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Non-uniform extensional motions of materially non-uniform simple solids

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NON-UNIFORM EXTENSIONAL MOTIONS of materially non-uniform simple solids are considered in greater detail. These motions may be useful as applied to quasi-elongational motions with temperature and structure variations. In particular, the constitutive equations are discussed for steady drawing processes of polymer fibres.

1. Introduction

IN OUR PREVIOUS PAPER [1] the results valid for uniform motions with constant stretch history (MCSH) have been generalized to the case of non-uniform stagnant motions (NUSM) of materially non-uniform incompressible simple fluids. The coresponding constitutive equations are very similar to those known for MCSH.

In the present paper we discuss in greater detail the non-uniform extensional motions (hereafter called NUEM) of materially non-uniform simple solids [2]. Such motions deserve more attention since in many practical situations met in the rheology of polymers (drawing of fibres, non-uniform elongations, etc.), thermal and structural effects as well as nonlinear viscoelastic properties are of major importance (cf. [3]) and can be taken into account through the assumption of the proper material non-uniformity. In solids, in contrast to fluids, the deformation energy cannot be neglected and may, through the corresponding dissipation mechanisms, lead to temperature variations and, in consequence, to variable material properties (cf. [3]).

In what follows the non-uniform extensional motions (NUEM) are defined in general and steady-state cases. Next, the corresponding constitutive equations are discussed for materially non-uniform simple locally isotropic solids.

2. Non-uniform extensional motion (NUEM)

Consider a class of isochoric motions for which the deformation gradient at the current time t, relative to a configuration at time 0, is of the following diagonal form:

$$\mathbf{F}_0(\mathbf{X},\tau) = \begin{bmatrix} \lambda^{-1/2} & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \qquad \det \mathbf{F}_0 = 1, \tag{2.1}$$

where the non-uniform stretch ratio $\lambda(\mathbf{X}, t)$ depends on time t as well as on the position $\mathbf{X}$ of a particle X in an arbitrarily chosen reference configuration $\boldsymbol{\kappa}$ (not necessarily at time 0). Thus, a non-uniformity of the quantities considered can be expressed either by $\mathbf{X}$ or X, $\mathbf{X} = \boldsymbol{\kappa}(X)$. Such a motion may be called the non-uniform extensional motion (NUEM).

In general, we obtain the velocity gradient in the form:

$$\mathbf{L}_1(\mathbf{X}, t) = \dot{\mathbf{F}}(\mathbf{X}, t)\mathbf{F}^{-1}(\mathbf{X}, t) = \begin{bmatrix} -\frac{1}{2}\frac{\dot{\lambda}}{\lambda} & 0 & 0 \\ 0 & -\frac{1}{2}\frac{\dot{\lambda}}{\lambda} & 0 \\ 0 & 0 & \frac{\dot{\lambda}}{\lambda} \end{bmatrix}. \tag{2.2}$$

If, in particular, the gradient (2.1) can be presented in an exponential form:

$$\mathbf{F}_0(\mathbf{X}, \tau) = \exp(\tau\mathbf{L}(\mathbf{X})), \qquad \mathbf{L} = \mathbf{L}^T, \tag{2.3}$$

where τ denotes any past time and the diagonal tensor $\mathbf{L}(\mathbf{X})$ depends only on the position $\mathbf{X}$, we arrive at the definition of steady NUEM.

From Eq. (2.3), the deformation gradient relative to a configuration at the current time t amounts to

$$\mathbf{F}_t(\mathbf{X}, t) = \mathbf{F}_0(\mathbf{X}, \tau)\mathbf{F}_0^{-1}(\mathbf{X}, t) = \exp(-s\mathbf{L}(\mathbf{X})), \qquad \tau = t - s, \tag{2.4}$$

leading to the following time-independent velocity gradient:

$$\mathbf{L}_1(\mathbf{X}) = \frac{\partial}{\partial\tau}\mathbf{F}(\mathbf{X}, \tau)\Big|_{\tau=t} = \mathbf{L}(\mathbf{X}). \tag{2.5}$$

Therefore, for steady NUEM we can write

$$\mathbf{L}_1(\mathbf{X}) \equiv \mathbf{L}(\mathbf{X}) = \begin{bmatrix} -\frac{1}{2}V' & 0 & 0 \\ 0 & -\frac{1}{2}V' & 0 \\ 0 & 0 & V' \end{bmatrix}, \tag{2.6}$$

where $V'(\mathbf{X})$ formally denotes the z-component of the velocity gradient.

Equations (2.3) and (2.4) lead to the following expressions for the left Cauchy-Green deformation tensor $\mathbf{B}$ and the history of right relative deformation tensor $\mathbf{C}_t^t$ (cf. [4]):

$$\mathbf{B}(\mathbf{X}, t) = \mathbf{F}_0(\mathbf{X}, t)\mathbf{F}_0^T(\mathbf{X}, t) = \exp(t\mathbf{L}(\mathbf{X})\exp(t\mathbf{L}^T(\mathbf{X})), \tag{2.7}$$

$$\begin{aligned} \mathbf{C}_t^t(\mathbf{X}, s) \equiv \mathbf{C}_t(\mathbf{X}, t-s) &= \mathbf{F}_t^T(\mathbf{X}, t-s)\mathbf{F}_t(\mathbf{X}, t-s) \\ &= \exp(-s\mathbf{L}^T(\mathbf{X}))\exp(-s\mathbf{L}(\mathbf{X})), \end{aligned} \tag{2.8}$$

respectively. The above expressions can be simplified a little since the tensors $\mathbf{L}(\mathbf{X})$ are diagonal by assumption.

3. Constitutive equations of materially non-uniform simple isotropic solids

According to our remarks, at the beginning we assume that *a priori* unknown temperature and structure distributions lead to a material non-uniformity, i.e. to the fact that all the functionals, functions and material constants depend on the position $\mathbf{X}$ and vary from particle to particle or from place to place.

The general constitutive equations of materially non-uniform simple isotropic solids (cf. [2, 4]) can be expressed as

$$(3.1) \qquad \mathbf{T}(\mathbf{X},t) = \mathop{\mathcal{H}_{\kappa}}_{s=0}^{\infty} \left(\mathbf{C}_t^t(\mathbf{X},s), \mathbf{B}(\mathbf{X},t), \mathbf{X} \right),$$

where $\mathbf{T}$ is the non-uniform stress-tensor, $\mathcal{H}_{\kappa}$ denotes the non-uniform constitutive functional depending on the reference configuration $\boldsymbol{\kappa}$ and the tensors $\mathbf{C}_t^t$ and $\mathbf{B}$ have been defined by Eqs. (2.7) and (2.8). In the case of incompressible materials, the stress tensor $\mathbf{T}$ should be replaced by the corresponding extra-stress tensor $\mathbf{T}_E$.

It can be proved that the constitutive equations (3.1) are in agreement with the principles of determinism and local action. They also satisfy the principle of objectivity (invariance with respect to the reference frame), if the group of material isotropy (symmetry) is equivalent to the full orthogonal group (cf. [2, 4]). A non-uniform material may be considered to be globally isotropic if there exists the configuration $\boldsymbol{\kappa}$ at which its isotropy group is the same for all the particles. In other words, in a globally isotropic non-uniform solid all possible directions of deformation are equivalent while its material properties vary from particle to particle.

For steady NUEM defined by Eq. (2.3), after introducing Eqs. (2.7), (2.8) into Eq. (3.1) and taking into account the properties of tensor exponentials, i.e.

$$(3.2) \qquad \exp \mathbf{A} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{A}^n,$$

we arrive at

$$(3.3) \qquad \mathbf{T}(\mathbf{X},t) = \mathop{\mathcal{H}_{\kappa}}_{s=0}^{\infty} (\exp(-2s\mathbf{L}(\mathbf{X})), \exp(2t\mathbf{L}(\mathbf{X})), \mathbf{X}) = \boldsymbol{h}(\mathbf{L}(\mathbf{X}), \mathbf{B}(\mathbf{X},t), \mathbf{X}),$$

where $\boldsymbol{h}$ denotes an isotropic function of the tensor arguments. In particular, instead of $\mathbf{L}(\mathbf{X})$, the first Rivlin – Ericksen kinematic tensor $\mathbf{A}_1 = 2\mathbf{L}$ can be used.

Various reprtesentations of Eqs. (3.3) can be constructed in the usual way. For instance, we have

$$
\begin{aligned}
(3.4)\qquad \mathbf{T}(\mathbf{X},t) = \boldsymbol{\alpha}(\mathbf{L}(\mathbf{X}),\mathbf{X},I_B)\mathbf{1} + \boldsymbol{\alpha}_1(\mathbf{L}(\mathbf{X}),\mathbf{X},I_B)\mathbf{B}(\mathbf{X},t)\\
+\mathbf{B}(\mathbf{X},t)\boldsymbol{\beta}_1(\mathbf{L}(\mathbf{X}),\mathbf{X},I_B) + \mathbf{B}^2(\mathbf{X},t)\boldsymbol{\beta}_2(\mathbf{L}(\mathbf{X}),\mathbf{X},I_B) + \boldsymbol{\alpha}_2(\mathbf{L}(\mathbf{X}),\mathbf{X},I_B)\mathbf{B}^2(\mathbf{X},t),
\end{aligned}
$$

where the material (tensor) coefficients depend on the velocity gradient $\mathbf{L}(\mathbf{X})$, the invariants of tensor $\mathbf{B}$, and explicitly on the position $\mathbf{X}$.

4. Application to steady non-uniform drawing of materially non-uniform polymer fibres

In the case of drawing of solid polymer fibres (cf. [3]), we may assume that, under a quasi-elongational approximation, the deformation gradient as well as the velocity gradient are of the form (2.1) and (2.6), respectively, with

$$
(4.1)\qquad \lambda = \frac{V}{V_0}, \qquad \varepsilon = \ln\lambda, \qquad \dot{\varepsilon} = \frac{\dot{\lambda}}{\lambda} = \frac{\lambda'}{\lambda}V = V',
$$

where $V(z)$ denotes the axial velocity depending on the spatial position z, and the primes denote differentiation with respect to z. The possibility of replacement of the particle position $\mathbf{X}$ by its place in space $\mathbf{x}$ (or rather z) results from the assumption that the motion considered is steady; then the reference configuration can be chosen at the current time t.

Under the above assumption, Eqs. (3.3) lead to the following stress difference:

$$
(4.2)\qquad T^{33} - T^{11} = \sigma(V,V';z) = \sigma_1(\lambda,\lambda';z) = \sigma_2(\varepsilon,\dot{\varepsilon};z).
$$

Thus, in the case of non-uniform drawing of solid polymer fibres, the corresponding elongational stress may depend at most on the velocity and its axial gradient or on the strain and its time-derivative, respectively.

It is worth noting that the constitutive equations desribing drawing processes of polymer fibres were also considered by COLEMAN [5]. He proposed the particular approximate form

$$
(4.3)\qquad T = \tau(\lambda) + \beta(\lambda)\lambda'^2 + \gamma(\lambda)\lambda'',
$$

where λ is the stretch ratio. A simple comparison of the above equation with $(4.2)_2$ shows that our equation is pretty general since it admits arbitrary dependence on λ and λ', and explicitly on z. Equation (4.3), however, shows a particular dependence on λ' and moreover on λ''.

5. Conclusions

The concept of non-uniform extensional motions (NUEM) of materially non-uniform simple locally isotropic solids leads, in the case of steady motions, to the constitutive equations in a form of isotropic function of the deformation gradient and the velocity gradient, depending explicitly on the position of a particle.

For drawing processes of solid polymer fibres, a simplified form of the constitutive equations depending on the velocity, its axial gradient and the place along the fibre axis may be very useful.

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