0 & c_{2_\eta}\theta k^{-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ z_1 \\ z_2 \end{bmatrix} := M_\nu(\lambda, \eta) \begin{bmatrix} u_1 \\ u_2 \\ z_1 \\ z_2 \end{bmatrix},$$

where $\mathcal{M}_\nu(\lambda, \eta) := \mathcal{M}(\lambda, \eta, U_\nu(\lambda, \eta))$ and $\mathcal{M}(\lambda, \eta, u)$ is defined in Lemma 2 by (5).

LEMMA 6. Let Assumptions 2 and 3 be fulfilled. Then, for all sufficiently large λ, $\theta \in (0, \infty)$ and all $\eta \in [0, 1]$, the matrix $M_\nu(\lambda, \eta)$, $\nu \in \{-, +\}$, has four real eigenvalues. Two of them are positive and two of them are negative. □

The proof of this lemma will be sketched in Appendix B. It is easy to note that the eigenvector corresponding to the eigenvalue q of the matrix $M_\nu(\lambda, \eta)$, $\nu \in \{-, 0, +\}$ has the form $(\xi_1, \xi_2, \zeta_1, \zeta_2)$, where $\zeta_i = q\xi_i$, $i = 1, 2$. (See [1] p. 335.)

LEMMA 7. Let Assumptions 2 and 3 be fulfilled. Then, for all sufficiently large λ, $\theta \in (0, \infty)$ and all $\eta \in [0, 1]$, the matrix $M_0(\lambda, \eta)$ has one negative, one positive eigenvalue and two complex conjugate eigenvalues with positive real parts. The components ξ_1, ξ_2 of the eigenvector corresponding to the negative eigenvalue satisfy the condition $\xi_2\xi_1^{-1} < 0$. □

The proof of this Lemma 7 will be given in Appendix B.

6. Isolating neighbourhood during continuation

In this section we construct an η-family of compact subsets $N_3(\eta)$, such that for each $\eta \in [0, 1] N_3(\eta)$ is an isolating neighbourhood for the flow generated by the first order system corresponding to $(1.\eta)$. Every $N_3(\eta)$ consists of a parallelepiped

$$N_1 = \Big\{(u_1, u_2, z_1, z_2) : \ U_{-i}(\lambda, \eta) \le u_i \le U_{+i}(\lambda, \eta), \ 0 \le z_i \le L, \ i = 1, 2\Big\},$$

plus "small" neighbourhoods of the singular points, which we want to connect $(N(\eta))$, minus a small neighbourhood of the remaining singular point $N_3(0, \kappa, \eta)$. This point can be excised according to Lemma 7 and the Lemma in 4.D in [1].

Let us denote:

$$\mathcal{F}_{1\eta}(u_1, u_2) - \lambda(u_1 - u_2) := \mathcal{H}_1(\lambda, \eta, u_1, u_2),$$
$$\mathcal{F}_{2\eta}(u_1, u_2) + \lambda(u_1 - u_2) := \mathcal{H}_2(\lambda, \eta, u_1, u_2).$$

As the proofs carried out below are the same for all $\eta \in [0, 1]$ and all sufficiently large λ, then, *to simplify notation, when there will be no danger of confusion, we will write* $\mathcal{H}_i(\xi)$ *instead of* $\mathcal{H}_i(\lambda, \eta, u_1(\xi), u_2(\xi))$.

According to Lemma 1 for fixed η and λ (sufficiently large), the zero sets of $\mathcal{H}_1$ and $\mathcal{H}_2$ near the points $U_-(\lambda, \eta) := (U_{-1}, U_{-2})(\lambda, \eta)$ and $U_+(\lambda, \eta) := (U_{+1}, U_{+2})(\lambda, \eta)$ have the graph like that in Fig. 1.

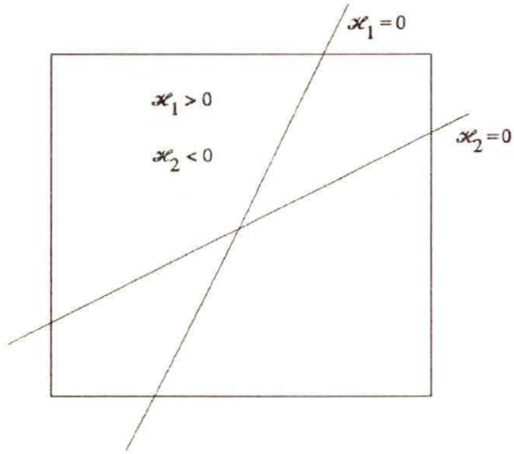

FIG. 1.

LEMMA 8. There exist smooth functions $\varepsilon_{2\nu}(\eta)$, such that for $\eta \in [0, 1]$, $\nu \in \{-, +\}$, $\delta \in (0, 1]$ and all sufficiently small $\Delta_1 > 0$, the set

$$N(\delta, \eta, \nu, \Delta_1) := \Big\{(u, z) : |u_1 - U_{\nu 1}(\lambda, \eta)| \le \delta\Delta_1,\ |u_2 - U_{\nu 2}(\lambda, \eta)| \le \delta\Delta_{2\nu}(\eta),$$
$$|z_i| < L,\ i = 1, 2\Big\},$$

where $\Delta_{2\nu}(\eta) = \Delta_1\varepsilon_{2\nu}(\eta)$ is an isolating neighbourhood. □

By means of Lemma 9 it may be proved that $(U_\nu, 0, 0)$ is the maximal invariant set in $N(\delta, \eta, \nu, \Delta_1)$, but we do not use this fact explicitly belowe.

Proof of Lemma 8. According to point c of Lemma 1 we have $\mu'_\eta(u_1) > \vartheta'_\eta(u_1)$ for $u_1 \in \{U_{-1}(\lambda, \eta), U_{+1}(\lambda, \eta)\}$. Thus, for all sufficiently small $\Delta_1 > 0$, we can find a smooth function $\varepsilon_{2\nu}(\eta)$ such that, if $\Delta_{2\nu}(\eta) = \varepsilon_{2\nu}(\eta)\Delta_1$, then the curve $\mathcal{H}_1 = 0$ intersects the upper and lower side of the rectangle $|u_1 - U_{\nu 1}| \le \delta\Delta_1$, $|u_2 - U_{\nu 2}| \le \delta\Delta_{2\nu}(\eta)$ and the curve $\mathcal{H}_2 = 0$ intersects the right and left side of this rectangle. $\mathcal{H}_2 < 0$ (> 0) at the upper (lower) side and $\mathcal{H}_1 < 0$ (> 0) at its right (left) side. Bounded solutions of the considered family of equations (according to Lemma 4) cannot touch the sets $|z_i| = L$. They can only touch the boundary of this neighbourhood at points whose projection onto the (u_1, u_2)-plane are contained in the sides of the considered rectangle (for fixed δ). However, this is impossible. Suppose for example, that u_2 has a maximum at the upper side of this rectangle. Then $z_2(\zeta) = 0$ and $z_2'(\zeta) \leq 0$ for some $\zeta \in (-\infty, \infty)$. Simultaneously $z_2'(\zeta) = -\mathcal{H}_2(\zeta)$. This contradicts the fact that $\mathcal{H}_2 < 0$ at that side. The remaining cases may be analyzed similarly. □

Let

$$N_1(\eta) := \Big\{(u_1, u_2, z_1, z_2):\ U_{-i}(\lambda, \eta) \leq u_i \leq U_{+i}(\lambda, \eta),\ \ 0 \leq z_i \leq L,\ \ i = 1, 2\Big\},$$

where L is the number appearing in Lemma 4. Then, let:

$$N(\delta, \eta) := N(\delta, \eta, -, \Delta_1)) \cup N(\delta, \eta, +, \Delta_1),$$

where Δ_1 *is fixed* and so small that $N(\delta, \eta, \nu, \Delta_1)$, $\nu \in \{-, +\}$, is an isolating neighbourhood of the point $(U_\nu(\lambda, \eta), 0, 0)$ and

$$N_3(0, \kappa, \eta) := \Big\{(u_1, u_2, z_1, z_2):\ \ |u_1 - U_{01}(\lambda, \eta)| + |z_1| < \kappa,$$
$$|u_2 - U_{02}(\lambda, \eta)| + |z_2| < \kappa\Big\},$$

where κ is a sufficiently small positive number. Finally, let

$$N_2(\delta, \eta) := N_1(\eta) \cup N(\delta, \eta), \qquad N_2(\eta) := N_2(1, \eta),$$
$$N_3(\delta, \eta) := N_2(\delta, \eta) \backslash N_3(0, \kappa, \eta), \qquad N_3(\eta) := N_3(1, \eta).$$

For any compact set Z comprised in the phase space $\mathcal{S}(Z)$ will denote the maximal invariant set comprised in Z.

LEMMA 9. Suppose that Assumptions 1–4 are fulfilled. Then, for sufficiently small $\kappa > 0$ and all $\eta \in [0, 1]$, the set $N_3(\eta)$ is an isolating neighbourhood for the flow determined by $(1.\eta)$. Furthermore, we have:

1. $\mathcal{S}(N_2(\eta)) = \mathcal{S}(N_1(\eta))$.
2. For any $\theta \in [\theta_0, \theta_1]$

$$\mathcal{S}(N_3(\eta)) = \{\text{singular points} \cup \text{(perhaps) connecting trajectories}\}. \quad \square$$

P r o o f. First, let us note that the following lemma is valid:

LEMMA 10. $\mathcal{S}(N_1(\eta)) \cap \partial N_1(\eta)$ consists only of singular points belonging to N_1. □

The proof of this lemma is given in Appendix C.

Let $N_2(0,\eta) := \cap\{N_2(\delta,\eta) : \delta \in (0,1]\}$. Arguing as in the proof of Lemma Sec. 4. in [1] let us note that

$$N_2(0,\eta)\backslash N_1(\eta) = \Big\{(u_1,u_2,z_1,z_2): \quad (u_1,u_2) = U_\nu(\lambda,\eta),$$
$$\nu \in \{-,+\}, \quad |z_1|, |z_2| < L, \quad z_1 < 0 \ \text{ or } \ z_2 < 0\Big\}.$$

Trajectories through the points belonging to this set leave the set $N_2(0)$ in appropriate direction. So, if $\mathcal{S}(N_2(\delta^*,\eta)) \neq \mathcal{S}(N_1(\eta))$ for some $\delta^* \in (0,1]$, then there must exist $\delta \in (0,\delta^*]$ such that $\mathcal{S}(N_2(\delta,\eta)) \cap \partial N_2(\delta,\eta)$ comprises a point not belonging to $\mathcal{S}(N_1(\eta))$. The set $\partial N_2(\delta,\eta)$ may be divided into three parts: $\partial N(\delta,\eta)\backslash N_1(\eta)$, $\partial N_1(\eta)\backslash N(\delta,\eta)$ and $\partial N_1(\eta) \cap \partial N(\delta,\eta)$. In Lemma 8 and the first part of proof of Lemma 10 (Appendix C) we showed that the intersection of $\mathcal{S}(N_2(\delta,\eta))$ with $\partial N(\delta,\eta)\backslash N_1(\eta)$ and $\partial N_1(\eta)\backslash N(\delta,\eta)$ is eiter empty or belongs to N_1. Thus, it suffices to show the following statement:

LEMMA 11. $\mathcal{S}(N_2(\delta,\eta)) \cap \partial N_1(\eta) \cap \partial N(\delta,\eta) = \emptyset$ for all $\delta \in (0,1]$. □

The proof of this lemma is given in Appendix C.

In view of this lemma, point 1 of Lemma 9 is proved.

Now, due to Assumption 3, Lemma 2 and Lemma 7 we infer that the set of points on trajectories comprised in $\mathcal{S}(N_2(\delta,\eta)) = \mathcal{S}(N_1(\eta))$ tending to the point $U_0(\lambda,\eta)$ as $\xi \to \infty$ is empty. Thus, according to Lemma in Sec. 4.D, for κ sufficiently small $N_3(0,\kappa,\eta)$ can be excised from $N_2(\delta,\eta)$, $\delta \in (0,1]$, $\eta \in [0,1]$. It follows that $N_3(\eta)$ is a good isolating neighbourhood. Point 2 of Lemma 9 follows straightforwardly from the definition of $N_1(\eta)$. □

7. Connection index for $\eta = 0$ and existence proof

Now, for $\eta \in [0,1]$, let

$$S'_\eta := (U_-(\lambda,\eta),0,0) \times [\theta_0,\theta_1], \qquad S''_\eta := (U_+(\lambda,\eta),0,0) \times [\theta_0,\theta_1].$$

Let S_η denote the maximal invariant set in the set $N_3(\eta)\times[\theta_0,\theta_1]$ with respect to the flow generated by (1.η) together with the equation $\theta' = 0$. Due to the results of the above sections, the connection triples (S'_0, S''_0, S_0) and (S'_1, S''_1, S_1) are related by continuation. By Theorem in Sec. 2.D of [1] these triples have the same (homotopic) connection indices. According to the definition (see [1] and the Appendix A), the connection index of the triple (S'_0, S''_0, S_0) is the Conley index of $N_3(0) \times [\theta_0,\theta_1]$ with respect to the flow generated by Eqs. (1.0) (by which we mean (1.η) with $\eta = 0$) written as a first order system, i.e. the system:

$$\begin{aligned} u_1' &= z_1\,, \\ u_2' &= z_2\,, \\ z_1' - c_1\theta z_1 + F_s - \lambda(u_1 - u_2) &= 0, \\ kz_2' - c_1 k\theta z_2 + F_s + \lambda(u_1 - u_2) &= 0, \end{aligned}$$

together with the equation

$$\theta' = \beta\phi(u, u')(\theta - 2^{-1}(\theta_0 + \theta_1)),$$

where β is a sufficiently small positive parameter. Let U' and U'' denote open neighbourhoods in $R^4 \times (\theta_0 - \varepsilon, \theta_1 + \varepsilon)$ of $S'(\theta_0) \cup S'(\theta_1)$ and $S''(\theta_0) \cup S''(\theta_1)$, respectively, having disjoint closures. The real-valued continuous function ϕ is arbitrary except for the fact that it is positive in U' and negative in U'' (see Definition A.4 of the Appendix A).

To analyze the connection index for the above system it is convenient to change the dependent variables, namely to consider the system:

(9.a) $$w' = z_w', \qquad z_w' - c_1\theta z_w + 2(1+k)^{-1}F_s(w) = 0;$$

(9.b) $$\Delta' = z_\Delta, \qquad z_\Delta' - c_1\theta z_\Delta - (1+k^{-1})\lambda\Delta = 0;$$

(9.c) $$\theta' = \beta\phi_\Delta\left(\theta - 2^{-1}(\theta_0 + \theta_1)\right),$$

where

$$(w, \Delta, z_w, z_\Delta) = \left[(1+k)^{-1}(u_1 + ku_2),\ u_1 - u_2,\ (1+k)^{-1}(z_1 + kz_2),\ z_1 - z_2\right]$$

and

$$\phi_\Delta(w, \Delta, w', \Delta') = \phi(u(w, \Delta), u'(z_w, z_\Delta)).$$

The transformation $(u_1, u_2, z_1, z_2, \theta) \to (w, \Delta, z_w, z_\Delta, \theta)$ is a linear homeomorphism which transforms $N_3(0) \times [\theta_0, \theta_1]$ to a compact set; let us write it as $N_{w\Delta} \times [\theta_0, \theta_1]$. The set of exit points are transformed into the set of exit points, so the invariant set comprised in $N_{w\Delta} \times [\theta_0, \theta_1]$ has the same Conley index as the invariant set contained in $N_3(0) \times [\theta_0, \theta_1]$. Let us denote:

$$P := \{(w, \Delta, z_w, z_\Delta) : \quad \Delta = 0, \quad z_\Delta = 0\}.$$

LEMMA 12. For $\lambda > 0$ the set $\mathcal{S}(N_{w\Delta} \times [\theta_0, \theta_1])$ is comprised in the set $P \times [\theta_0, \theta_1]$ □

P r o o f. For any finite values of θ all nonconstant trajectories of solutions to system (9.b) lie either on stable or unstable manifold of the singular point $(0, 0, 0, 0)$, so it leaves $N_{w\Delta}$ in positive or negative "time" direction. □

So, in variables $(w, \Delta, z_w, z_\Delta, \theta)$ the sets S_0' and S_0'' are contained in $P \times [\theta_0, \theta_1]$. Moreover, without losing generality we may assume that the function $\phi_\Delta(w, \Delta, z_w, z_\Delta)$ is constant with respect to (Δ, z_Δ) in some open neighbourhood of the plane P. It is clear that the set $[(\bigcup_{x \in N_{w\Delta} \cap P} Z_{x\varepsilon}) \cap N_{w\Delta}] \times [\theta_0, \theta_1]$, where $Z_{x\varepsilon}$ is the set of points of a plane perpendicular to P at x (in $(w, \Delta, z_w, z_\Delta)$-space) and whose distance from P is not greater than ε, is a good isolating neighbourhood of the considered invariant set. Moreover, according to the robustness of the Conley index theory for sufficiently small (positive) ε, the set

$$(\bigcup_{x \in \mathcal{J}} Z_{x\varepsilon}) \times [\theta_0, \theta_1],$$

where $\mathcal{J}$ is the subset of $N_{w\Delta} \cap P$, such that $Z_{x\varepsilon}$ is completely comprised in $N_{w\Delta}$ for $x \in \mathcal{J}$, retains this property *only if ε is taken sufficiently small.*

Obviously this set can be written as a Cartesian product

$$N^* := N_w \times N_{\Delta\varepsilon} \times [\theta_0, \theta_1] := N_w \times \{(\Delta, z_\Delta) : \ \operatorname{dist}[(\Delta, z_\Delta), (0,0)] \le \varepsilon\} \times [\theta_0, \theta_1],$$

where N_w is equivalent to $\mathcal{J}$ (defined in terms of w and z_w). Note, that the system consisting of (9.a) and (9.c) does not depend, for sufficiently small ε, on Δ. According to this fact the trajectories belonging to the invariant set do not change, if the second equation in (9.b) is replaced by any of the family of equations $z_\Delta' - \alpha c_1 \theta z_\Delta - (1 + k^{-1})\lambda\Delta = 0$, where $\alpha \in [0, 1]$. Thus, for all $\alpha \in [0, 1]$, the set N^* is a good isolating neighbourhood and we can replace (9.b) by

$$\Delta' = z_\Delta, \qquad z_\Delta' - (1 + k^{-1})\lambda\Delta = 0. \tag{10}$$

In this way the system for (Δ, z_Δ) is completely decoupled from the rest of the system as the equations of (9.a) and (9.c) do not depend on (Δ, z_Δ). Due to the known properties, the Conley index of N^* is homotopic to $h_\Delta \wedge h_{w\theta}$, where h_Δ is the Conley index of $N_{\Delta\varepsilon}$ with respect to (10) and $h_{w\theta}$ is the Conley index of $N_w \times [\theta_0, \theta_1]$ with respect to the flow generated by (9.a), (9.c).

Now, according to Assumptions 2–4 there exists $\theta^s \in (\theta_0, \theta_1)$ such that (9.a) has a heteroclinic solution connecting the points $(w, z_w) = (0, 0)$ and $(w, z_w) = (1, 0)$. Let T_1 denote the trajectory of (9.a) for $\theta = \theta^s$, crossing the z_w axis at a point, say $(0, 1)$. Let $\theta_c \in (0, \theta_0)$ be so small that for $\theta = \theta_c$ the eigenvalues of the linearization matrix of the sytem (9.a) at $(w_0, 0)$ are complex conjugate (and have positive real part). Let T_2 denote a (connected) segment of the spiral trajectory of (9.a) with $\theta = \theta_c$ which lies in the halfplane $z \ge 0$ sufficiently close to $(w_0, 0)$. One can see that (without changing the Conley index) $N_w \times [\theta_0, \theta_1]$ can be deformed to the region bounded by T_1, T_2, the lines $w = -\omega$, $w = 1 + \omega$, $\omega > 0$ small, the boundaries of small diamonds consisting of the points $(1, 0)$ and $(0, 0)$ and the line $z_w = 0$ as it is done in [1]. (During the deformation the invariant trajectory, if it exists, does not touch the boundary of the deformed region). Thus the Conley index of $N_w \times [\theta_0, \theta_1]$ with respect to the flow generated by (9.a), (9.c) can be computed to be homotopic to $\overline{0}$. Consequently, the connection index of N^* i.e. $h_\Delta \wedge \overline{0} \cong \Sigma^1 \wedge \overline{0} \cong \overline{0}$. On the other hand, according to the results of Sec. 5, for any $\eta \in [0,1]$, the singular points $(U_+(\lambda,\eta),0,0)$ and $(U_-(\lambda,\eta),0,0)$ are isolated invariant sets and the Conley index of them is homotopic to Σ^2. As $(\Sigma^1 \wedge \Sigma^2) \vee \Sigma^2 = \Sigma^3 \vee \Sigma^2$ is not in the homotopy class of $\overline{0}$, then according to Theorem in Ses. 2.F of [1], it follows that $S_1' \cup S_1' \neq S_1$. Consequently in view of Lemma 9 point 2 we infer (by letting $\beta \to 0$) that the following theorem is true:

THEOREM. *Let Assumptions 1–4 be satisfied. Then there exists $\theta^* \in [\theta_0, \theta_1]$ such that for $\theta = \theta^*$ and all sufficiently large $\lambda > 0$, there exists a heteroclinic solution to system* (1.1) *connecting the constant states $U_-(\lambda,1)$ and $U_+(\lambda,1)$.* □

8. Discussion

It is possible to estimate the minimal value of λ which is sufficient to prove existence of a heteroclinic solution to the system (1.1), which was rather impossible in the method chosen in [5]. An example of such an estimation will be given below. It is worthwhile to note, that this value of λ depends only on the functions F_i and their first derivatives. Especially, as one could foresee, this value does not depend on the other coefficients i.e. k, c_1, c_2. Finally, let us stress that from the mathematical point of view Assumption 4 is not necessary. This condition, which reflects the physical situation described by the system (0), was assumed only for definiteness.

To see, how the minimal value of λ can be estimated, let us take for example a quite realistic situation, when $F_2 \equiv 0$ and $F_1 = F(u_1)$ (which corresponds to the assumption that the energy is gained and radiated out only by the electron component).

LEMMA 13. For $F_2 \equiv 0$ the solutions to system (4) are independent of η. □

P r o o f. The system (4) takes the form:

$$-\lambda\Delta + \eta F(u_1) + (1-\eta)F_s(w) = 0,$$
$$\lambda\Delta + (1-\eta)F_s(w) = 0,$$

where $\Delta = (u_1 - u_2)$ and $w = (u_1 + ku_2)(1+k)^{-1}$. We have $F(u_1) = F_s(w) + F'(w^*)k(k+1)^{-1}\Delta$, where $w^* \in [u_1, w]$. Suppose that, for a fixed $\eta \in [0,1]$, this system has a solution (u_1, u_2), for which $\Delta \neq 0$. Multiplying the second equation by $(1-\eta F'(w^*)k(k+1)^{-1}\lambda^{-1})$ and adding it to the first one we obtain an equation $[1 + (1-\eta)(1-\eta F'(w^*)k(k+1)^{-1}\lambda^{-1}]F_s = 0$. Consequently, for λ sufficiently large $F_s = 0$, and from the second equation we infer that $\Delta = 0$. □

To find the estimation we will verify in turn all the conditions imposed on λ in the text above.

First, the positiveness of the determinant at U_ν, $\nu \in \{,-,+\}$, its negativeness at U_0 and the conditions $(-\lambda(u_1-u_2)+\eta F+(1-\eta)F_s)_{,2}>0$, $(\lambda(u_1-u_2)+k(1-\eta)F_s)_{,1}>0$ (which were necessary in the proof of Lemmas 7 and 10) are guaranteed by $\lambda > \widehat{F} := \max\limits_{u\in[0,1]} |F'(u)|$.

Now, according to the proof of Lemma 3, $|u_1-u_2|$ is *a priori* smaller than

$$d = 6\lambda^{-1}(\max_{u\in[0,1]} |F(u)|) := 6\lambda^{-1}F_m\,.$$

Let

$$\int\limits_0^1 F(u)\,du = I > 0, \qquad \min_{u\in[0,1]}\int\limits_0^u F(u)\,du = J = \int\limits_0^j F(u)\,du < 0.$$

We have

$$\begin{aligned}\mathcal{I} := (1+k)^{-1}\int\limits_{-\infty}^{\infty}(\mathcal{F}_{1\eta}+\mathcal{F}_{2\eta})(u_1'\,d\xi + ku_2'\,d\xi)\\ = (1+k)^{-1}\int\limits_{-\infty}^{\infty}[\eta F(u_1)+2(1-\eta)F_s(w)]\,(u_1'\,d\xi+ku_2'\,d\xi)\\ = 2(1-\eta)\int\limits_0^1 F(w)\,dw + (1+k)^{-1}\eta\int\limits_0^1 F(u_1)\,du_1\\ +(1+k)^{-1}\eta\left\{\int\limits_{-\infty}^{\infty}F(u_2)ku_2'\,d\xi + \int\limits_{-\infty}^{\infty}F'(u^*)dku_2'\,d\xi\right\},\end{aligned}$$

where $u^*(\xi)\in[u_1(\xi),u_2(\xi)]$.

The sum of the first three terms is equal to $[2-\eta]I$ and the module of last term is estimated by the number $\widehat{F}dk = 6\lambda^{-1}\widehat{F}F_m$. Thus, for $\lambda > 6\widehat{F}F_mI^{-1}$ the integral $\mathcal{I}$ is larger than 0. Likewise, we can *a priori* estimate the minimum over ξ of the integral

$$\mathcal{J}(\xi) := (1+k)^{-1}\int\limits_{-\infty}^{\xi}(\mathcal{F}_{1\eta}+\mathcal{F}_{2\eta})(u_1'\,d\xi + ku_2'\,d\xi).$$

So, acting as before we can write $\mathcal{J}$ as:

$$\begin{aligned}2(1-\eta))\int\limits_0^{w(\xi)}F(w)\,dw + (1+k)^{-1}\eta\int\limits_0^{u_1(\xi)}F(s)\,ds\\ +(1+k)^{-1}\eta\left\{\int\limits_0^{u_2(\xi)}F(u_2)ku_2'\,d\xi + \int\limits_0^{u_2(\xi)}F'(u^*)\,[u_2(\xi)-u_1(\xi)]\,ku_2'\,d\xi\right\},\end{aligned}$$

where $u^*(\xi)\in[u_1(\xi),u_2(\xi)]$.

Let us choose ξ in such a way that $u_1(\xi) = j$. Then, $\mathcal{J}$ can be estimated from above by:

$$2(1-\eta)J+\eta(1+k)^{-1}(1+k)J+2(1-\eta)dF_m+\eta(1+k)^{-1}k\left\{F_m d + \widehat{F}d(j+d)\right\}$$
$$= [2-\eta]J + [2(1-\eta)+\eta(1+k)^{-1}k]F_m d + \widehat{F}dk\eta(1+k)^{-1}(j+d).$$

As the trajectory must stay in the rectangle $[0,1]\times[0,1]$, then $(j+d)\le 1$ and $\mathcal{J}$ is smaller than zero if

$$6\lambda^{-1}F_m < d < (2-\eta)|J|\left[2(1-\eta)F_m + \eta(1+k)^{-1}kF_m + \eta k(1+k)^{-1}\widehat{F}\right]^{-1}.$$

This condition is satisfied for

$$\lambda > 6F_m\left[2(1-\eta)F_m + \eta(1+k)^{-1}k\left(F_m+\widehat{F}\right)\right](2-\eta)^{-1}|J|^{-1}.$$

The right-hand side of this inequality is smaller than $6F_m(F_m+\widehat{F})|J|^{-1}$, independently of η and k. Putting everything together we can say that the heteroclinic trajectory for some finite $\theta = \theta^* > 0$ exists if only

$$\lambda > \max\left\{\widehat{F}, 6F_m(F_m+\widehat{F})|J|^{-1}, 6F_m\widehat{F}I^{-1}\right\}.$$

As $|J| < F_m$, $I < F_m$, then

$$\lambda > 6F_m(F_m+\widehat{F})\left(\min\{|J|, I\}\right)^{-1}.$$

In a general case the evaluation can be carried out in principle in the same way, though it would be a little bit more laborious.

Appendix A

Let us recall the basic facts of the connection triple theory taken from [1] (see also [2, 3]), which are used to prove existence of heteroclinic orbits. Suppose that we are given a system of n first order ordinary differential equations (in R^n) parametrized (continuously) by a parameter θ belonging to some nonempty closed interval $[\theta_0, \theta_1]$. Let $X = R^n \times [\theta_0, \theta_1]$. Let S', S'' and S be isolated invariant sets for the flow on X determined by this family of equations and let $S'(\theta)$, $S''(\theta)$, $S(\theta)$ be the set of points in S', S'', S with parameter value θ.

DEFINITION A.1. *The triple S', S'', S is called a connection triple if the following conditions are satisfied:*

a. $S' \cup S'' \subset S$,

b. $S' \cap S'' = \emptyset$,

c. *for* $\theta = \theta_0$ *and* $\theta = \theta_1$, $S(\theta) = S'(\theta) \cup S''(\theta)$. □

Now, suppose that we are given a family of local flows on a space X parametrized in a continuous way by a parameter $\eta \in [0, 1]$. Suppose that S_0 and S_1 are isolated invariant sets for the flows on X corresponding to $\eta = 0$ and $\eta = 1$.

DEFINITION A.2. *We say that S_0 and S_1 are related by continuation, if there exists a compact set N in the space $X \times [0, 1]$ such that N_η (i.e. the set of points in N with parameter value η) is an isolating neighbourhood and such that N_0 and N_1 are isolating neighbourhoods for S_0 and S_1, respectively.* □

DEFINITON A.3. *Suppose that for each $\eta \in [0, 1]$ there exist compact sets N', N'', N such that N'_η, N''_η, N_η are isolating neighbourhoods for the isolated invariant sets S'_η, S''_η and S_η, respectively. Suppose then, that for each $\eta \in [0, 1]$ $(S'_\eta, S''_\eta, S_\eta)$ is a connection triple. Then, we say that the triples (S'_0, S''_0, S_0) and (S'_1, S''_1, S_1) are related by continuation.* □

With a connection triple an index $h(S', S'', S)$ may be connected. Its definition may be found for instance in [1] (see Lemma, p. 325).

DEFINITION A.4. *Let (S', S'', S) be a connection triple for a family of differential equations on R^n parametrized by θ in the interval $[\theta_0, \theta_1]$. Assume the equations are defined for $\theta \in (\theta_0 - \varepsilon, \theta_1 + \varepsilon)$ for some $\varepsilon > 0$ (this is no real restriction – they can be extended to such an interval). Let U' and U'' be open neighbourhoods in $R^n \times (\theta_0 - \varepsilon, \theta_1 + \varepsilon)$ of $S'(\theta_0) \cup S'(\theta_1)$ and $S''(\theta_0) \cup S''(\theta_1)$ (respectively); choose these to have disjoint closures. Let ϕ be a continuous real-valued function on R^n which is positive on U' and negative U''. Append to the given family of equations the equation $\theta' = \mu\phi(x)[\theta - 2^{-1}(\theta_0 + \theta_1)]$, where μ is a small positive parameter. Let N be a compact neighbourhood in $R^n \times (\theta_0 - \varepsilon, \theta_1 + \varepsilon)$ such that $N(\theta)$ is an isolating neighbourhood of $S(\theta)$ for each θ. Then, there is a $\mu_0 > 0$ such that if $\mu \in (0, \mu_0)$ then N is an isolating neighbourhood for the "enlarged system". Let h_μ be the index of $S(N)$, $\mu \in (0, \mu_0)$. Then h_μ is independent of μ, and in fact depends only on the triple (S', S'', S).* □

Now, let us assume (as it is in our case) that for every $\theta \in [0, 1]$ $S'(\theta)$, $S''(\theta)$ are fixed hyperbolic singular points and their indices are constant. Let us denote them by h' and h'', respectively. Now, if there was no connection between S' and S'', then due to point b of Definition A.1, and Definition A.4 we would have $h(S', S'', S) = (\Sigma^1 \wedge h') \vee h''$, where h' and h'' are the Conley indices of $S'(\theta)$ and $S''(\theta)$, $\theta \in [\theta_0, \theta_1]$. So, if $h(S', S'', S) \neq (\Sigma^1 \wedge h') \vee h''$, then $S \neq S' \cup S''$.

The final theorem necessary for our proof is stated in [1] Section D p. 326.

PROPOSITION. The index of a connection triple is constant on equivalence classes under the continuation relation. □

Appendix B

In this appendix we prove Lemmas 6 and 7 concerning the eigenvalues of the linearized system.

Let us fix λ, η and ν and denote for simplicity:

$$\mathcal{M} := \mathcal{M}_\nu(\lambda,\eta) := \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad M := M_\nu(\lambda,\eta), \qquad t := c_1\theta, \qquad Tt := c_{2\eta}k^{-1}\theta.$$

Let $(\xi_1, \xi_2, \zeta_1, \zeta_2)$ be the eigenvector of M corresponding to an eigenvalue q of M. As we mentioned in Sec. 5, it follows from the structure of $M(\lambda,\eta)$ and considerations in [1] p. 335 that $(\xi_1, \xi_2, \zeta_1, \zeta_2)$ and q are coupled simultaneously by the following three relations:

$$\begin{aligned} \zeta_i &= q\xi_i\,, \qquad i = 1,2, \\ q &= 2^{-1}t \pm \sqrt{4^{-1}t^2 - (a + b\xi_2\xi_1^{-1})}\,, \\ q &= 2^{-1}Tt \pm \sqrt{4^{-1}(Tt)^2 - (d + c\xi_2^{-1}\xi_1)}\,. \end{aligned} \tag{B.1}$$

The eigenvalues of $M(\lambda,\eta)$ are the roots of the equation:

$$\det(\mathcal{M}) + q(-dt - atT) + q^2(a + d + t^2T) + q^3(-t - tT) + q^4 = 0. \tag{B.2}$$

Using this fact we can prove the following lemma.

LEMMA B.1. Suppose that $\det\mathcal{M} \neq 0$, $a < 0$, $d < 0$, $t > 0$, $tT > 0$. Then the real part of the eigenvalues of M is different from 0. □

P r o o f. As we have noticed, these eigenvalues are given by the roots of Eq. (B.2). It is obvious that 0 is not a solution of it. So, suppose that there is a pair of eigenvalues q_1, q_2 such that $q_1 = iL$, $q_2 = -iL$, $L \neq 0$. Substituting in (B.2) first $q = iL$ and next $q = -iL$ and subtracting the obtained equations, we arrive at the equation $2iLt(-aT - d + (1 + T)L^2) = 0$. However, according to the assumptions of the lemma we infer that the expression in the bracket is not equal to zero. □

Now, let us note that according to Lemma 2 and Assumption 3 for sufficiently large λ the following conditions are fulfilled (independently of $\eta \in [0,1]$):

$$\det(\mathcal{M}) \neq 0, \qquad t > 0, \qquad tT > 0, \qquad a < 0, \qquad d < 0, \qquad bc > 0. \tag{B.3}$$

P r o o f of Lemma 6. Let us note that, if we put $\nu \in \{-,+\}$ in the definition of $\mathcal{M}$, then, due to Assumption 3, Lemma 1 and Lemma 2, $\det(\mathcal{M}) > 0$ for sufficiently large λ (independently of $\eta \in [0,1]$). Lemma 6 follows straightforwardly from the following more general lemma.

LEMMA B.2 Assume (B.3) and that $\det(\mathcal{M}) > 0$. Then the matrix $M(\lambda,\eta)$ has four real eigenvalues. Two of them are positive and two are negative. Moreover, $\xi_2\xi_1^{-1} > 0$ only for one of the positive and one of the negative eigenvalues. □

Proof. The condition $\det(\mathcal{M}) > 0$ implies that either $a^2 - b^2 > 0$ or $d^2 - c^2 > 0$ (or both). However, as one can easily see by renumbering the components of u, we may always assume that the first one is satisfied, i.e. $(a^2 - b^2) > 0$.

As a starting point of our analysis we will take the situation characterized by the equalities: $T = 1$, $a = d$, $b = c$. In this case (B.2) takes the form:

$$(-a - b - q^2 + qt)(-a + b - q^2 + qt) = 0.$$

As $(a^2 - b^2) > 0$, then this equation has four real solutions:

$$q_{1\pm} = 2^{-1}\left(t \pm \sqrt{-4a - 4b + t^2}\right), \qquad q_{2\pm} = 2^{-1}\left(t \pm \sqrt{-4a + 4b + t^2}\right).$$

Two of them are positive and two are negative.

For $\varrho \in [0, 1]$ let us make the following deformation:

$$\text{(B.4)} \qquad \begin{aligned} &a(\varrho) = a, \qquad d(\varrho) = a + (d - a)\varrho, \qquad b(\varrho) = b,\\ &c(\varrho) = b + (c - b)\varrho, \qquad T(\varrho) = 1 + (T - 1)\varrho. \end{aligned}$$

Then we have:

$$\text{(B.5)} \qquad \det(\mathcal{M}(\varrho)) = (1 - \varrho)\det(\mathcal{M}(0)) + \varrho\det(\mathcal{M}(1)).$$

Thus, it follows that the deformed coefficients retain assumptions (B.3). For simplicity we will omit the explicit dependence of c, d and T on ϱ, if it does not cause confusion.

Now, as b and c $(= c(\varrho))$ are both nonzero, then, according to relations (B.1) and to the structure of eigenvectors, we can assume that ξ_1 is nonzero (the ratio $\xi_2\xi_1^{-1}$ must stay finite). Thus without losing generality we can assume for definiteness that $\xi_1 = 1$ and the eigenvector corresponding to an eigenvalue q has the form $(1, \xi_2, q, q\xi_2)$. So (B.1) can be written as:

$$\text{(B.6)} \qquad \begin{aligned} q &= 2^{-1}t \pm \sqrt{4^{-1}t^2 - (a + b\xi_2)},\\ q &= 2^{-1}Tt \pm \sqrt{4^{-1}(Tt)^2 - (d + c\xi_2^{-1})}. \end{aligned}$$

Now, it can be easily proved that during the above deformation (with respect to ϱ) the eigenvalues of M stay real (and according to Lemma 8 two of them are positive). For $\varrho = 0$ the eigenvectors of M take the form $\{1, 1, q_{1\pm}, q_{1\pm}\}$, $\{1, -1, q_{2\pm}, q_{2\pm}\}$, where $q_{1\pm} = 2^{-1}(t \pm \sqrt{-4a - 4b + t^2})$ and $q_{2\pm} = 2^{-1}(t \pm \sqrt{-4a + 4b + t^2})$. (Remind that we have set $\xi_1 = 1$ for definiteness).

As ξ_2 never becomes 0, then its sign will not change during the deformation. The proof of Lemmat B.2 is thus completed. □

Proof of Lemma 7. Let us note, that, if we put $\nu = 0$ in the definition of $\mathcal{M}$, then, due to Assumption 3, Lemma 1 and Lemma 2, $\det(\mathcal{M}) < 0$ for sufficiently large λ (independently of $\eta \in [0,1]$). In this proof we will also use the fact that for sufficiently large λ both b and c are positive. Note, that during the deformation (B.4), the condition $\det(\mathcal{M}(\varrho)) < 0$ retains its validity according to (B.5). Using arguments as in the proof of Lemma 6 we may assume without losing generality that $(a^2 - b^2) < 0$. According to Lemma B.1, the eigenvalues of M cannnot cross the imaginary axis. The number of eigenvalues with positive or negative real parts is constant during the deformation. As $(a+b)(a-b) < 0$, then one of these factors is positive and the other is negative. Thus for $\varrho = 0$ there exists only one eigenvalue with negative real part and three ones with $\operatorname{Re} q > 0$. As b and c are positive, then the sign of ξ_2 for negative eigenvalue is the same as its sign for $\varrho = 0$, as it cannot become 0 (see the proof of Lemma 6). As $(a-b) < (a+b)$ and $(a-b) < 0$, then, for $\varrho = 0$ the negative eigenvalue is equal to $q = 2^{-1}(t - \sqrt{4a + 4b + t^2})$. Comparing it with $(B.6)_1$ we obtain the claim of the lemma. □

Appendix C

Proof of Lemma 10. To prove Lemma 10 we will show that a trajectory in the closure of N_1 cannot touch ∂N_1 and then return to its interior unless at singular points. As, according to Lemma 4, all bounded solutions of our system have its derivatives estimated in their absolute value by a common finite constant, then it suffices to examine the following cases:

1. $u_i(\zeta) = U_{-i}(\lambda,\eta)$ or $u_i(\zeta) = U_{+i}(\lambda,\eta)$ for some $\zeta \in (-\infty,\infty)$.

a. Let $z_i(\zeta) \neq 0$. Then the trajectory leaves N_1 immediately.

b. Let $z_i(\zeta) = 0$. Due to Lemma 1, μ'_η and ϑ'_η cannot achieve nonpositive values, so the lines $\mathcal{H}_1 = 0$, $\mathcal{H}_2 = 0$ cannot intersect the sides $N_1(\eta) \cap \{(u_1,u_2)\}$ except at the singular points. Moreover, at the upper (lower) side of this rectangle we have $\mathcal{H}_2 < 0$ (> 0) and at the right (left) side $\mathcal{H}_1 < 0$ (> 0), except for the singular points. The proof that such a trajectory leaves $N_1(\eta)$ if it does not reach singular points is carried out as in Lemma 8.

2. $z_i(\zeta) = 0$ for some $\zeta \in (\infty,\infty)$.

a. Let $z'_i(\zeta) \neq 0$. Then the trajectory leaves N_1 immediately.

b. Let $z'_i(\zeta) = 0$. Then also $[-\mathcal{F}_{i\eta} + \lambda(u_1 - u_2)(-1)^{i-1}](\zeta) = 0$. Let j denote the index complementary to i. Then one obtains by differentiation:

$$z''_i(\zeta) = \{-(\mathcal{F}_{i\eta,j})(\zeta) - \lambda\} z_j(\zeta).$$

Thus, if $\lambda > 0$ is sufficiently large (larger than nondiagonal entries of the matrix $\mathcal{F}_{i\eta,j}$) and $z_j(\zeta) > 0$, then $z''_i(\zeta) < 0$, so that near this point $z_i < 0$ and the trajectory lies outside N_1. Now, let us assume that $z_j(\zeta) = 0$. The trajectory leaves N_1 (in appropriate direction), unless $z'_j(\zeta) = 0$. Then, however, we would have also $[-\mathcal{F}_{j\eta} + \lambda(u_1 - u_2)(-1)^{j-1}](\zeta) = 0$, so this point would be a singular point. □

P r o o f of Lemma 11. First, let us note that $\partial N_1(\eta)$ consists of the following sets:

$$\{z_i = 0\} \cap N_1(\eta), \qquad \{z_i = L\} \cap N_1(\eta),$$
$$\{u_i = U_{-i}(\lambda,\eta)\} \cap N_1(\eta), \qquad \{u_i = U_{+i}(\lambda,\eta)\} \cap N_1(\eta),$$

$i \in \{1,2\}$. The second pair of sets cannot comprise points lying on bounded trajectories. The first one intersecting $\partial N(\delta,\eta)$ gives us eight sets, namely:

$$\{z_i = 0,\ u_1 - U_{-1}(\lambda,\eta) = \delta\Delta_1,\ 0 \le z_j < L\},$$
$$\{z_i = 0,\ u_2 - U_{-2}(\lambda,\eta) = \delta\Delta_{-2}(\eta),\ 0 \le z_j < L\}$$
$$\{z_i = 0,\ u_1 - U_{+1}(\lambda,\eta) = \delta\Delta_1,\ 0 \le z_j < L\},$$
$$\{z_i = 0,\ u_2 - U_{+2}(\lambda,\eta) = \delta\Delta_{+2}(\eta),\ 0 \le z_j < L\},$$

where $i \in \{1,2\}$ and j is the index complementary to i and $\nu \in \{-,+\}$. Below, we will show that a trajectory touching one of the above sets cannot belong to $\mathcal{S}(N_2(\delta,\eta))$, i.e. it leaves $N_2(\delta,\eta)$ when continued in appropriate direction. Let us consider particular cases.

1. $u_1(\zeta) - U_{+1} = -\delta\Delta_1$ for some $\zeta \in (-\infty,\infty)$. If $z_1(\zeta) = 0$, then we arrive at the case analyzed in Lemma 8. So, let us suppose that $z_1(\zeta) > 0$ and $z_2(\zeta) = 0$. We can distinguish the three possibilities:

a. $\mathcal{H}_2(\zeta) < 0$, $z_2'(\zeta) = -\mathcal{H}_2(\zeta) > 0$. Then, for decreasing "times" the trajectory leaves $N(\delta,\eta)$ (as u_1 decreases) and $N_1(\eta)$ (as z_2 becomes negative).

b. $\mathcal{H}_2(\zeta) > 0$, $z_2'(\zeta) = -\mathcal{H}_2(\zeta) < 0$, so u_2 achieves a maximum. Consider increasing "times". Then z_2 becomes negative and never achieves the value 0 again while staying in $N(\delta,\eta)$. For, suppose to the contrary, that there exists $\zeta_1 \in (\zeta,\infty]$, such that $z_2(\zeta_1) = 0$ and $z_2(\xi) < 0$ for $\xi \in (\zeta,\zeta_1)$. Then $z_2'(\zeta_1) \ge 0$. But, simultaneously $z_2'(\zeta_1) = -\mathcal{H}_2(\zeta_1) < 0$, as the curve $\mathcal{H}_2 = 0$ lies above the starting point P_1 (see Fig. 2) and it has positive slope in $N(\delta,\eta)$. The trajectory can reach the curve $\mathcal{H}_2 = 0$ only outside $N(\delta,\eta)$. But leaving $N(\delta,\eta)$ would imply leaving also $N_1(\eta)$, as $z_2(\xi) < 0$ for $\xi < \zeta$.

c. $\mathcal{H}_2(\zeta) = 0$. Then $z_2'(\zeta) = -\mathcal{H}_{2,1}(\zeta)z_1 < 0$. Thus, for increasing "times" this case is the same as case b.

2. $u_1(\zeta) - U_{-1} = \delta\Delta_1$ for some $\zeta \in (-\infty,\infty)$. If $z_1(\zeta) = 0$, then we arrive at the case considered in Lemma 8. So, let us suppose that $z_1(\zeta) > 0$ and $z_2(\zeta) = 0$. As before, some particular cases are to be distinguished:

a. $\mathcal{H}_2(\zeta) > 0$, $z_2'(\zeta) = -\mathcal{H}_2(\zeta) < 0$. Then for increasing "times" z_2 becomes negative and the trajectory "immediately" leaves $N(\delta,\eta)$ (u_1 grows) and $N_1(\eta)$ (z_2 decreases).

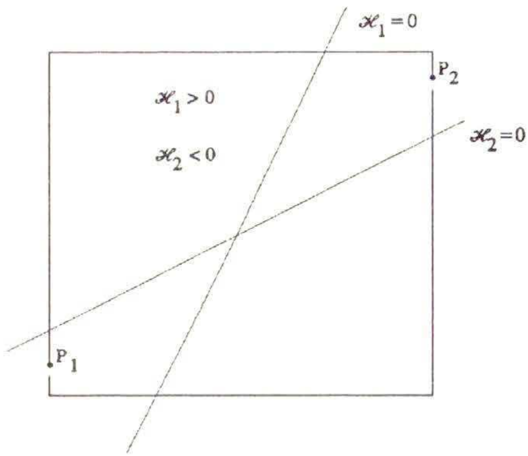

FIG. 2.

b. $\mathcal{H}_2(\zeta) < 0$, $z_2'(\zeta) = -\mathcal{H}_2(\zeta) > 0$, so u_2 achieves a minimum. Consider decreasing "times". Then z_2 becomes negative and never achieves the value 0 again while staying in $N(\delta, \eta)$. For, suppose to the contrary, that there exists $\zeta_1 \in [-\infty, \zeta)$, such that $z_2(\zeta_1) = 0$ and $z_2(\xi) < 0$ for $\xi \in (\zeta_1, \zeta)$. Then $z_2'(\zeta_1) \le 0$. But, simultaneously $z_2'(\zeta_1) = -\mathcal{H}_2(\zeta_1) > 0$, as the curve $\mathcal{H}_2 = 0$ lies below the starting point P_2 (see Fig. 2) and it has positive slope in $N(\delta, \eta)$. The trajectory can reach the curve $\mathcal{H}_2 = 0$ only outside $N(\delta, \eta)$. But leaving $N(\delta, \eta)$ would imply leaving also $N_1(\eta)$, as $z_2(\xi) < 0$ for $\xi < \zeta$.

c. $\mathcal{H}_2(\zeta) = 0$, $z_2'(\zeta) = -\mathcal{H}_{2,1} z_1 < 0$. Thus for increasing "times" this case is the same as case a.

3. The remaining cases are considered similarly.

Now, the intersection of the sets: $\{u_i = U_{-i}(\lambda, \eta)\} \cap N_1(\eta)$, $\{u_i = U_{+i}(\lambda, \eta)\} \cap N_1(\eta)$ with $\partial N(\delta, \eta)$ gives us the following sets:

$$\{u_1 - U_{-1}(\lambda, \eta) = \delta \Delta_1,\ u_2 = U_{-2}(\lambda, \eta),\ 0 \le z_k < L,\ k = 1, 2\},$$
$$\{u_2 - U_{-2}(\lambda, \eta) = \delta \Delta_{-2},\ u_1 = U_{-1}(\lambda, \eta),\ 0 \le z_k < L,\ k = 1, 2\},$$
$$\{u_1 - U_{+1}(\lambda, \eta) = -\delta \Delta_1,\ u_2 = U_{+2}(\lambda, \eta),\ 0 \le z_k < L,\ k = 1, 2\},$$
$$\{u_2 - U_{+2}(\lambda, \eta) = -\delta \Delta_{+2},\ u_1 = U_{+1}(\lambda, \eta),\ 0 \le z_k < L,\ k = 1, 2\}.$$

Let us take, for example, the first set. Let us look at the projection of the trajectory onto the (u_1, u_2)-space. This projection starts at the point

$$X_s = (U_{-1}(\lambda, \eta) + \delta \Delta_1,\ U_{-2}(\lambda, \eta)) := (u_1(\xi_s), u_2(\xi_s)).$$

As $\mathcal{H}_1(\xi_s) > 0$, then, $z_1(\xi) > 0$ for all $\xi < \xi_s$ sufficiently close to ξ_s. (If $z_1(\xi_s) = 0$, then $z_1'(\xi_s) < 0$.) So that the backward trajectory could stay in the set $N_-(\delta, \eta) \cup N_1(\eta)$, for sufficiently small $\xi < \xi_s$ we should have $z_1'(\xi) < 0$ i.e. $z_1(\xi^*) = 0$ and

$z_1(\xi^*) = -\mathcal{H}_1(\xi^*) \geq 0$ for some $\xi^* < \xi_s$. Such a situation could happen only below the curve $\mathcal{H}_1 = 0$ (or just on it). However, the trajectory arriving at the curve $\mathcal{H}_1 = 0$ must come below the curve $\mathcal{H}_2 = 0$, first. But, below that curve we would have $z_2 \geq 0$, due to the fact that $z_2 = 0$ implies $z_2' = -\mathcal{H}_2 < 0$ (we consider the backward trajectory). Consequently, the projection of the backward trajectory must cross the boundary of $N_-(\delta, \eta)$ at the point not belonging to $\partial N_1(\eta)$. Thus this trajectory does not stay in $N_2(\delta, \eta)$ (see Fig. 1).

The proof that the trajectory (in appropriate time direction) starting at a point belonging to the other three of the sets written down below does not stay in the set $N_2(\delta, \eta)$, is carried out almost verbatim in the same way as above. So, the proof of Lemma 11 is completed. □

Acknowledgments

The author wishes to express his gratitude to Prof. Z. PERADZYŃSKI for pointing out the problem. This work was supported by grant KBN No 7T07A00109.

References

1. C. CONLEY and R. GARDNER, *An application of the generalized Morse index to travelling wave solutions of a competitive reaction diffusion model*, Indiana University Mathematical J., **33**, 3, 1984.
2. C. CONLEY and J. SMOLLER, *Isolated invariant sets of parametrized systems of differential equations*, [in:] The Structure of Attractors in Dynamical Systems, Lecture Notes in Math., **668**, Springer-Verlag, Berlin 1978.
3. C. CONLEY, *A qualitative singular perturbation theorem*, [in:] Global Theory of Dynamical Systems, Lecture Notes in Math., **819**, Springer-Verlag, Berlin 1981.
4. R. GARDNER, *Existence of travelling wave solutions of predator-prey systems via the connection index*, SIAM J. Appl. Math., **44**, 1, 1984.
5. B. KAŹMIERCZAK and Z. PERADZYŃSKI, *Heteroclinic solutions for a system of strongly coupled ODE's*, Math. Meth. in Appl. Scien., **19**, 6, pp. 451-461.

POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

Received February 16, 1996.