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Boundary value problems for Poisson's equation in a multi-wedge – multi-layered region Part II. General type of interfacial conditions

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THE BOUNDARY VALUE problems for Poisson's equation in the plane domains represented by wedges and layers are considered. Conditions of a general form along all the interior and exterior boundaries are prescribed. The analysis is significantly simplified by incorporating the geometrical features of the layers and wedges: they present chain-like systems. The essence of the method applied consists in using the Fourier and Mellin transforms for the corresponding regions, and in combining the transformations of respective functions along the common boundaries. The problems are reduced to systems of functional or functional-difference equations, and later to systems of singular integral equations with fixed point singularities. The results, concerning the solvability of the obtained systems of the integral equations are presented. In the Appendix the formulae are also given making it possible to use directly the results obtained from this and the previous paper to solve the boundary value problems for linear partial-differential equations of divergence form in a similar domain, corresponding to physical problems for anisotropic nonhomogeneous bodies.

1. Introduction

IN THE PREVIOUS PAPER [12] we have considered the boundary value problems for Poisson's equation in the plane domains represented by wedges and layers. Linear conditions of general form have been prescribed on the exterior boundaries and all the interfaces except the one between the regions of different geometry (layers and wedges). Along these interior boundaries $\Gamma_\pm$ we assume now general interfacial conditions in the form: $\left[\mu\dfrac{\partial u}{\partial n}\right]\Big|_{\Gamma_\pm} = f_\pm$, $\left([u] - (\tau_\pm r + \tau)\mu\dfrac{\partial u}{\partial n}\right)\Big|_{\Gamma_\pm} = g_\pm$ $(\tau_\pm, \tau \geq 0)$. These relations generalize the usual "ideal" contact conditions ($\tau, \tau_\pm = 0$) considered in the previous paper [12]. They appear, for example, if we presuppose that there are special thin intermediate regions between the layered part and the wedge parts of the domain, and which are represented in turn by a thin layer and two thin wedges. Thus in the case of Mode III problem it can be proved that $\tau = h_a/\mu_a$, $\tau_\pm = \theta_a^\pm/\mu_a^\pm$. Here μ_a, $\mu_a^\pm$ are the shear moduli and h_a, $\theta_a^\pm$ are the respective geometric parameters of these thin elastic adhesive regions (μ is a piecewise constant function prescribed for the shear modulus of the materials). Moreover, from the assumptions (the intermediate regions are thin) it follows that $\tau, \tau_\pm \ll 1$.

These general conditions can be independently considered on the particular model of a thin interconnecting adhesive surface. Then the parameters τ, $\tau_\pm$ can be interpreted as a measure of flexibility of the adhesive. The mentioned models have been discussed and investigated in details in [13]. Particularly, it is shown that when a crack terminates at the bimaterial interface prescribed by the "nonideal" contact, the asymptotic behaviour of the stresses is different in comparison with the case of the "ideal" contact and essentially depends on the parameters $\tau_{\pm}$, τ. Consequently, *a priori* estimations of the solutions in the general case ($\tau_{+}^{2} + \tau_{-}^{2} + \tau^{2} > 0$) should be corrected. Moreover, in spite of the fact that the method of investigation is similar to that proposed in [12], all the problems can be reduced (using a common scheme) to systems of functional ($\tau = 0$) or functional-difference ($\tau > 0$) equations, contrary to [12], where only the systems of the functional equations appear. However, even if we deal only with the systems of functional equations ($\tau = 0$) and reduce them (following [12]) to the systems of integral equations, then some of the systems obtained lead to ill-posed (incorrect) problems. If this takes place (for certain values of the remaining nonzero parameters $\tau_{\pm}$ and the exterior boundary conditions), there are two possibilities: the symbols of the corresponding singular integral operators with fixed point singularities are degenerate at infinity, or the systems of integral equations degenerate from the second kind to the first one at zero point. Hence, the respective systems are incorrect problems, in general.

Returning to the systems of the functional-difference equations ($\tau > 0$), they cannot be uniquelly transformed, in the general case, to the systems of integral equations. The process depends essentially on the external boundary conditions, and the parameters $\tau_{\pm}$, τ. Nevertheless, all the systems of functional-difference equations for all values of the parameters are reduced to a similar class of systems of singular integral equations with fixed point singularities investigated in [10, 11]. In the majority of cases the systems obtained are degenerate. Taking this fact into account, other procedures to reduce the systems of the functional-difference equations to the systems of integral equations for certain cases are also proposed. For all cases of the boundary conditions under consideration and all values of the parameters $\tau_{\pm}$, τ characterizing the "nonideal" interfacial contact, the systems of the integral equations are investigated. So the indices of the nondegenerate operators in Banach spaces of summable functions with a weight are calculated for different parameters of the spaces. In the cases when the operators are degenerate, the theories developed in [18, 19] are used to investigate the corresponding systems, and the indices of normalized operators are calculated.

In the first section we formulate the problems. In the next one, all the problems are reduced to certain systems of functional-difference equations. In the third section, the systems obtained are transformed to systems of singular integral equations for such values of parameters for which the initial systems are of functional type only ($\tau = 0$). The symbols of the corresponding integral operators are investigated and theorems concerning the solvability of the systems of equations are presented. Separately we consider those systems for which the corresponding integral operators are not normally solvable. In the fourth section, the general functional-difference systems ($\tau > 0$) are reduced to systems of integral equations and the symbols of the corresponding operators are investigated for nondegenerate operators as well as in opposite cases.

So, all problems of Poisson's equations under different exterior and interior boundary conditions have been solved. In the Appendix the formulae are given which make it possible to use the results of this paper and [12] in solving the boundary value problems for linear partial-differential equations of divergence form. Such equations prescribe Mode III problems or similar physical problems (e.g. heat conduction and mass diffusion in solids, theory of consolidation and the like [16]) in anisotropic nonhomogeneous bodies.

2. Problem formulation

Let us consider the infinite domain presented in Fig. 1 consisting of a layered part $\Omega_L = \bigcup_{i=1}^{n} \Omega_i$ and two wedge parts $\Omega^+ = \bigcup_{j=1}^{l} \Omega_j^+$, $\Omega^- = \bigcup_{k=1}^{m} \Omega_k^-$.

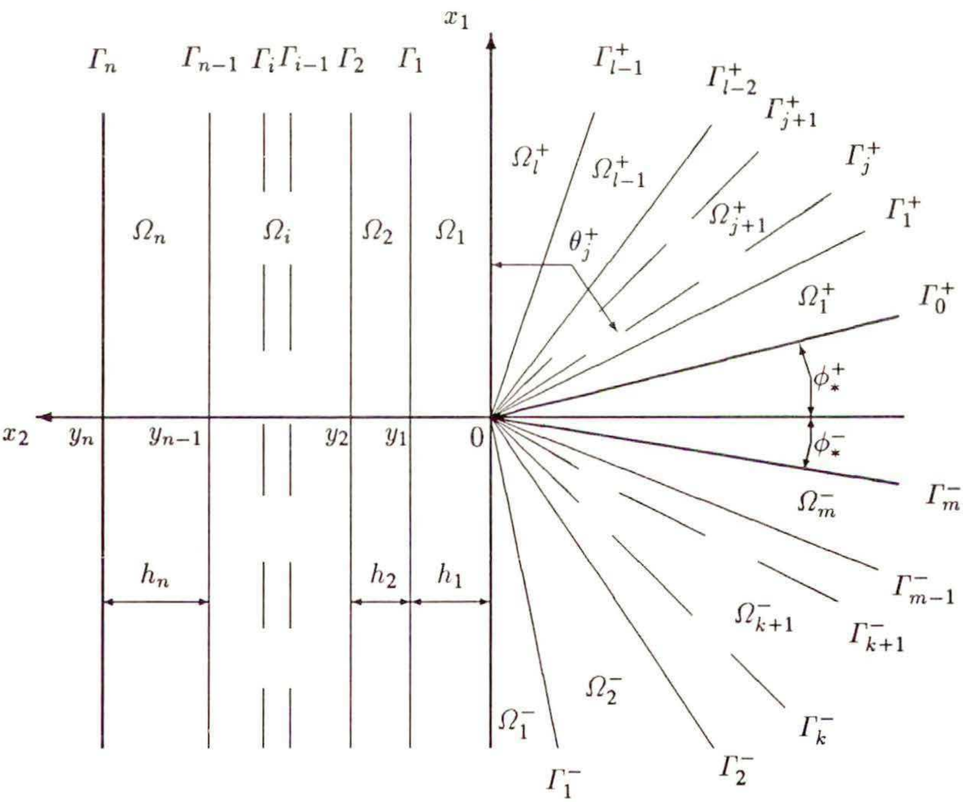

FIG. 1. Domain Ω under consideration.

By Γ_i ($i = 1, 2, ..., n-1$) we denote interior boundaries between the regions Ω_i and Ω_{i+1}. Similarly, Γ_j^+ ($j = 1, 2, ..., l-1$) and Γ_k^- ($k = 1, 2, ..., m-1$) are the interior boundaries between the corresponding wedge regions. Thus, by Γ_n,

Γ_0^+ and Γ_m^- we denote the exterior boundaries of the layered region (Ω_L), or the wedges ($\Omega^{\pm}$), respectively. Besides, let $\Gamma_0 = \Gamma_l^+ \cup \Gamma_0^-$ denote the interior boundary between the different parts of the domain Ω.

We shall seek the function $u(x_1, x_2)$ which satisfies Poisson's equation (2.1) inside the corresponding regions Ω_i, Ω_j^+, Ω_k^-:

$$\begin{aligned} -\mu_i \, \Delta u_i &= W_i\,, & (x_1, x_2) &\in \Omega_i\,, \\ -\mu_j^+ \, \Delta u_j^+ &= W_j^+\,, & (r,\theta) &\in \Omega_j^+\,, \\ -\mu_k^- \, \Delta u_k^- &= W_k^-\,, & (r,\theta) &\in \Omega_k^-\,, \end{aligned} \tag{2.1}$$

with certain positive constants μ_i, μ_j^+, μ_k^-.

Along the interior boundaries of the layered domain Ω_L the conditions hold:

$$\begin{aligned} \left(u_{i+1} - u_i - \mu_i \tau_i \frac{\partial}{\partial x_2} u_i\right)\Big|_{\Gamma_i} &= \delta u_i(x_1), \qquad x_1 \in \mathbb{R}, \\ \frac{\partial}{\partial x_2}\left(\mu_{i+1} u_{i+1} - \mu_i u_i\right)\Big|_{\Gamma_i} &= \delta q_i(x_1), \qquad x_1 \in \mathbb{R}, \qquad i = 1, 2, ..., n-1. \end{aligned} \tag{2.2}$$

Analogous relations for the interior boundaries of wedged domains $\Omega^{\pm}$ are given in the form:

$$\begin{aligned} \left(u_{j+1}^+ - u_j^+ - \mu_j^+ \tau_j^+ \frac{\partial}{\partial \theta} u_j^+\right)\Big|_{\Gamma_j^+} &= \delta u_j^+(r), \qquad r \in \mathbb{R}_+, \\ \frac{1}{r}\frac{\partial}{\partial \theta}\left(\mu_{j+1}^+ u_{j+1}^+ - \mu_j^+ u_j^+\right)\Big|_{\Gamma_j^+} &= \delta q_j^+(r), \qquad r \in \mathbb{R}_+, \qquad j = 1, 2, ..., l-1; \end{aligned} \tag{2.3}$$

$$\begin{aligned} \left(u_{k+1}^- - u_k^- - \mu_k^- \tau_k^- \frac{\partial}{\partial \theta} u_k^-\right)\Big|_{\Gamma_k^-} &= \delta u_k^-(r), \qquad r \in \mathbb{R}_+, \\ \frac{1}{r}\frac{\partial}{\partial \theta}\left(\mu_{k+1}^- u_{k+1}^- - \mu_k^- u_k^-\right)\Big|_{\Gamma_k^-} &= \delta q_k^-(r), \qquad r \in \mathbb{R}_+, \qquad k = 1, 2, ..., m-1, \end{aligned} \tag{2.4}$$

where τ_i, τ_j^+, $\tau_k^- \geq 0$ are certain constants.

Finally, the last of the interior conditions between the regions of different geometry (along the boundaries Γ_l^+, Γ_0^-) are of the general form:

$$\begin{aligned} \left(u_1 - u_l^+ - \mu_1 \tau \frac{\partial}{\partial x_2} u_1 - \mu_l^+ \tau_+ \frac{\partial}{\partial \theta} u_l^+\right)\Big|_{\Gamma_l^+} &= \delta u^+(x_1), \\ \frac{\partial}{\partial x_2}\left(\mu_1 u_1 - \mu_l^+ u_l^+\right)\Big|_{\Gamma_l^+} &= \delta q^+(x_1), \qquad x_1 > 0; \end{aligned} \tag{2.5}$$

$$\begin{aligned} \left(u_1 - u_1^- - \mu_1 \tau \frac{\partial}{\partial x_2} u_1 + \mu_1^- \tau_- \frac{\partial}{\partial \theta} u_1^-\right)\Big|_{\Gamma_0^-} &= \delta u^-(x_1), \\ \frac{\partial}{\partial x_2}\left(\mu_1 u_1 - \mu_1^- u_1^-\right)\Big|_{\Gamma_0^-} &= \delta q^-(x_1), \qquad x_1 < 0, \end{aligned} \tag{2.6}$$

with the constants τ, $\tau_{\pm} \geq 0$.

Now we define the exterior boundary conditions for the domain Ω. So, on the wedge boundaries Γ_0^+, Γ_m^-, one of the following relations holds:

$$(2.7)\quad \begin{aligned} &\text{(a)} && u_1^+|_{\Gamma_0^+} = \delta u_0^+(r), && r \in \mathbb{R}_+,\\ &\text{(b)} && \mu_1^+ \frac{1}{r}\frac{\partial}{\partial\theta} u_1^+|_{\Gamma_0^+} = \delta q_0^+(r), && r \in \mathbb{R}_+, \end{aligned}$$

$$(2.8)\quad \begin{aligned} &\text{(a)} && u_m^-|_{\Gamma_m^-} = -\delta u_m^-(r), && r \in \mathbb{R}_+,\\ &\text{(b)} && \mu_m^- \frac{1}{r}\frac{\partial}{\partial\theta} u_m^-|_{\Gamma_m^-} = -\delta q_m^-(r), && r \in \mathbb{R}_+. \end{aligned}$$

On the exterior boundary Γ_n we shall consider conditions (a), (b) analogous to (2.7), (2.8) and the relation (c):

$$(2.9)\quad \begin{aligned} &\text{(a)} && u_{n|\Gamma_n} = -\delta u_n(x_1), && x_1 \in \mathbb{R},\\ &\text{(b)} && \mu_n \frac{\partial}{\partial x_2} u_{n|\Gamma_n} = -\delta q_n(x_1), && x_1 \in \mathbb{R},\\ &\text{(c)} && \lim_{x_2\to\infty} u_{n+1} = 0. \end{aligned}$$

In the case (c) we assume that the last region Ω_{n+1} is a half-plane. Then the condition $(2.9)_a$ means that the solution of the problem tends to zero both at $x_2 \to \infty$ and $x_1 \to \infty$. Consequently, we have here nine different combinations of exterior conditions. The corresponding problems (2.1) – (2.6) with the boundary conditions (2.7) – (2.9) are denoted by $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$, where $(\mathcal{J}^+ = 1,2;\ \mathcal{J}^- = 1,2;\ \mathcal{J} = 1,2,3)$. Here the value of $\mathcal{J}^+$ is 1 (or 2) if the condition $(2.7)_a$ (or $(2.7)_b$) holds. In an analogous way, one can define the values of $\mathcal{J}^-$, $\mathcal{J}$ from the conditions (2.8) and (2.9), respectively.

We assume that all known functions which appear in the equations and the boundary conditions are sufficiently smooth and their behaviour near zero and infinity points is specifically defined (for details see (1.10) in [12]). In the opposite case (when the defined functions are not smooth and have some singularities), it is easy to find special solutions of the problems accounting for these singularities. Then due to the linearity of the problems, the solution of the initial problem can be represented as a sum of the solutions.

We shall seek the regular solutions of the problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ in the class of functions $\mathbf{LW}(\Omega)$ such that $u \in \mathbf{LW}(\Omega)$ if the following relations are true:

$$(2.10)_1\qquad u_{|G} \in \mathbf{C}^2(G);$$

$$(2.10)_2\qquad \begin{cases} u(x_1, x_2) = O(r^{-\gamma_1}),\\ r\,\mathbf{grad}\, u(r) = O(r^{-\gamma_2}), \qquad (x_1, x_2) \in G, \quad r = \sqrt{x_1^2 + x_2^2} \to \infty; \end{cases}$$

$$(2.10)_3 \qquad \begin{cases} u(x_1, x_2) = u_* + O(r^{\gamma_0} \ln^k r), & (x_1, x_2) \in \Omega_L, \\ u(x_1, x_2) = v_\pm + O(r^{\gamma_0}), & (x_1, x_2) \in \Omega^\pm, \quad r \to 0. \end{cases}$$

Here G denotes all regions of Ω, and $\gamma_0, \gamma_1, \gamma_2$ ($0 < \gamma_0 \leq 1; \gamma_1, \gamma_2 > 0$), $k+1 \in \mathbb{N}$ are certain constants which will be found by solving the problem. Besides, in the cases of the first type boundary condition, at least on one exterior boundary of the wedge ($\mathcal{J}^+\mathcal{J}^- < 4$), additional relation corresponding to the respective notch surface holds:

$$(2.11) \qquad v_\pm(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J}) = 0 \qquad (\mathcal{J}^\pm = 1).$$

It has been shown in [12] for the case of the "ideal" contact conditions along the interfacial boundary Γ_0 ($\tau, \tau_\pm = 0$) that $v_\pm = u_*$, $k = 0$. In spite of the fact that the values of parameters $\gamma_0, \gamma_1, \gamma_2$ are different for the "ideal" contact and the "nonideal" one, they are positive. Therefore, all problems (2.1)–(2.9) in the class $\mathbf{LW}(\Omega)$ have unique solutions, because functions of that class belong to "energetic spaces" ([14]) of the respective boundary value problems. The fact that $v_\pm = 0$ in the same problems follows from the corresponding boundary conditions and from the properties of the functions belonging to $\mathbf{LW}(\Omega)$.

3. Reduction of the problems to systems of functional-difference equations

Applying the Fourier and Mellin transforms to the Poisson's equation (2.1) and to the exterior and interior boundary conditions (2.2)–(2.9) in each respective composite domains Ω_L, $\Omega^\pm$, and using the sweep method [7], we obtain the following relations between the transformations of unknown functions and their derivatives along the boundary Γ_0 (see Eqs. (A.22), (A.45), (A.46) in Appendix A [12]):

$$(3.1) \qquad \mathbf{u}_b^1(\lambda) = M_p(\lambda)\mathbf{p}_b^1(\lambda) + m_p^+(\lambda) + m_p^-(\lambda),$$

$$(3.2) \qquad \mathbf{v}_t^l(s) = M_q(s)\mathbf{q}_t^l(s) + m_q(s),$$

$$(3.3) \qquad \mathbf{w}_b^1(s) = M_r(s)\mathbf{r}_b^1(s) + m_r(s),$$

where

$$\mathbf{u}_b^1(\lambda) = \overline{u}_1|_{\Gamma_0}, \qquad \mathbf{v}_t^l(s) = \widetilde{u}_l^+|_{\Gamma_l^+}, \qquad \mathbf{w}_b^1(s) = \widetilde{u}_1^-|_{\Gamma_0^-},$$

$$\mathbf{p}_b^1(\lambda) = \mu_1 \frac{\partial}{\partial x_2}\overline{u}_1|_{\Gamma_0}, \qquad \mathbf{q}_t^l(s) = \mu_l^+ \frac{\partial}{\partial \theta}\widetilde{u}_l^+|_{\Gamma_l^+}, \qquad \mathbf{r}_b^1(s) = \mu_1^- \frac{\partial}{\partial \theta}\widetilde{u}_1^-|_{\Gamma_0^-}.$$

Here, the Fourier transformation $\overline{f}(\lambda)$ and Mellin transformation $\widetilde{f}(s)$ of a function f are defined in the usual way (see (A.2), (A.28) [12]). Functions M_p, $m_p^\pm$, M_q, m_q, M_r, m_r are obtained in [12] (Appendix A (A.23), (A.47)). Their behaviour depends essentially on the exterior boundary conditions (2.7)–(2.9) (see Lemma A1, Lemma A2 of the mentioned paper).

Define the unknown odd and even functions z_-, z_+ by the relation:

$$z_+(x_1) + z_-(x_1) = \mu_1 \frac{\partial}{\partial x_2} u_1|_{\Gamma_0}; \tag{3.4}$$

then, applying the line of reasoning used in Sec. 2 [12], the remaining contact conditions (2.5), (2.6) can be reduced to the following systems of functional-difference equations:

$$\widehat{\mathbf{Y}}(s) - \mu_1 \tau \widehat{\mathbf{Z}}(s-1) = \mathbf{\Phi}(s)\widehat{\mathbf{Z}}(s) + F(s), \qquad \max\{0, 1-\gamma_0\} < \Re s < \gamma_\infty, \tag{3.5}$$

where we introduce the symbols: $\widehat{u}(s) = \widetilde{u}(-s)$, $d_*(s) = z_*^+ \pi \Gamma(s)$, $\gamma_\infty = \min\{1, \gamma_1, \gamma_2\}$,

$$\mathbf{Y}(\lambda) = \mu_1 \lambda M_p \mathbf{Z}(\lambda) + H_Z(\lambda), \qquad \mathbf{Z}(\lambda) = \begin{bmatrix} \overline{z}_+^*(\lambda) \\ i\overline{z}_-(\lambda) \end{bmatrix},$$

$$H_Z(\lambda) = \mu_1 \lambda \begin{pmatrix} \dfrac{M_p z_*^+}{1+\lambda^2} + m_p^+ \\ i m_p^- \end{pmatrix},$$

$$F(s) = F(s, t_+, t_-, \tau) = \frac{\mu_1}{\Gamma(s)\sin \pi s} \begin{pmatrix} (d_+(s) + [sM_+ + st_- + \tau] d_*(s)) \sin \dfrac{\pi s}{2} \\ (d_-(s) - s[M_- + t_+] d_*(s)) \cos \dfrac{\pi s}{2} \end{pmatrix},$$

$$\mathbf{\Phi}(s) = \mathbf{\Phi}(s, t_+, t_-) = \mu_1 \begin{pmatrix} -s[M_-(s) + t_+] \operatorname{tg} \dfrac{\pi s}{2} & -s[M_+(s) + t_-] \\ s[M_+(s) + t_-] & s[M_-(s) + t_+] \operatorname{ctg} \dfrac{\pi s}{2} \end{pmatrix},$$

$$\overline{z}_+^*(\lambda) = \overline{z}_+(\lambda) - z_*^+ (1+\lambda^2)^{-1}, \qquad z_+^*(x_1) = z_+(x_1) - z_*^+ \pi \exp(-|x_1|),$$

$$2M_\pm(s) = M_q(s) \pm M_r(s), \qquad 2t_\pm = \tau^+ \pm \tau^-,$$

$$2d_\pm(s) = [M_r - \tau^-] \widetilde{\delta q}^-(s+1) \mp [M_q + \tau^+] \widetilde{\delta q}^+(s+1)$$
$$+ m_r \pm m_q + \widetilde{\delta u}^-(s) \pm \widetilde{\delta u}^+(s).$$

The unknown constant $z_*^+ = \overline{z}_+(0)$ for some types of the boundary conditions can be defined from *a priori* estimates (see (A.24), (A.49) [12]):

$$z_*^+ = \begin{cases} \dfrac{1}{2\pi} \Xi_W, & \mathcal{J}^+ = \mathcal{J}^- = 2, \quad \mathcal{J} = 1,2,3, \\ -\Xi_L, & \mathcal{J} = 2,3, \quad \mathcal{J}^\pm = 1,2, \\ \text{unknown}, & \text{for remaining problems.} \end{cases} \tag{3.6}$$

Here $2\pi \Xi_L$, $\Xi_W = \Xi_W^+ + \Xi_W^-$ are the resultant vectors of all the exterior forces in the respective regions Ω_L, $\Omega^\pm$, and are defined in Lemma A1, Lemma A2 [12]. Besides, an additional condition should be satisfied

$$2\pi \Xi_L + \Xi_W = 0 \tag{3.7}$$

for the solvability of the problems $(\mathcal{J}, \mathcal{J}^+, \mathcal{J}^-)$, $\mathcal{J}^\pm = 2$; $\mathcal{J} = 2, 3$ (see Remark A1 [12]). But, for the remaining problems (1,1,1) and (1,2,1) the value of z_*^+ can not be calculated from *a priori* estimates and will be obtained by solving the problems.

A priori estimates (A24) [12] following from the properties of the functions from the class $\mathbf{LW}(\Omega)$ lead to the result that the vector-functions $\widehat{\mathbf{Y}}(s)$, $\widehat{\mathbf{Z}}(s)$ are analytic in the strips $-\gamma_0 < \Re s < \gamma_1$ and $-\gamma_0 < \Re s < \gamma_2$, respectively. Using Lemma A1 and Eq. (2.17) from [12] it can be seen that

$$\mathbf{Y}(\lambda) + \mathbf{Z}(\lambda) = O(\lambda^{-2}), \qquad \lambda \to \infty. \tag{3.8}$$

Taking this fact into account, we rewrite the systems of functional-difference equations (3.5) inside the strip $\max\{0, 1 - \gamma_0\} < \Re s < \gamma_\infty$, in the form:

$$[\widehat{\mathbf{Y}} + \widehat{\mathbf{Z}}](s) = \mu_1 \tau \widehat{\mathbf{Z}}(s-1) + \mathbf{\Phi}_*(s)\widehat{\mathbf{Z}}(s) + F(s), \qquad \mathbf{\Phi}_*(s) = \mathbf{I} + \mathbf{\Phi}(s); \tag{3.9}$$

then the left-hand side of (3.9) is an analytic vector-function in the strip $-2 < \Re s < \gamma_\infty$, which is wider than the analyticity strips of $\widehat{\mathbf{Y}}(s)$, $\widehat{\mathbf{Z}}(s)$.

These systems for the case $\tau, \tau_\pm = 0$ have been investigated in [12]. Note that in the general case $\tau, \tau_\pm > 0$ not only there exists the term with the shifted argument, but the behaviour of the matrix-functions $\mathbf{\Phi}_*(s)$ (depending on the values of $\tau_\pm$) is different from that in [12].

4. Analysis of the system of equations (3.9) in the case $\tau = 0$, $t_+ > 0$

First of all let us note, that the system of Eqs. (3.9) in this case is not a difference system, but a functional system only:

$$[\widehat{\mathbf{Y}} + \widehat{\mathbf{Z}}](s) = \mathbf{\Phi}_*(s)\widehat{\mathbf{Z}}(s) + F(s), \qquad 0 < \Re s < \gamma_\infty. \tag{4.1}$$

We need the following Lemma generalizing the corresponding one from [12]:

LEMMA. For each problem $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ there exists $\nu_\infty = \nu_\infty(\mathcal{J}^+, \mathcal{J}^-)$ $(0 < \nu_\infty < 1)$ such that a matrix-function $\mathbf{\Phi}_*^{-1}(s)$ inverse to $\mathbf{\Phi}_*$ is analytic in region $|\Re s| < \nu_\infty(\mathcal{J}^+, \mathcal{J}^-)$, and satisfies the estimates:

1.

$$\mathbf{\Phi}_*^{-1}(s) = \left\{ \begin{array}{ll} \chi_+ \mathbf{I} + \chi_- \mathbf{E}\,\mathrm{tg}(\pi s/2), & t_- = t_+ = 0, \\ (2\varpi_\pm)^{-1}[\mathbf{I} \mp \mathbf{E}\,\mathrm{tg}(\pi s/2)], & t_- = \mp t_+, \quad t_+ > 0, \\ \mathbf{T}(s), & t_+^2 \neq t_-^2, \quad t_+ > 0, \end{array} \right\} + \mathbf{\Phi}_{**}(s);$$

$$\mathbf{\Phi}_{**}(s) = \left\{ \begin{array}{ll} O(e^{-\varepsilon|\Im s|}), & t_- = t_+ = 0, \\ O(|\Im s|^{-1}), & t_- = \mp t_+, \quad t_+ > 0, \\ O(|\Im s|^{-2}), & t_+^2 \neq t_-^2, \quad t_+ > 0, \end{array} \right\}, \qquad |\Im s| \to \infty;$$

$$\det \mathbf{\Phi}_*^{-1}(s) = \left\{\begin{array}{lll} O(1), & t_- = t_+ = 0, & \\ O(|\Im s|^{-1}), & t_- = \mp t_+, & t_+ > 0, \\ O(|\Im s|^{-2}), & t_+^2 \neq t_-^2, & t_+ > 0, \end{array}\right\}, \qquad |\Im s| \to \infty;$$

for all problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$, $\mathcal{J}^\pm = 1,2$; $\mathcal{J} = 1,2,3$;

2.

$$\mathbf{\Phi}_*^{-1}(s) = \left\{\begin{array}{lll} \mathbf{A}_1 + a_1 s\mathbf{E}, & \mathcal{J}^\pm = 1, & \mathcal{J} = 1,2,3, \\ \mathbf{A}_2 + a_2 s\mathbf{E}, & \mathcal{J}^+\mathcal{J}^- > 1, & \mathcal{J} = 1,2,3, \end{array}\right\} + O(s^2), \qquad s \to 0;$$

$$\det \mathbf{\Phi}_*^{-1}(s) = \left\{\begin{array}{lll} b_1, & \mathcal{J}^\pm = 1, & \mathcal{J} = 1,2,3, \\ 0, & \mathcal{J}^+\mathcal{J}^- > 1, & \mathcal{J} = 1,2,3, \end{array}\right\} + O(s^2), \qquad s \to 0.$$

Here the constants and the matrices are calculated by the relations:

$$\begin{aligned} \chi_\pm &= \frac{\varpi_- \pm \varpi_+}{\varpi_-^2 + \varpi_+^2}, \qquad b_1 = \frac{\pi}{\pi + \mu_1(\eta_+ + \eta_- + 2t_+)}, \\ a_1 &= \frac{\mu_1 b_1}{2}(\eta_+ - \eta_- + 2t_-), \qquad b_2 = \frac{c_+ + c_-}{c_+ + c_- + \pi\mu_1 c_- c_+}, \\ a_2 &= \frac{\pi}{2}\frac{c_+ - c_-}{c_+ + c_-} b_2, \qquad \varpi_- = 1 + \frac{\mu_1}{\mu_1^-}, \qquad \varpi_+ = 1 + \frac{\mu_1}{\mu_l^+}, \\ c_+ &= (\mathcal{J}^+ - 1)\zeta_l^+, \qquad c_- = (\mathcal{J}^- - 1)\zeta_1^-, \qquad \varepsilon = \min\{\phi_l^+, \phi_1^-\}, \\ \mathbf{T}(s) &= \frac{1}{s\mu_1(t_-^2 - t_+^2)} \begin{pmatrix} -t_+ \mathrm{tg}(\pi s/2) & t_- \\ -t_- & -t_+ \mathrm{tg}(\pi s/2) \end{pmatrix}, \\ \mathbf{E} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{A}_1 = \begin{pmatrix} 1 & 0 \\ 0 & b_1 \end{pmatrix}, \qquad \mathbf{A}_2 = \begin{pmatrix} b_2 & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

but the values of constants ζ_l^+, ζ_1^-, η_+, η_- are defined in Lemma A2 from [12].

As one can see, the behaviour of the matrix-function $\mathbf{\Phi}_*^{-1}(s)$ at infinity depends on the type of the interfacial contact conditions (on the values of the parameters t_+, t_-). The corresponding three cases (see 1) we shall denote by the upper index $j = 1,2,3$. However, the behaviour near the zero point depends on the conditions along the boundaries of the exterior wedges (on the values of the parameters $\mathcal{J}^\pm$). The respective two cases (see 2) we shall denote by the lower index $k = 1,2$.

REMARK 1. Let us note that the function $\det \mathbf{\Phi}_*(s)$ has in the strip $(0 < \Re s < 1)$ one zero in the first case of Lemma $(t_+ = t_- = 0)$ only, and this zero is real (see [12]). In the remaining cases $(t_+ > 0)$ the determinant has two zeros with different real parts in this strip. It means that the gradient of the solution of the corresponding boundary value problem will have two singularity terms near the wedge tip.

REMARK 2. When all geometrical and mechanical parameters of the boundary value problem are symmetrical with respect to the OX_2-axis (see Corollary A2 [12]), the systems of the equations (3.9), (4.1) split into two independent equations, because the matrix-function $\mathbf{\Phi}_*(s)$ is diagonal in such situations. Then one can conclude that $\nu_\infty(\mathcal{J}^+,\mathcal{J}^+) = \min\{\omega_\infty(1,\mathcal{J}^+),\omega_\infty(2,\mathcal{J}^+)\}$, where $\omega_\infty(1,\mathcal{J}^+)$, $\omega_\infty(2,\mathcal{J}^+)$ are zeros of the corresponding diagonal elements of the matrix-function $\mathbf{\Phi}_*(s)$.

Typical graphs of the function $\det\mathbf{\Phi}_*(s)$ in the interval (0,1) for the problems $(2,2,\mathcal{J})$ $(\mathcal{J} = 1,2,3)$ when the wedge regions $\Omega^\pm$ are represented by two symmetrical wedges with angles $\pi/2$, and the mechanical parameters are symmetrical also with respect to OX_2-axis ($\mu_1^+ = \mu_1^-$, $\tau^+ = \tau^-$), are presented in the Fig. 2 a, b.

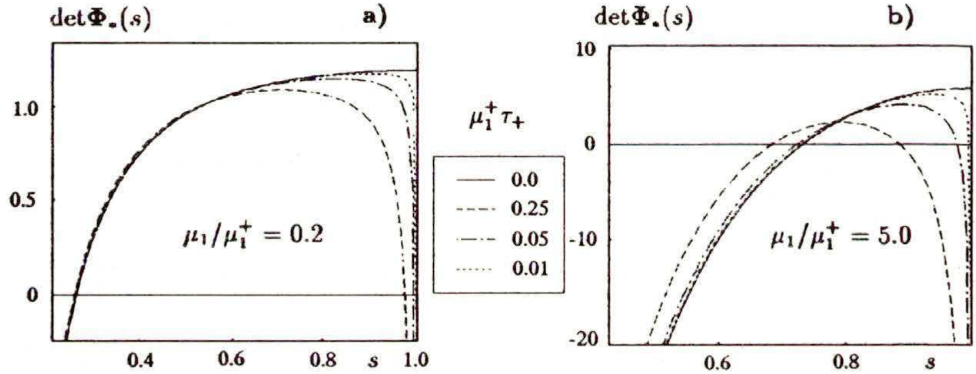

FIG. 2. Graphs of the function $\det\mathbf{\Phi}_*(s)$ in the interval (0,1) in the case $\mu_1^+ = \mu_1^-$, $\tau_+ = \tau_-$.

Here continuous lines correspond to the "ideal" ($\tau_\pm = 0$) contact, but dashed and dotted lines correspond to "non-ideal" contact with respective values of dimensionless parameter $\mu_1^+\tau_+ = 0.01, 0.05, 0.25$.

Let us note that the values of the first zero $\nu_\infty(2,2)$ for small magnitudes of $\mu_1^+\tau_+ < 0.1$ differ but little from the values of the unique zero for the "ideal" contact condition ($\tau_+ = 0$). Numerical results for the values of the mentioned two zeros of the function $\mathbf{\Phi}_*(s)$, for certain geometry and exterior boundary conditions, are presented in [13].

Taking into account the results of the Lemma, we can rewrite the systems of the functional equations (4.1) in an equivalent form:

$$(4.2)\qquad \mathbf{\Phi}_*^{-1}(s)\left[\widehat{\mathbf{Y}}+\widehat{\mathbf{Z}}\right](s) - \mathbf{\Phi}_*^{-1}(s)F(s) = \widehat{\mathbf{Z}}(s), \qquad 0 < \Re s < \min\{\nu_\infty,\gamma_\infty\}.$$

Note that the vector-functions $\mathbf{\Phi}_*^{-1}(s)[\widehat{\mathbf{Y}}+\widehat{\mathbf{Z}}](s)$ and $\widehat{\mathbf{Z}}(s)$ are analytic in the strip $-\min\{\nu_\infty,\gamma_0\} < \Re s < \min\{\gamma_\infty,\nu_\infty\}$, at least. However, the vector-function $\mathbf{\Phi}_*^{-1}(s)F(s)$ can have, in general, a pole at the point $s = 0$. By investigating the behaviour of the vector-function $F(s)$ near the zero point in a similar way as in [12] (we do not present the respective results in this paper) it can be shown that the vector-function $\mathbf{\Phi}_*^{-1}(s)F(s)$ is analytic in the strip $-\nu_\infty < \Re s < \nu_\infty$ for the value of the parameter z_*^+ defined in (3.6). Besides, $\mathbf{\Phi}_*^{-1}(s)F(s)$ has also no pole in this strip in the problems for which this parameter can not be known from (3.6). Finally, this vector-function tends to zero in the strip along any line parallel to the imaginary axis for all the considered problems.

Further, it is evident that the first pole of the vector-function $\widehat{\mathbf{Z}}(s)$ which is the nearest to the imaginary axis in half-plane $\Re s < 0$ coincides with the corresponding pole of the vector-functions $\mathbf{\Phi}_*^{-1}(s)[\widehat{\mathbf{Y}} + \widehat{\mathbf{Z}}](s)$, $\mathbf{\Phi}_*^{-1}(s)F(s)$, hence:

$$\gamma_0 = \nu_\infty(\mathcal{J}^+, \mathcal{J}^-). \tag{4.3}$$

The other parameters from the definition of the class $\mathbf{L}\mathbf{W}(\Omega)$ can be also found,

$$\begin{aligned} v_\pm &= u_*, \\ u_* &= \frac{2}{\mu_1}\int_0^\infty \mathbf{y}_1(\lambda)\frac{d\lambda}{\lambda} \equiv 2\int_0^\infty \left[M_p(\lambda)\mathbf{z}_1(\lambda) + (\mu_1\lambda)^{-1}h_Z^{(1)}(\lambda)\right] d\lambda, \end{aligned} \tag{4.4}$$

where $\mathbf{y}_1$, $\mathbf{z}_1$, $h_Z^{(1)}$ are the first components of the vector-functions $\mathbf{Y}$, $\mathbf{Z}$, H_Z (see (3.5)).

4.1. Reduction of the systems of functional equations to systems of integral equations

Let us recall that the system (4.1) under the first assumptions $t_+ = 0$ ($\tau^\pm = 0$) has been investigated in [12]. For the cases $t_+ > 0$ these systems can be also reduced to systems of singular integral equations, taking into account the behaviour of the matrix-functions $\mathbf{\Phi}_*^{-1}(s)$ at the infinity point.

Thus, in the case $t_- = \mp t_+$, $t_+ > 0$ ($j = 2$ see Lemma), system (4.2) is written in the form:

$$\mathbf{\Phi}_{**}(s)\left[\widehat{\mathbf{Y}} + \widehat{\mathbf{Z}}\right](s) + \left[\frac{1}{2\varpi_\pm}\widehat{\mathbf{Y}} + \left(\frac{1}{2\varpi_\pm} - 1\right)\widehat{\mathbf{Z}}\right](s) \mp \frac{1}{2\varpi_\pm}\operatorname{tg}\frac{\pi s}{2}\mathbf{E}\left[\widehat{\mathbf{Y}} + \widehat{\mathbf{Z}}\right](s) = \mathbf{\Phi}_*^{-1}(s)F(s).$$

Then, applying the inverse Mellin transform to this system, and using a line of reasoning similar to that used in Sec. 4 [12], we obtain a system of singular integral equations:

$$\begin{aligned} \mathcal{B}_Z^{(2)}(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})\mathbf{Z} &= \mathbf{G}_Z^{(2)}, \qquad \mathcal{J} = 1,3, \quad \mathcal{J}^\pm = 1,2; \\ \mathcal{B}_Y^{(2)}(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})\mathbf{Y} &= \mathbf{G}_Y^{(2)}, \qquad \mathcal{J} = 2,3, \quad \mathcal{J}^\pm = 1,2; \end{aligned} \tag{4.5}$$

where

$$[\mathcal{B}^{(2)}_{Z(Y)}\mathbf{u}](\lambda) = \mathbf{u}(\lambda) + \int_0^\infty K^{(2)}_{Z(Y)}(\lambda,\xi)\Psi^{(2)}(\lambda,\xi)\mathbf{u}(\xi)\,d\xi \pm \frac{1}{\pi\varpi_\pm}\mathbf{E}\int_0^\infty K^{(2)}_{Z(Y)}(\lambda,\xi)\mathbf{u}(\xi)\frac{\lambda\,d\xi}{\lambda^2-\xi^2},$$

$$K^{(2)}_Z(\lambda,\xi) = \frac{2\varpi_\pm(1+\mu_1\xi M_p(\xi))}{\lambda\mu_1 M_p(\lambda)+1-2\varpi_\pm},$$

$$K^{(2)}_Y(\lambda,\xi) = \frac{2\varpi_\pm(1+(\mu_1\xi M_p(\xi))^{-1})}{1+(1-2\varpi_\pm)(\lambda\mu_1 M_p(\lambda))^{-1}},$$

$$\mathbf{G}^{(2)}_Z(\lambda) = \frac{2\varpi_\pm}{\lambda\mu_1 M_p(\lambda)+1-2\varpi_\pm} \times\left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\lambda^s\mathbf{\Phi}_*^{-1}(s)F(s)\,ds - \left[\mathcal{B}^{(2)}_0 H_Z\right](\lambda)\right),$$

$$\mathbf{G}^{(2)}_Y(\lambda) = \frac{2\varpi_\pm}{(1-2\varpi_\pm)(\lambda\mu_1 M_p(\lambda))^{-1}+1} \times\left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\lambda^s\mathbf{\Phi}_*^{-1}(s)F(s)\,ds - \left[\mathcal{B}^{(2)}_0 H_Y\right](\lambda) + H_Y(\lambda)\right),$$

$$\Psi^{(2)}(\lambda,\xi) = \frac{1}{2\pi i\xi}\int_{-i\infty}^{i\infty}\mathbf{\Phi}_{**}(s)\left(\frac{\lambda}{\xi}\right)^s ds, \qquad H_Y(\lambda) = -\frac{1}{\mu_1\lambda M_p(\lambda)}H_Z(\lambda),$$

$$[\mathcal{B}^{(2)}_0\mathbf{u}](\lambda) = \frac{1}{2\varpi_\pm}\mathbf{u}(\lambda) + \int_0^\infty \Psi^{(2)}(\lambda,\xi)\mathbf{u}(\xi)\,d\xi \pm \frac{1}{\pi\varpi_\pm}\mathbf{E}\int_0^\infty \mathbf{u}(\xi)\frac{\lambda\,d\xi}{\lambda^2-\xi^2}.$$

In the third case ($t_+^2 \neq t_-^2$, $t_+ > 0$, see Lemma) the inverse Mellin transform can be directly applied to the system (4.2). Consequently, the systems of the integral equations are found:

$$\text{(4.6)}\qquad \begin{aligned} &\mathcal{B}^{(3)}_Z(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})\mathbf{Z} = \mathbf{G}^{(3)}_Z, \qquad \mathcal{J}=1,3, \quad \mathcal{J}^\pm = 1,2;\\ &\mathcal{B}^{(3)}_Y(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})\mathbf{Y} = \mathbf{G}^{(3)}_Y, \qquad \mathcal{J}=2,3, \quad \mathcal{J}^\pm = 1,2; \end{aligned}$$

where

$$[\mathcal{B}^{(3)}_{Z(Y)}\mathbf{u}](\lambda) = f_{Z(Y)}(\lambda)\mathbf{u}(\lambda) + \int_0^\infty K^{(3)}_{Z(Y)}(\xi)\Psi^{(3)}(\lambda,\xi)\mathbf{u}(\xi)\,d\xi,$$

$$f_Z(\lambda) = 1, \qquad f_Y(\lambda) = \frac{1}{\lambda\mu_1 M_p(\lambda)},$$

$$\Psi^{(3)}(\lambda,\xi) = \frac{1}{2\pi i\xi}\int_{-i\infty}^{i\infty} \Phi_*^{-1}(s)\left(\frac{\lambda}{\xi}\right)^s ds,$$

$$K_Z^{(3)}(\xi) = -(1+\mu_1\xi M_p(\xi)), \qquad K_Y^{(3)}(\xi) = -[1+(\mu_1\xi M_p(\xi))^{-1}],$$

$$\mathbf{G}_Z^{(3)}(\lambda) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \lambda^s\Phi_*^{-1}(s)F(s)\,ds + \int_0^{i\infty}\Psi^{(3)}(\lambda,\xi)H_Z(\xi)\,d\xi,$$

$$\mathbf{G}_Y^{(3)}(\lambda) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \lambda^s\Phi_*^{-1}(s)F(s)\,ds + \int_0^{i\infty}\Psi^{(3)}(\lambda,\xi)H_Y(\xi)\,d\xi - H_Y(\lambda).$$

Basing on the results from [2, 10, 11] it can be shown that the obtained operators $\mathcal{B}_{Z(Y)}^{(2)}(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$, $\mathcal{B}_{Z(Y)}^{(3)}(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ for all of the problems $(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ are bounded in the spaces $\mathbf{L}_2^{p,\alpha,\beta}(\mathbb{R}_+)$ [10] with any values of the parameters $-\nu_\infty(\mathcal{J}^+,\mathcal{J}^-) < \alpha \le \beta < \nu_\infty(\mathcal{J}^+,\mathcal{J}^-)$, $1 \le p < \infty$. The right-hand sides of systems (4.5), (4.6) belong to the spaces $\mathbf{W}_{2(m)}^{p,\alpha,\beta}(\mathbb{R}_+)$ for any $m \in \mathbb{N}$. Besides, all these systems of the integral equations are of the second kind, but the operators $\mathcal{B}_Y^{(3)}(\mathcal{J}^+,\mathcal{J}^-,2)$ are degenerate to the first kind in the point $\lambda = 0$, in view of the behaviour of the function $M_p(\lambda)$ (see Lemma A1 from [12]).

From *a priori* estimates for the solutions of class $\mathbf{LW}(\Omega)$ it follows that the inclusions should be true:

$$\mathbf{Y} \in \mathbf{W}_{2(1)}^{1,\alpha_1,\beta}(\mathbb{R}_+), \qquad \mathbf{Z} \in \mathbf{W}_{2(1)}^{1,\alpha_2,\beta}(\mathbb{R}_+), \qquad -\gamma_i < \alpha_i < 0, \quad 0 < \beta < \gamma_0.$$

Moreover, taking into account the smoothness of the kernels of the integral operators and the reasons given in Appendix B [12], it is sufficient to assume that for arbitrary $p \in [1,\infty)$:

$$\mathbf{Y} \in \mathbf{L}_2^{p,\alpha_1,\beta}(\mathbb{R}_+), \qquad \mathbf{Z} \in \mathbf{L}_2^{p,\alpha_2,\beta}(\mathbb{R}_+), \qquad -\gamma_i < \alpha_i < 0, \quad 0 < \beta < \gamma_0. \tag{4.7}$$

Let the matrix-function $\mathbf{B}^{(j)}(\alpha - it,\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ ($t \in \overline{\mathbb{R}}$, $j = 2,3$, $\mathcal{J}^\pm = 1,2$, $\mathcal{J} = 1,2,3$) denote the symbol of the corresponding operator $\mathcal{B}^{(j)}(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ in the respective space (for definition of the symbol of singular operator see, for example, [3, 18]). Then, basing on the results from [10] one can conclude that:

$$\begin{aligned}
&\mathbf{B}_Z^{(j)}(\alpha - it,\mathcal{J}^+,\mathcal{J}^-,1) = \mathbf{I} - \Phi_*^{-1}(\alpha - it),\\
&\mathbf{B}_{Z(Y)}^{(j)}(\alpha - it,\mathcal{J}^+,\mathcal{J}^-,3) = \mathbf{I} - \left(1 - \frac{\mu_1}{\mu_{n+1}}\right)\Phi_*^{-1}(\alpha - it),\\
&\mathbf{B}_Y^{(2)}(\alpha - it,\mathcal{J}^+,\mathcal{J}^-,2) = \Phi_*^{-1}(\alpha - it).
\end{aligned} \tag{4.8}$$

Taking into account the fact that formulae of symbols of the operators $\mathcal{B}^{(2)}_{Z(Y)}$ $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ and $\mathcal{B}^{(3)}_{Z(Y)}(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$, $(\mathcal{J} = 1, 3)$ are of similar form, we will not use the upper indices $(j = 2, 3)$ when it does not involve difficulties. Note only that the matrix-function $\Phi_*^{-1}(s)$ depends on j (on the values of $\tau_\pm$).

REMARK 3. Strictly speaking, all the operators $\mathcal{B}(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ (as well as the operators from [12]) are isometrically equivalent (with the accuracy to compact operators) to some pair systems of integral equations on the axis with the kernels depending on the difference of the arguments [10]. Their symbols are represented in the forms ($t \in \overline{\mathbb{R}}$, $\theta = \pm 1$):

$$\operatorname{Symb} \mathcal{B}(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})|_{\mathbf{L}_2^{p,\alpha,\beta}}(t, \theta) = \mathbf{B}(\alpha - it, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J})\frac{1+\theta}{2} + \mathbf{I}\frac{1-\theta}{2}.$$

Hence, it is sufficient to investigate only the matrix-functions $\mathbf{B}(\alpha - it, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$. Thus we have denoted the symbol of the operator $\mathcal{B}(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ by the corresponding matrix-function $\mathbf{B}(\alpha - it, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ instead of that written above. Besides, these matrix-functions are continuous in $\mathbb{R}$, but can have a point of discontinuity at infinity. Hence, they are not the symbols, but presymbols, in general (for details see [2, 8, 18]).

Note that the operators $\mathcal{B}_Z^{(j)}(\mathcal{J}^+, \mathcal{J}^-, 3)$ $(j = 2, 3)$ are isometrically equivalent to the operators $\mathcal{B}_Y^{(j)}(\mathcal{J}^+, \mathcal{J}^-, 3)$ (see Remark B2 [12]). Consequently, it is sufficient to investigate only the first of them. Moreover, in the case $\mu_1 = \mu_{n+1}$ these operators are the Fredholm ones (they can be represented in the form $I + \mathcal{K}$, where $\mathcal{K}$ is a compact operator), and we will not consider such situation below.

One can see that the symbols $\mathbf{B}_Y^{(2)}(\alpha - it, \mathcal{J}^+, \mathcal{J}^-, 2)$ of the operators $\mathcal{B}_Y^{(2)}(\mathcal{J}^+, \mathcal{J}^-, 2)$ are degenerate at the infinity point for any values of α. Hence, these operators are not normally solvable in the considered spaces (see [18]) and the corresponding systems of integral equations are ill-posed problems [19]. The theory of such singular integral equations in classical spaces is constructed in [18].

4.2. Investigation of symbols of the nondegenerate operators

Let $\alpha = 0$, then by $\nu_0(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$, $(\mathcal{J} = 1, 3)$ we denote the real parts of zeros of the determinants of the matrix-functions $\mathbf{B}_Z(-it, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ $(\mathcal{J} = 1, 3)$, which are the nearest to the imaginary axis (inside half-plane $\Re s \geq 0$). Besides, by $\nu_*(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ we denote the real parts of the next zeros $(\nu_* > \nu_0)$. It can be shown that

$$(4.9) \qquad 0 < \nu_0(2, 2, 1), \qquad \nu_0(\mathcal{J}^+, \mathcal{J}^-, 3) < 1, \qquad \mathcal{J}^\pm = 1, 2,$$

and all zeros are real and simple. For other problems

$$\nu_0(1, 1, 1) = \nu_0(1, 2, 1) = 0,$$

and the orders of multiplicity of these real zeros are equal to two. Thus the problems with nondegenerate symbols are divided into two groups, depending on the values of the respective zeros.

First of all consider the first group (all of the problems for which $\nu_0(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J}) > 0$). Denote by $\alpha_*(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J}) = \min\{\nu_0(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J}), \nu_\infty(\mathcal{J}^+, \mathcal{J}^-)\}$. Then it is easy to see that for all values of $|\alpha| < \alpha_*(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ the indices of the respective operators are equal to zero:

$$\kappa = -\text{ind}\ \det\mathbf{B}_{Z(Y)}(\alpha - it, \mathcal{J}^+, \mathcal{J}^-, \mathcal{J}) = 0, \qquad |\alpha| < \alpha_*(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J}). \tag{4.10}$$

However, when we deal with the systems of integral equations, the partial indices κ_1, κ_2 play also an important role [4]. Using a line of reasoning similar to [12] it can be shown that the symbols of operators are definite matrix-functions [4] for these problems. Hence, we can prove the following theorem:

THEOREM 1. *Let* $1 \le p < \infty$, $m \in \mathbb{N}$, $\nu_0(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J}) > 0$, $\beta < \nu_\infty(\mathcal{J}^+, \mathcal{J}^-)$, $\beta - \alpha \ge 0$, $|\alpha| < \alpha_*(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ *then*;

1) *the operators* $\mathcal{B}_{Z(Y)}(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$, *in the spaces* $\mathbf{L}_2^{p,\alpha,\beta}(\mathbb{R}_+)$ *are normally solvable, and their indices and all partial (left-hand and right-hand) indices are equal to zero*;

2) *there exist the unique solutions of the corresponding systems of equations from* $\mathbf{W}^{p,\alpha,\beta}_{(m),2}(\mathbb{R}_+) \subset \mathbf{L}_2^{p,\alpha,\beta}(\mathbb{R}_+)$.

Results concerning asymptotics of the solutions near zero and infinity points, and the convergence of numerical method can be obtained analogously to those presented in [12].

Now, consider the operators for the problems (1,1,1) and (1,2,1) when $\nu_0 = 0$. In these cases the index and partial (left-hand and right-hand) indices are calculated:

$$\kappa = -\text{ind}\ \det\mathbf{B}_{Z(Y)}(\alpha - it, 1, \mathcal{J}^-, 1) = \pm 1,$$
$$\kappa_1(1, \mathcal{J}^-, 1) = \pm 1, \qquad \kappa_2(1, \mathcal{J}^-, 1) = 0,$$

depending on the value $0 < \pm\alpha < \min\{\nu_*(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J}), \nu_\infty(\mathcal{J}^+, \mathcal{J}^-)\}$. For these problems the values of z_*^+ are unknown (see (3.6)). Moreover, the right-hand sides of the systems (4.5), (4.6) can be represented in the form $\mathbf{G}_Z = \mathbf{G}_Z^1 + z_*^+ \mathbf{G}_Z^2$, where the vector-functions $\mathbf{G}_Z^1$ and $\mathbf{G}_Z^2$ belong to the spaces $\mathbf{W}^{p,\alpha,\beta}_{(m),2}(\mathbb{R}_+)$. So we can prove the following theorems:

THEOREM 2. *Assume* $1 \le p < \infty$, $-\nu_* < \alpha < 0$, $\beta < \nu_\infty$, $\beta - \alpha \ge 0$, $m \in \mathbb{N}$; *then*

1) *the operators* $\mathcal{B}_Z(1,1,1)$, $\mathcal{B}_Z(1,2,1)$ *in the spaces* $\mathbf{L}_2^{p,\alpha,\beta}(\mathbb{R}_+)$ *are normally solvable with the index* $\kappa = -1$ *and the partial (left-hand and right-hand) indices* $\kappa_1 = -1$, $\kappa_2 = 0$;

2) *for these problems there exist unique values of* z_*^+ *for which the systems of equations* (4.5), (4.6) *have (unique) solutions* $\mathbf{Z}(\lambda)$ *in the spaces* $\mathbf{W}^{p,\alpha,\beta}_{(m),2}(\mathbb{R}_+) \subset \mathbf{L}^{p,\alpha,\beta}_2(\mathbb{R}_+)$.

Let us note that the systems of the integral equations in these cases can not be solved by applying numerical methods directly to the systems, as it has been stated in Theorem 1. To remedy this, the systems should be regularized (see [3, 9, 18]). Then the systems obtained will have unique solutions for arbitrary right-hand sides (for any values of z_*^+). Thus, solving the regularized systems for the right-hand sides corresponding to the individual vector-functions $\mathcal{G}^1_Z$ and $\mathcal{G}^2_Z$, the unique values of z_*^+ can be found from the conditions (2.11) and relations (4.4). For these values of z_*^+ the right-hand sides of the equations belong to kernels of the corresponding conjugate operators.

THEOREM 3. *Let* $1 \le p < \infty$, $0 < \alpha < \nu_*$, $\beta < \nu_\infty$, $\beta - \alpha \ge 0$, $m \in \mathbb{N}$, *then*

1) *the operators* $\mathcal{B}_Z(1,1,1)$, $\mathcal{B}_Z(1,2,1)$ *in the spaces* $\mathbf{L}^{p,\alpha,\beta}_2(\mathbb{R}_+)$ *are normally solvable with the index* $\kappa = 1$ *and the partial (left-hand and right-hand) indices are* $\kappa_1 = 1$, $\kappa_2 = 0$;

2) *for these problems there exist unique nontrivial solutions* $\mathbf{Z}_0$ *of the homogeneous systems* (4.5), (4.6) *which belong to any spaces* $\mathbf{W}^{p,\alpha,\beta}_{(m),2}(\mathbb{R}_+)$:

$$\mathbf{Z}_0 \in \bigcap_{p,\alpha,\beta} \mathbf{W}^{p,\alpha,\beta}_{(m),2}(\mathbb{R}_+) .$$

The asymptotics of the solutions from the Theorems 2–3 can be obtained analogously to [12]. Note that nontrivial solutions of homogeneous boundary value problems which can be constructed from the nontrivial solutions of the corresponding homogeneous systems of the integral equations (Theorem 3) do not belong to class $\mathbf{LW}(\Omega)$. They tend to infinity (as $\ln r$) when $r \to \infty$. Such solutions play an important role in the asymptotic method theory (see [15]).

REMARK 4. For the symmetrical problem (1,1,1) the operator $\mathcal{B}_Z(1,1,1)$ splits into two scalar operators (Remark 2). Then, one of them has the index which is equal to zero (see the values of partial indices) and for the corresponding singular integral equation the Theorem 1 holds also true.

4.3. Investigation of the degenerate problems

Now we consider the operators $\mathcal{B}^{(2)}_Y(\mathcal{J}^+,\mathcal{J}^-,2)$ ($\mathcal{J}^\pm = 1,2$) which are not normally solvable in the spaces $\mathbf{L}^{p,\alpha,\beta}_2(\mathbb{R}_+)$ (the symbols are degenerate at infinity). They can be presented in the form:

$$\mathcal{B}^{(2)}_Y(\mathcal{J}^+,\mathcal{J}^-,2) = \mathcal{B}_2\mathcal{P} + \mathcal{Q} + \mathcal{K}.$$

Here $\mathcal{P}$, $\mathcal{Q}$ are complementary projectors ($\mathcal{P} + \mathcal{Q} = I$) of multiplying by the characteristic functions of the sets $(0,1)$ and $(1,\infty)$, respectively. The operators $\mathcal{B}_2$ are isometrically equivalent to the Wiener-Hopf integral operators in the classical spaces $\mathbf{L}_2^p(\mathbb{R})$ with the symbols $\mathbf{B}_Y^{(2)}(\alpha - it, \mathcal{J}^+, \mathcal{J}^-, 2)$, but $\mathcal{K}$ are compact operators. We shall "normalize" the corresponding systems of integral equations following for the theory developed in [18]. First of all let us note, that the matrix-functions $\mathbf{B}_Y^{(2)}(\alpha - it, \mathcal{J}^+, \mathcal{J}^-, 2)$ can be represented depending on the value $t_- = \mp t_+$ (see Lemma) in the following manner:

$$\mathbf{B}_Y^{(2)}(\alpha - it, \mathcal{J}^+, \mathcal{J}^-, 2) = \mathbf{A}_2(t) \begin{pmatrix} (t+i)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pm i & 1 \end{pmatrix},$$

where the matrix-functions $\mathbf{A}_2(t)$ are not degenerate at infinity.

Let us consider the operators:

$$\mathcal{D} = \begin{pmatrix} \mathcal{D}_1\mathcal{P} + \mathcal{Q} & 0 \\ \pm i\mathcal{P} & I \end{pmatrix}, \qquad \mathcal{G} = \begin{pmatrix} \mathcal{P}\mathcal{G}_1\mathcal{P} + \mathcal{Q} & 0 \\ \mp i\mathcal{P}\mathcal{G}_1\mathcal{P} & I \end{pmatrix}, \tag{4.11}$$

in the spaces $\mathbf{L}_2^{p,\alpha,\beta}(\mathbb{R}_+)$, where the scalar operators $\mathcal{D}_1$, $\mathcal{G}_1$ are of the form

$$[\mathcal{D}_1 u](\lambda) = i\int_0^\lambda \frac{\varrho_{\alpha-2,\beta-2}(\xi)}{\varrho_{\alpha-1,\beta-1}(\lambda)} u(\xi)\, d\xi, \qquad [\mathcal{G}_1 u](\lambda) = i\left[[1 - \varrho^*_{\alpha,\beta}(\lambda)]u(\lambda) - \lambda u'(\lambda)\right].$$

By u' we denote the distributional derivative of a function $u \in \mathbf{L}^{p,\alpha,\beta}(\mathbb{R}_+)$, but functions connected with the weight of the spaces are defined as follows:

$$\begin{aligned} \varrho_{\alpha,\beta}(\lambda) &= \begin{cases} \lambda^\alpha, & 0 < \lambda < 1, \\ \lambda^\beta, & 1 < \lambda < \infty; \end{cases} \\ \varrho^*_{\alpha,\beta}(\lambda) &= \frac{\lambda\varrho'_{\alpha,\beta}(\lambda)}{\varrho_{\alpha,\beta}(\lambda)} = \begin{cases} \alpha, & 0 < \lambda < 1, \\ \beta, & 1 < \lambda < \infty. \end{cases} \end{aligned} \tag{4.12}$$

Introduce spaces $\widetilde{\mathbf{L}}_2^{p,\alpha,\beta}(\mathbb{R}_+) = \mathcal{G}\left(\mathbf{L}_2^{p,\alpha,\beta}(\mathbb{R}_+)\right)$, $\mathbf{L}_2^{p,\alpha,\beta}(\mathbb{R}_+) \subset \widetilde{\mathbf{L}}_2^{p,\alpha,\beta}(\mathbb{R}_+)$. One can directly verify that the relations are true: $\mathcal{G}\mathcal{D} = I$, $\mathcal{D}\mathcal{G} = I$, and the spaces $\widetilde{\mathbf{L}}_2^{p,\alpha,\beta}(\mathbb{R}_+)$ with the norm:

$$\|u\|\Big|_{\widetilde{\mathbf{L}}_2^{p,\alpha,\beta}} = \|\mathcal{D}u\|\Big|_{\mathbf{L}_2^{p,\alpha,\beta}},$$

become the Banach spaces.

Represent the initial operators $\mathcal{B}_Y^{(2)}(\mathcal{J}^+, \mathcal{J}^-, 2)$ from $\widetilde{\mathbf{L}}_2^{p,\alpha,\beta}(\mathbb{R}_+)$ to $\mathbf{L}_2^{p,\alpha,\beta}(\mathbb{R}_+)$ in the form:

$$\mathcal{B}_Y^{(2)}(\mathcal{J}^+, \mathcal{J}^-, 2) = \overline{\mathcal{B}}_Y^{(2)}(\mathcal{J}^+, \mathcal{J}^-, 2)\mathcal{D}, \qquad \overline{\mathcal{B}}_Y^{(2)}(\mathcal{J}^+, \mathcal{J}^-, 2) = \mathcal{A}_2\mathcal{P} + \mathcal{Q} + \mathcal{K}_*.$$

Here the operators $\mathcal{A}_2 = \mathcal{A}_2(\mathcal{J}^+, \mathcal{J}^-, 2)$ are isometrically equivalent to the Wiener-Hopf operators with the symbols $\mathbf{A}_2(t)$. Besides, we can prove that the operators $\mathcal{K}_*$: $\mathbf{L}_2^{p,\alpha,\beta}(\mathbb{R}_+) \to \mathbf{L}_2^{p,\alpha,\beta}(\mathbb{R}_+)$ are also compact. By investigating the symbols of the operators $\overline{\mathcal{B}}_Y^{(2)}(\mathcal{J}^+, \mathcal{J}^-, 2)$ it is found that they are normally solvable in the spaces $\mathbf{L}_2^{p,\alpha,\beta}(\mathbb{R}_+)$ with the indices $\kappa(\mathcal{J}^+, \mathcal{J}^-, 2)$ and partial indices κ_1, κ_2:

$$\kappa(1,1,2) = 0, \qquad \kappa_1 = \kappa_2 = 0; \qquad 0 < |\alpha| < \min\{\nu_0(1,1,2), \nu_\infty(1,1)\};$$
$$\kappa(\mathcal{J}^+, \mathcal{J}^-, 2) = \pm 1, \quad \kappa_1 = 0, \ \kappa_2 = \pm 1, \qquad \mathcal{J}^+\mathcal{J}^- > 1,$$

depending on the value $0 < \pm\alpha < \min\{\nu_*(\mathcal{J}^+, \mathcal{J}^-, 2), \nu_\infty(\mathcal{J}^+, \mathcal{J}^-)\}$.

Now we can solve the normalized systems of equations:

$$\overline{\mathcal{B}}_Y^{(2)}(\mathcal{J}^+, \mathcal{J}^-, 2)\overline{\mathbf{Y}} = \mathbf{G}_Y^{(2)},$$

instead of systems (4.5). Theorems which are similar to those proved above can be formulated for these systems. Then relation (3.7) is the usual condition of solvability of the corresponding boundary value problems. Recall that $\mathbf{G}_Y^{(2)} \in \mathbf{W}_{(m),2}^{p,\alpha,\beta}(\mathbb{R}_+)$ and consequently, the solutions $\overline{\mathbf{Y}}$ belong to spaces $\mathbf{W}_{(1),2}^{p,\alpha,\beta}(\mathbb{R}_+)$, at least. Then the solutions $\mathbf{Y} = \mathcal{G}\overline{\mathbf{Y}}$ of the initial systems (4.5) belong to the spaces $\mathbf{L}_2^{p,\alpha,\beta}(\mathbb{R}_+) \subset \widetilde{\mathbf{L}}_2^{p,\alpha,\beta}(\mathbb{R}_+)$, because the operators $\mathcal{G} : \mathbf{W}_{(m),2}^{p,\alpha,\beta}(\mathbb{R}_+) \to \mathbf{W}_{(m-1),2}^{p,\alpha,\beta}(\mathbb{R}_+)$ are bounded for any $m \in \mathbb{N}$. Consequently, condition (4.7) has been satisfied.

Taking into account the volume of the paper we shall not present here the integral form of the operators $\overline{\mathcal{B}}_Y^{(2)}(\mathcal{J}^+, \mathcal{J}^-, 2)\mathcal{K}_*$, and the analytic structure of the spaces $\widetilde{\mathbf{L}}_2^{p,\alpha,\beta}(\mathbb{R}_+)$.

The remaining degenerate operators $\mathcal{B}_Y^{(3)}(\mathcal{J}^+, \mathcal{J}^-, 2)$ $(\mathcal{J}^\pm = 1, 2)$ will be investigated in the Hilbert spaces $\mathbf{L}_2^{2,\alpha,\beta}(\mathbb{R}_+)$. To this end we apply the method of solution of ill-posed (incorrect) problems [19]. Consider the Tikhonov functional $(a > 0)$:

$$\mathcal{F}_a[\mathbf{Y}, \mathbf{G}_Y^{(3)}] = \|\mathcal{B}_Y^{(3)}\mathbf{Y} - \mathbf{G}_Y^{(3)}\|^2_{\mathbf{L}_2^{2,\alpha,\beta}} + a\|\mathbf{Y}\|^2_{\mathbf{L}_2^{2,\alpha,\beta}}. \tag{4.13}$$

Let $\mathbf{Y}_a$ be the minimal element of the functional $\mathcal{F}_a$ in the space $\mathbf{L}_2^{2,\alpha,\beta}(\mathbb{R}_+)$ with the parameters $-\nu_*(\mathcal{J}^+, \mathcal{J}^-, 2) < \alpha < 0$, $\alpha \le \beta$, $\beta < \nu_\infty(\mathcal{J}^+, \mathcal{J}^-)$. As it has been shown above, the equation $\mathcal{B}_Y^{(3)}\mathbf{Y} = \mathbf{G}_Y^{(3)}$ can have a unique solution only in the mentioned spaces. Consequently, $\mathbf{Y}_a \to \mathbf{Y}$ weakly when $a \to 0$ (see [19]). The minimal element $\mathbf{Y}_a$ of the functional $\mathcal{F}_a$ for any $a > 0$ can be calculated by any standard variational methods [8]. Moreover, we can also write Euler equation for this functional:

$$[\mathcal{A}_3\mathbf{Y}_a](\lambda) = \frac{1}{a + f_Y^2(\lambda)}[\mathcal{B}^*\mathbf{G}_Y^{(3)}](\lambda), \tag{4.14}$$

where $\mathcal{B}^*$ is the formal operator conjugate to the operator $\mathcal{B}_Y^{(3)}$:

$$[\mathcal{B}^*\mathbf{u}](\lambda) = f_Y(\lambda)\mathbf{u}(\lambda) + \int_0^\infty \frac{w(\xi)}{w(\lambda)} K_Y^{(3)}(\lambda)\Big(\Psi^{(3)}(\xi,\lambda)\Big)^\top \mathbf{u}(\xi)\,d\xi,$$

$$w(\lambda) = \varrho_{2\alpha-1,2\beta-1}(\lambda),$$

$$[\mathcal{A}_3\mathbf{u}](\lambda) = \mathbf{u}(\lambda) + \int_0^\infty \mathbf{Q}(\lambda,\xi)\mathbf{u}(\xi)\,d\xi,$$

$$\mathbf{Q}(\lambda,\xi) = \frac{1}{a + f_Y^2(\lambda)}\Bigg\{f_Y(\lambda)K_Y^{(3)}(\xi)\Psi^{(3)}(\lambda,\xi) + \frac{w(\xi)}{w(\lambda)} f_Y(\xi)K_Y^{(3)}(\lambda)\Big(\Psi^{(3)}(\xi,\lambda)\Big)^\top + \int_0^\infty \frac{w(t)}{w(\lambda)} K_Y^{(3)}(\lambda)K_Y^{(3)}(\xi)\Big(\Psi^{(3)}(t,\lambda)\Big)^\top \Psi^{(3)}(t,\xi)\,dt\Bigg\}.$$

Here the functions $f_Y(\lambda)$, $K_Y^{(3)}(\lambda)$, $\varrho_{\alpha,\beta}(\lambda)$ are defined in (4.6), (4.12).

Basing on the results of [10], it can be shown that the symbol of the operator $\mathcal{A}_3$ in the space $\mathbf{L}_2^{p,\alpha,\beta}(\mathbb{R}_+)$ is of the form (see Remark 3):

$$\operatorname{Symb}\mathcal{A}_3\big|_{\mathbf{L}_2^{p,\alpha,\beta}} = \mathbf{I} + a^{-1}\Big(\Phi_*^{-1}(\alpha + it)\Big)^\top \Phi_*^{-1}(\alpha - it),$$

and for $\alpha = 0$ it is the real matrix-function. Moreover, its determinant is the even real function which is not equal to zero along $\overline{\mathbb{R}}$. Consequently, the index of the operator $\mathcal{A}_3$ is equal to zero for any $|\alpha| < \nu_0$. Further note that for $\alpha = 0$ the symbol of the operator is the Hermitian matrix-function (the transposed matrix-function is equal to the complex conjugate one). Then, taking into account the fact that the symbol is the definite matrix-function in the point $t = 0$ (or at infinity), we can conclude that it is definite in any point (see the corresponding theorem from [4]). Hence, for the system of equations (4.14) all partial (left-hand and right-hand) indices are equal to zero and the Theorem 1 holds true. Note only that the value of the first zero of the determinant of the operator symbol $\nu_0 = \nu_0(a)$ depends essentially on the value of $a > 0$. Besides, we should choose only negative value of α; then the convergence of the solution $\mathbf{Y}_a$ to the solution of system (4.6) has been justified.

So, the systems of integral equations (4.5), (4.6) which are obtained under the assumption $\tau = 0$ have been investigated for all problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ $(\mathcal{J}^\pm = 1,2,\ \mathcal{J} = 1,2,3)$ and for all values of the parameters $\tau_\pm \geq 0$. The values of the unknown parameters $\gamma_\infty(= \min\{\gamma_1,\gamma_2\})$, $k = 0$, γ_0, $u_*(= v_\pm)$ of class $\mathbf{LW}(\Omega)$ have been obtained (see the Theorems, *a priori* assumptions (4.7) and relations (4.3), (4.4)). Besides, the relation between the values of the parameters γ_1, γ_2 are given in Corollary A.1 [12].

5. Analysis of the system of functional-difference equations (3.9) in the case $\tau > 0$

It is easy to prove by contradiction that the terms of systems (3.9) can not have any pole, the real part of which lies between 0 and γ_∞ and, consequently, $\gamma_0 \geq 1$. Consider the equivalent systems

$$(5.1) \qquad \mathbf{\Phi}_*^{-1}(s)[\widehat{\mathbf{Y}} + \widehat{\mathbf{Z}}](s) = \mu_1 \tau \mathbf{\Phi}_*^{-1}(s)\widehat{\mathbf{Z}}(s-1) + \widehat{\mathbf{Z}}(s) + \mathbf{\Phi}_*^{-1}(s)F(s)$$

in the strip $0 < \Re s < \min\{\nu_\infty, \gamma_\infty\}$. Taking into account the results of the Lemma and *a priori* estimates for the vector-functions $[\widehat{\mathbf{Y}} + \widehat{\mathbf{Z}}](s)$, $\widehat{\mathbf{Z}}(s)$ (see arguments before (4.2)), one can easily see that the vector-function $\widehat{\mathbf{Z}}(s-1)$ can only have a simple pole in the point $s = 0$, and for some $\delta > 0$

$$(5.2) \qquad \mathbf{Z}(\lambda) = \lambda^{-1}\begin{pmatrix} 0 \\ z_-^* \end{pmatrix} + O(\lambda^{-1-\delta}), \qquad \lambda \to \infty.$$

Here the constant z_-^* is defined for some of the problems as follows:

$$(5.3) \qquad z_-^* = \begin{cases} 0, & \mathcal{J}^\pm = 1, \qquad \mathcal{J} = 1,2,3, \\ \text{unknown}, & \mathcal{J}^+\mathcal{J}^- > 1, \qquad \mathcal{J} = 1,2,3. \end{cases}$$

For the remaining problems $\mathcal{J}^+\mathcal{J}^- > 1$, $\mathcal{J} = 1,2,3$, this constant will be calculated below from an additional condition.

Introduce a vector-function $\mathbf{Z}_*(\lambda)$ by the relation:

$$(5.4) \qquad \mathbf{Z}_*(\lambda) = \mathbf{Z}(\lambda) - \frac{z_-^*\lambda}{1+\lambda^2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note that the inverse Fourier transformation of $\mathbf{Z}_*(\lambda)$ is of the form:

$$\mathcal{F}^{-1}[\mathbf{Z}_*](x_1) = \begin{pmatrix} z_+(x_1) \\ iz_-(x_1) \end{pmatrix} + \pi e^{-|x_1|}\begin{pmatrix} -z_*^+ \\ iz_*^- \operatorname{sign}(x_1) \end{pmatrix},$$

where the functions $z_+(x_1)$, $z_-(x_1)$ and the constants z_*^+, z_*^- are defined in (3.4), (3.6), (5.3). Using *a priori* estimates of the solutions belonging to the class $\mathbf{LW}(\Omega)$, and properties of the Mellin and Fourier transforms, we can obtain the values of the parameters from the definition of $\mathbf{LW}(\Omega)$:

$$(5.5) \qquad v_\pm = u_* - 2\tau \int_0^\infty \mathbf{z}_1(\lambda)\, d\lambda - \pi\tau(z_*^+ \mp z_*^-),$$

where the value of u_* is given by (4.4), but the integral of the first component of the vector-function $\mathbf{Z}$ (or $\mathbf{Z}_*$) is bounded in view of (5.2).

Rewrite the systems of equations (5.1) as follows:

$$(5.6)\qquad \mathbf{\Phi}_*^{-1}(s)[\widehat{\mathbf{Y}}+\widehat{\mathbf{Z}}](s)=\mu_1\tau\mathbf{\Phi}_*^{-1}(s)\widehat{\mathbf{Z}}_*(s-1)+\widehat{\mathbf{Z}}_*(s)+\mathbf{F}_Z^*(s).$$

Here the vector-function

$$\mathbf{F}_Z^*(s)=\mathbf{\Phi}_*^{-1}(s)F(s)+\frac{\mu_1\tau\pi z_*^-}{2\sin(\pi s/2)}\mathbf{\Phi}_*^{-1}(s)\begin{pmatrix}0\\1\end{pmatrix}+\frac{z_*^-\pi}{2\cos(\pi s/2)}\begin{pmatrix}0\\1\end{pmatrix}$$

tends to zero at infinity, but systems (5.6) are true in the strip $-\delta<\Re s<\min\{\nu_\infty,\gamma_\infty\}$. Note that $\mathbf{F}_Z^*(s)=\mathbf{F}_1(s)+z_*^+\mathbf{F}_2(s)+z_*^-\mathbf{F}_3(s)$, in general. Besides, the vector-functions multiplied by the unknown constants z_*^+, z_*^- are always bounded in the zero point.

Now we can reduce the systems of functional-difference equations (5.6) to systems of singular integral equations. The way to do that essentially depends on the behaviour of the matrix-function $\mathbf{\Phi}_*^{-1}(s)$ at infinity. Using the Lemma, let us rewrite the systems for the first case ($t_+=t_-=0$) in the form:

$$[\chi_+\mathbf{I}+\chi_-\mathbf{E}\,\mathrm{tg}(\pi s/2)+\mathbf{\Phi}_{**}(s)][\widehat{\mathbf{Y}}(s)+\widehat{\mathbf{Z}}(s)-\mu_1\tau\widehat{\mathbf{Z}}_*(s-1)]=\widehat{\mathbf{Z}}_*(s)+\mathbf{F}_Z^*(s).$$

Then, applying the inverse Mellin transform, we obtain

$$(5.7)\qquad [\chi_+(\mathbf{Y}+\mathbf{Z}-\mu_1\tau\lambda\mathbf{Z}_*)-\mathbf{Z}_*](\lambda)$$
$$-\frac{2\chi_-}{\pi}\mathbf{E}\int_0^\infty[\mathbf{Y}+\mathbf{Z}-\mu_1\tau\xi\mathbf{Z}_*](\xi)\frac{\lambda\,d\xi}{\lambda^2-\xi^2}+\int_0^\infty\Psi^{(1)}(\lambda,\xi)[\mathbf{Y}+\mathbf{Z}-\mu_1\tau\xi\mathbf{Z}_*](\xi)\,d\xi$$
$$=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\mathbf{F}_Z^*(s)\lambda^s\,ds,$$

where

$$\Psi^{(1)}(\lambda,\xi)=\frac{1}{2\pi\xi i}\int_{-i\infty}^{i\infty}\mathbf{\Phi}_{**}(s)\left(\frac{\lambda}{\xi}\right)^s ds.$$

It remains to leave in systems (5.7) only one of the unknown vector-functions using relations (3.5), (5.4) between $\mathbf{Y}(\lambda)$, $\mathbf{Z}(\lambda)$, $\mathbf{Z}_*(\lambda)$. For the exterior boundary conditions along Γ_n of the first and the third type ($\mathcal{J}=1,3$ see (2.9)), it is convenient to leave the vector-function $\mathbf{X}_Z(\lambda)=\mathbf{Z}_*(\lambda)(1+\lambda)$. This is because the matrix-functions belonging to the kernels of the obtained operators (which are different from the homogeneous matrix-functions of the degree -1) should be bounded at zero and infinity. The corresponding systems of integral equations are of the form:

$$(5.8)\qquad \mathcal{C}_Z^{(1)}(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})\mathbf{X}_Z=\mathbf{Q}_Z^{(1)},\qquad \mathcal{J}=1,3,\quad \mathcal{J}^\pm=1,2,$$

where

$$[\mathcal{C}_Z^{(1)}\mathbf{u}](\lambda) = \mathbf{u}(\lambda) + \int_0^\infty L_Z^{(1)}(\lambda,\xi)\Psi^{(1)}(\lambda,\xi)\mathbf{u}(\xi)\,d\xi - \frac{2}{\pi}\chi_-\mathbf{E}\int_0^\infty L_Z^{(1)}(\lambda,\xi)\mathbf{u}(\xi)\frac{\lambda\,d\xi}{\lambda^2-\xi^2},$$

$$L_Z^{(1)}(\lambda,\xi) = \frac{(1+\lambda)[1+\mu_1\xi M_p(\xi)-\mu_1\tau\xi]}{(1+\xi)[\chi_+(1+\mu_1\lambda M_p(\lambda)-\mu_1\tau\lambda)-1]},$$

$$H_Z^* = H_Z + \frac{z_-^*\lambda(1+\mu_1\lambda M_p)}{1+\lambda^2}\begin{pmatrix}0\\1\end{pmatrix},$$

$$\mathbf{Q}_Z^{(1)}(\lambda) = \frac{1+\lambda}{\chi_+(1+\mu_1\lambda M_p(\lambda)-\mu_1\tau\lambda)-1} \times\left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\lambda^s\Phi_*^{-1}(s)\mathbf{F}_Z^*(s)\,ds - [\mathcal{C}_0^{(1)}H_Z^*](\lambda)\right),$$

$$[\mathcal{C}_0^{(1)}\mathbf{u}](\lambda) = \chi_+\mathbf{u}(\lambda) + \int_0^\infty \Psi^{(1)}(\lambda,\xi)\mathbf{u}(\xi)\,d\xi - \frac{2}{\pi}\chi_-\mathbf{E}\int_0^\infty \mathbf{u}(\xi)\frac{\lambda d\xi}{\lambda^2-\xi^2}.$$

However, when we deal with the problem $(\mathcal{J}^+,\mathcal{J}^-,2)$, systems (5.8) are not suitable, because in this case the function $M_p(\lambda) = O(\lambda^{-2})$ as $\lambda\to 0$, and consequently the corresponding integral operators are not bounded in the spaces $\mathbf{L}_2^{p,\alpha,\beta}(\mathbb{R}_+)$ under consideration. For these problems the method of reducing the systems of functional-difference equations (5.1) to systems of integral equations should be similar, but slightly different.

Namely, from (3.8) and (5.2) it follows that

$$\mathbf{Y}(\lambda) = -\lambda^{-1}\begin{pmatrix}0\\z_-^*\end{pmatrix} + O(\lambda^{-1-\delta}),\qquad \lambda\to\infty. \tag{5.9}$$

Then denote

$$\mathbf{Y}_*(\lambda) = \mathbf{Y}(\lambda) + \frac{z_-^*\lambda}{1+\lambda^2}\begin{pmatrix}0\\1\end{pmatrix},$$

and rewrite systems (5.1) in the strip $-\delta < \Re s < \min\{\nu_\infty,\gamma_\infty\}$ in an equivalent form:

$$\Phi_*^{-1}(s)\left[\widehat{\mathbf{Y}}_* + \widehat{\mathbf{Z}}\right](s) = \mu_1\tau\Phi_*^{-1}(s)\widehat{\mathbf{Z}}(s-1) + \widehat{\mathbf{Z}}(s) + \mathbf{F}_Y^*(s), \tag{5.10}$$

Here the vector-function

$$\mathbf{F}_Y^*(s) = \Phi_*^{-1}(s)F(s) + \frac{z_*^-}{2\cos(\pi s/2)}\Phi_*^{-1}(s)\begin{pmatrix}0\\1\end{pmatrix}$$

is analytic in the mentioned strip in view of the Lemma and tends to zero at infinity.

Repeating the former line of reasoning we are led to systems of integral equations with respect to vector-function $\mathbf{X}_Y(\lambda) = \mathbf{Y}_*(\lambda)(1+\lambda)$:

$$\mathcal{C}_Y^{(1)}(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})\mathbf{X}_Y = \mathbf{Q}_Y^{(1)}, \qquad \mathcal{J} = 2,3, \quad \mathcal{J}^\pm = 1,2, \tag{5.11}$$

where

$$[\mathcal{C}_Y^{(1)}\mathbf{u}](\lambda) = \mathbf{u}(\lambda) + \int_0^\infty L_Y^{(1)}(\lambda,\xi)\Psi^{(1)}(\lambda,\xi)\mathbf{u}(\xi)\,d\xi - \frac{2}{\pi}\chi_-\mathbf{E}\int_0^\infty L_Y^{(1)}(\lambda,\xi)\mathbf{u}(\xi)\frac{\lambda\,d\xi}{\lambda^2-\xi^2},$$

$$L_Y^{(1)}(\lambda,\xi) = \frac{(1+\lambda)[1+(\mu_1\xi M_p(\xi))^{-1} - \tau(M_p(\xi))^{-1}]}{(1+\xi)(\chi_+[1+(\mu_1\lambda M_p(\lambda))^{-1} - \tau(M_p(\lambda))^{-1}] - (\mu_1\lambda M_p(\lambda))^{-1})},$$

$$\mathbf{Q}_Y^{(1)}(\lambda) = \frac{1+\lambda}{\chi_+[1+(\mu_1\lambda M_p(\lambda))^{-1} - \tau(M_p(\lambda))^{-1}] - (\mu_1\lambda M_p(\lambda))^{-1}} \times\left(\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\lambda^s\Phi_*^{-1}(s)\mathbf{F}_Y^*(s)\,ds - [\mathcal{C}_0^{(1)}H_Y^*](\lambda) + H_Y^*(\lambda)\right),$$

$$H_Y^* = H_Y - \frac{z_-^*}{\mu_1(1+\lambda^2)M_p(\lambda)}\begin{pmatrix}0\\1\end{pmatrix}.$$

For the second case ($t_+ > 0$, $t_- = \pm t_+$), the systems of integral equations are analogously obtained, because the behaviour of the matrix-function at infinity is similar to that in the first case ($t_+ = 0$). Then the corresponding operators and systems of integral equations can be obtained from (5.8), (5.11) by replacing the upper indices 1 with 2, and the constants χ_+, χ_- with the constants $(2\varpi_\pm)^{-1}$, $\mp(2\varpi_\pm)^{-1}$, respectively.

In the third case ($t_+ > 0$, $t_-^2 \neq t_+^2$), the procedures of reducing the systems (5.1) to systems of integral equations are the same as in proving (4.6), (5.8) and (5.11). The corresponding systems are of the form:

$$\begin{aligned} &\mathcal{C}_Z^{(3)}(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})\mathbf{X}_Z = \mathbf{Q}_Z^{(3)}, \qquad \mathcal{J} = 1,3, \quad \mathcal{J}^\pm = 1,2;\\ &\mathcal{C}_Y^{(3)}(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})\mathbf{X}_Y = \mathbf{Q}_Y^{(3)}, \qquad \mathcal{J} = 2,3, \quad \mathcal{J}^\pm = 1,2;\end{aligned} \tag{5.12}$$

where

$$[\mathcal{C}_{Z(Y)}^{(3)}\mathbf{u}](\lambda) = g_{Z(Y)}(\lambda)\mathbf{u}(\lambda) + \int_0^\infty L_{Z(Y)}^{(3)}(\xi)\Psi^{(3)}(\lambda,\xi)\mathbf{u}(\xi)\,d\xi,$$

$$g_Z(\lambda) = \frac{1}{1+\lambda},$$

$$g_Y(\lambda) = \frac{1}{\lambda\mu_1 M_p(\lambda)(1+\lambda)},$$

$$L_Z^{(3)}(\xi) = -\frac{1+\mu_1\xi M_p(\xi) - \mu_1\tau\xi}{1+\xi},$$

$$L_Y^{(3)}(\xi) = -\frac{1+(\mu_1\xi M_p(\xi))^{-1} - \tau(M_p(\xi))^{-1}}{1+\xi},$$

$$\mathbf{Q}_Z^{(3)}(\lambda) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \lambda^s \mathbf{\Phi}_*^{-1}(s)\mathbf{F}_Z^*(s)\,ds + \int_0^{i\infty} \Psi^{(3)}(\lambda,\xi)H_Z^*(\xi)\,d\xi,$$

$$\mathbf{Q}_Y^{(3)}(\lambda) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \lambda^s \mathbf{\Phi}_*^{-1}(s)\mathbf{F}_Y^*(s)\,ds + \int_0^{i\infty} \Psi^{(3)}(\lambda,\xi)H_Y^*(\xi)\,d\xi - H_Y^*(\lambda).$$

Here the matrix-function $\Psi^{(3)}(\lambda,\xi)$ and the vector-functions $H_Z(\lambda)$, $H_Y(\lambda)$ have been previously defined.

Basing on the results presented in [2, 10, 11] one can show that the obtained operators $\mathcal{C}_{Z(Y)}^{(j)}(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ $(j=1,2,3)$ for all of the problems $(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ are bounded in the spaces $\mathbf{L}_2^{p,\alpha,\beta}(\mathbb{R}_+)$ with the parameters $-\nu_\infty(\mathcal{J}^+,\mathcal{J}^-) < \alpha \le \beta < \nu_\infty(\mathcal{J}^+,\mathcal{J}^-)$, $1 \le p < \infty$. As before, the right-hand sides of the corresponding systems of integral equations belong to the spaces $\mathbf{W}_{(m),2}^{p,\alpha,\beta}(\mathbb{R}_+)$ for any $m \in \mathbb{N}$. All these systems of integral equations are of the second kind, but the operators $\mathcal{C}_Z^{(3)}(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ $(\mathcal{J}=1,3)$, $\mathcal{C}_Y^{(3)}(\mathcal{J}^+,\mathcal{J}^-,3)$ are degenerate to the first kind at infinity, and the operators $\mathcal{C}_Y^{(3)}(\mathcal{J}^+,\mathcal{J}^-,2)$ are degenerate at zero and at infinity.

Note that the vector-functions $\mathbf{X}_{Z(Y)}(\lambda)$ should belong to the spaces:

$$\begin{gathered}\mathbf{X}_Y \in \mathbf{L}_2^{p,\alpha_1,\beta}(\mathbb{R}_+), \qquad \mathbf{X}_Z \in \mathbf{L}_2^{p,\alpha_2,\beta}(\mathbb{R}_+),\\ -\gamma_i < \alpha_i < 0, \qquad 0 < \beta < \delta,\end{gathered} \tag{5.13}$$

for arbitrary $p \in [1,\infty)$ and some $\delta > 0$, in view of *a priori* estimates (5.2), (5.9) for the vector-functions $\mathbf{Z}(\lambda)$ and $\mathbf{Y}(\lambda)$ and the choice of $\mathbf{Z}_*(\lambda)$ and $\mathbf{Y}_*(\lambda)$.

REMARK 5. By assuming $z_*^- = 0$ in the systems obtained in this section, one can equivalently investigate all these systems in the spaces (5.13), however with the negative values of β $(-\delta < \beta < 0)$ only.

Using the results from [10] we can write the symbols $\mathbf{C}_{Z(Y)}^{(j)}(t,\theta,\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ $(t \in \overline{\mathbb{R}}$, $\theta = \pm 1)$ of the nondegenerate operators $\mathcal{C}_{Z(Y)}^{(j)}(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ $(j=1,2)$, which are represented in the form $\mathcal{C} = \mathcal{A}\mathcal{P} + \mathcal{B}\mathcal{Q} + \mathcal{K}$.

$$\mathbf{C}_Z^{(j)}(t,\theta,\mathcal{J}^+,\mathcal{J}^-,1) = \left[\mathbf{I} - \mathbf{\Phi}_*^{-1}(\alpha - it)\right]\frac{1+\theta}{2} + \mathbf{\Phi}_*^{-1}(\beta - it)\frac{1-\theta}{2},$$

$$\text{(5.14)}\quad \mathbf{C}_{Z(Y)}^{(j)}(t,\theta,\mathcal{J}^+,\mathcal{J}^-,3) = \left[\mathbf{I} - \left(1 - \frac{\mu_1}{\mu_{n+1}}\right)\mathbf{\Phi}_*^{-1}(\alpha - it)\right]\frac{1+\theta}{2} + \mathbf{\Phi}_*^{-1}(\beta - it)\frac{1-\theta}{2},$$

$$\mathbf{C}_Y^{(j)}(t,\theta,\mathcal{J}^+,\mathcal{J}^-,2) = \mathbf{\Phi}_*^{-1}(\alpha - it)\frac{1+\theta}{2} + \mathbf{\Phi}_*^{-1}(\beta - it)\frac{1-\theta}{2}.$$

Note that the symbols of the operators $\mathcal{C}_{Z(Y)}^{(2)}(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ in the spaces $\mathbf{L}_2^{p,\alpha,\beta}(\mathbb{R}_+)$ are degenerate for any values of β ($\det \mathbf{\Phi}_*^{-1}(\beta - it)$ tends to zero as $t \to \infty$). Hence we can directly investigate the operators $\mathcal{C}_{Z(Y)}^{(1)}(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ only. Thus the indices and the partial indices of the operators $\mathcal{C}_{Z(Y)}^{(1)}(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ in the spaces $\mathbf{L}_2^{p,\alpha,\beta}(\mathbb{R}_+)$ for some $|\alpha| < \alpha_*$, $|\beta| < \beta_*$ are calculated as follows:

$$\kappa(\alpha,\beta,\mathcal{J}^+,\mathcal{J}^-,1) = \begin{cases} \operatorname{sign}\alpha; & (\kappa_1 = \operatorname{sign}\alpha,\ \kappa_2 = 0), \quad \mathcal{J}^+\mathcal{J}^- = 1,\\ \operatorname{sign}\alpha - \operatorname{sign}\beta; & (\kappa_1 = \operatorname{sign}\alpha,\ \kappa_2 = -\operatorname{sign}\beta), \quad \mathcal{J}^+\mathcal{J}^- = 2,\\ -\operatorname{sign}\beta; & (\kappa_1 = 0, \kappa_2 = -\operatorname{sign}\beta), \quad \mathcal{J}^+\mathcal{J}^- = 4;\end{cases}$$

$$\kappa(\alpha,\beta,\mathcal{J}^+,\mathcal{J}^-,2) = \begin{cases} 0; & (\kappa_1, \kappa_2 = 0), \quad \mathcal{J}^+\mathcal{J}^- = 1,\\ \operatorname{sign}\alpha - \operatorname{sign}\beta; & (\kappa_1 = 0,\ \kappa_2 = \operatorname{sign}\alpha - \operatorname{sign}\beta), \quad \mathcal{J}^+\mathcal{J}^- > 1;\end{cases}$$

$$\kappa(\alpha,\beta,\mathcal{J}^+,\mathcal{J}^-,3) = \begin{cases} 0; & (\kappa_1, \kappa_2 = 0), \quad \mathcal{J}^+\mathcal{J}^- = 1,\\ -\operatorname{sign}\beta; & (\kappa_1 = 0,\ \kappa_2 = -\operatorname{sign}\beta), \quad \mathcal{J}^+\mathcal{J}^- > 1.\end{cases}$$

After eliminating, when the occasion requires, the index and the partial indices of the operators $\mathcal{C}_{Z(Y)}^{(1)}(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ (and the constituent operators $\mathcal{A}$, $\mathcal{B}$) by the methods presented in [18], we can solve the corresponding systems of equations (5.8), (5.11). The unknown constants z_+^*, z_-^* (if they are presented in the respective systems) are obtained from the conditions of solvability of the systems. For example, if the parameters α, β of the spaces $\mathbf{L}_2^{p,\alpha,\beta}(\mathbb{R}_+)$ satisfy the conditions $\alpha < 0 < \beta$ (see (5.13)), then we have $\kappa(\alpha,\beta,1,1,1) = -1$ ($\kappa_1 = -1, \kappa_2 = 0$), and the corresponding system contains the unknown constant z_+^* only. But in the problem (1,2,1) $\kappa(\alpha,\beta,1,2,1) = -2$ ($\kappa_1 = \kappa_2 = -1$) and two constants z_+^*, z_-^* are presented. However, if we choose the values of the space parameters in a different way: $\alpha < 0, \beta < 0$, then for the mentioned problem (1,2,1) $\kappa(\alpha,\beta,1,2,1) = 0$ ($\kappa_1 = -1, \kappa_2 = 1$) and there is only one constant z_+^* (see Remark 5) in the corresponding system.

5.1. Investigation of the degenerate problems

The degenerate systems with the operators $\mathcal{C}^{(2)}_{Z(Y)}(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ and $\mathcal{C}^{(3)}_{Z(Y)}(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ can be analogously transformed, and investigated as it has been done in the previous section for the operators $\mathcal{B}^{(2)}_Y(\mathcal{J}^+,\mathcal{J}^-,2)$ and $\mathcal{B}^{(3)}_Y(\mathcal{J}^+,\mathcal{J}^-,2)$. But we shall investigate them in a different way.

Namely, return to systems of functional-difference equations (5.1) and denote by $\mathbf{Z}_0(\lambda)$ a new vector-function, using the relation similar to (5.4) with the constant z_0^*:

$$\mathbf{Z}_0(\lambda) = \mathbf{Z}(\lambda) - \frac{z_0^*\lambda}{(1+\lambda^2)^2}\begin{pmatrix}0\\1\end{pmatrix}, \qquad z_0^* = \begin{cases}\text{unknown}, & \mathcal{J}^+\mathcal{J}^- = 1,\\ -4\pi^{-1}\widehat{\mathrm{f}}_2(0), & \mathcal{J}^+\mathcal{J}^- > 1,\end{cases} \tag{5.15}$$

such that the additional condition

$$\begin{aligned}(0,1)\,\widehat{\mathbf{Z}}_0(-1) &= 0, \qquad \mathcal{J}^+\mathcal{J}^- = 1, \qquad \mathcal{J} = 1,2,3,\\ (0,1)\,\widehat{\mathbf{Z}}_0(0) &= 0, \qquad \mathcal{J}^+\mathcal{J}^- > 1, \qquad \mathcal{J} = 1,2,3,\end{aligned} \tag{5.16}$$

is true for the problems $(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$. Here $\widehat{\mathrm{f}}_2(s)$ is the second component of the vector $\mathbf{\Phi}_*^{-1}(s)F(s)$. In the case $\mathcal{J}^+,\mathcal{J}^- = 1$ the unknown constant z_0^* will be calculated below. Note that the vector-functions $\mathbf{Z}_0(\lambda)$ and $\mathbf{Z}(\lambda)$ are of a similar behaviour (see (5.2)). It means that the systems:

$$\mathbf{\Phi}_*^{-1}(s)[\widehat{\mathbf{Y}} + \widehat{\mathbf{Z}}](s) = \mu_1\tau\mathbf{\Phi}_*^{-1}(s)\widehat{\mathbf{Z}}_0(s-1) + \widehat{\mathbf{Z}}_0(s) + \mathbf{F}_0(s) \tag{5.17}$$

are true in the strip $0 < \Re s < \min\{\nu_\infty, \gamma_\infty\}$, in general, but the vector-function

$$\mathbf{F}_0(s) = \mathbf{\Phi}_*^{-1}(s)F(s) + \frac{\mu_1\tau\pi z_0^- s}{4\sin(\pi s/2)}\mathbf{\Phi}_*^{-1}(s)\begin{pmatrix}0\\1\end{pmatrix} + \frac{z_0^-\pi(1+s)}{4\cos(\pi s/2)}\begin{pmatrix}0\\1\end{pmatrix}$$

is analytic in the strip $|\Re s| < \nu_\infty$, and its second component is equal to zero when $s = 0$ for the problems $(\mathcal{J}^+,\mathcal{J}^-,\mathcal{J})$ $(\mathcal{J}^+\mathcal{J}^- > 1)$ in view of (5.15).

Now introduce a new vector-function $\mathbf{V}(\lambda)$ by the relation:

$$\begin{aligned}\widehat{\mathbf{Z}}_0(s) &= \mathbf{R}^{-1}_{j,k}(s)\widehat{\mathbf{V}}(s),\\ \mathbf{R}_{3,k}(s) &= \Gamma(s+1)\cos\frac{\pi s}{2}\begin{pmatrix}1 & 0\\ 0 & x_k(s)\end{pmatrix},\\ \mathbf{R}_{2,k}(s) &= \frac{1}{2}\begin{pmatrix}\Gamma(s+1)\cos\dfrac{\pi s}{2} & \pm\Gamma(s+1)\cos\dfrac{\pi s}{2}x_k(s)\\ 1 & \mp x_k(s)\end{pmatrix},\\ \mathbf{R}_{1,k}(s) &= \begin{pmatrix}1 & 0\\ 0 & x_k(s)\end{pmatrix},\\ x_k(s) &= \begin{cases}-\mathrm{tg}(\pi s/2), & k = 1 \Leftrightarrow \mathcal{J}^+\mathcal{J}^- = 1,\\ \mathrm{ctg}(\pi s/2), & k = 2 \Leftrightarrow \mathcal{J}^+\mathcal{J}^- > 1,\end{cases}\end{aligned} \tag{5.18}$$

where choice of $j = 1,2,3$ depends on the behaviour of the matrix-function $\mathbf{\Phi}_*^{-1}(s)$ at infinity (see Lemma). Besides, in the case $j = 2$ the sign is defined from relation $t_- = \mp t_+$. The value of k $(k = 1,2)$ depends in turn on the behaviour of $\mathbf{\Phi}_*^{-1}(s)$ in zero point (see Lemma).

One can see that the vector-function $\widehat{\mathbf{V}}(s)$ has no poles in points $s = 0$ and $s = -1$ in view of (5.2) and (5.16). Consequently one can assume that

$$\mathbf{V} \in \mathbf{L}_2^{p,\alpha,\beta}(\mathbb{R}_+), \qquad -\gamma_1 < \alpha < 0, \quad 0 < \beta < 1 + \delta. \tag{5.19}$$

Besides, for the problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ $(\mathcal{J}^+\mathcal{J}^- = 1$ what is equivalent to $k = 1$, see (5.18)) the additional condition should be satisfied to calculate the unknown constant z_0^*:

$$((-1)^{j+1} - 1, 2)\,\widehat{\mathbf{V}}(0) = 0, \qquad \mathcal{J}^+\mathcal{J}^- = 1, \quad \mathcal{J} = 1,2,3, \quad j = 1,2,3. \tag{5.20}$$

From (5.18) one can obtain the relations between the vector-functions $\mathbf{Z}_0$ and $\mathbf{V}$:

$$\mathbf{Z}_0(\lambda) = [\mathcal{R}_{j,k}\mathbf{V}](\lambda), \tag{5.21}$$

$$\mathcal{R}_{1,k} = \begin{pmatrix} I & 0 \\ 0 & \mathcal{S}_k \end{pmatrix}, \qquad \mathcal{R}_{2,k} = \begin{pmatrix} \mathcal{T}_3 & I \\ \pm\mathcal{T}_{2,k} & \mp\mathcal{S}_k \end{pmatrix}, \qquad \mathcal{R}_{3,k} = \begin{pmatrix} \mathcal{T}_3 & 0 \\ 0 & \mathcal{T}_{2,k} \end{pmatrix},$$

where I is the unity operator, but the other scalar integral operators are defined as follows:

$$[\mathcal{S}_1 u](\lambda) = -\frac{2}{\pi}\int_0^\infty \frac{\xi u(\xi)\,d\xi}{\xi^2 - \lambda^2}, \qquad [\mathcal{S}_2 u](\lambda) = -\frac{2}{\pi}\int_0^\infty \frac{\lambda u(\xi) d\xi}{\xi^2 - \lambda^2},$$

$$[\mathcal{T}_3 u](\lambda) = \frac{2}{\pi}\int_0^\infty \sin(\lambda/\xi) u(\xi)\frac{d\xi}{\xi}, \qquad [\mathcal{T}_{2,1} u](\lambda) = \frac{2}{\pi}\int_0^\infty \cos(\lambda/\xi) u(\xi)\frac{d\xi}{\xi}, \tag{5.22}$$

$$[\mathcal{T}_{2,2} u](\lambda) = \frac{4}{\pi^2}\int_0^\infty [\mathrm{Si}\,(\lambda/\xi)\cos(\lambda/\xi) + \mathrm{ci}\,(\lambda/\xi)\sin(\lambda/\xi)]\, u(\xi)\frac{d\xi}{\xi}.$$

Here the singular integral operators $\mathcal{S}_2$, $\mathcal{T}_{2,2}$, $\mathcal{T}_3 : L^{p,\alpha,\beta}(\mathbb{R}_*) \to L^{p,\alpha,\beta}(\mathbb{R}_*)$ and $\mathcal{S}_1$, $\mathcal{T}_{2,1} : \overline{L}^{p,\alpha,\beta}(\mathbb{R}_*) \to L^{p,\alpha,\beta}(\mathbb{R}_*)$ are bounded. But $\overline{L}^{p,\alpha,\beta}(\mathbb{R}_*) \subset L^{p,\alpha,\beta}(\mathbb{R}_*)$ is the set of functions from $L^{p,\alpha,\beta}(\mathbb{R}_*)$ which satisfy the respective condition (5.20).

Rewrite the systems (5.17) in an equivalent form:

$$\mathbf{N}_{j,k}(s)[\widehat{\mathbf{Y}} + \widehat{\mathbf{Z}}](s) = \mu_1\tau\mathbf{M}_{j,k}(s)\widehat{\mathbf{V}}(s-1) + \widehat{\mathbf{V}}(s) + \mathbf{R}_{j,k}(s)\mathbf{F}_Z^0(s), \tag{5.23}$$

in the strip $-\delta < \Re s < \min\{\nu_\infty, \gamma_\infty\}$ for some value $\delta > 0$. Here we denote

$$\begin{aligned} \mathbf{N}_{j,k}(s) &= \mathbf{R}_{j,k}(s)\mathbf{\Phi}_*^{-1}(s) &&= \mathbf{N}_j^{(1)}(s) + \mathbf{N}_{j,k}^{(2)}(s), \\ \mathbf{M}_{j,k}(s) &= \mathbf{R}_{j,k}(s)\mathbf{\Phi}_*^{-1}(s)\mathbf{R}_{j,k}^{-1}(s-1) &&= \mathbf{M}_j^{(1)} + \mathbf{M}_{j,k}^{(2)}(s). \end{aligned} \tag{5.24}$$

Note that the matrix-functions of these representations satisfy the estimates $\mathbf{N}^{(2)}_{j,k}(it) = o(t^{-1/2})$, $\mathbf{M}^{(2)}_{j,k}(it) = o(t^{-1})$ as $t \to \infty$, but

$$\mathbf{N}^{(1)}_1(s) = \begin{pmatrix} \chi_+ & \chi_- \operatorname{tg}(\pi s/2) \\ -\chi_- & -\chi_+ \operatorname{tg}(\pi s/2) \end{pmatrix}, \qquad \mathbf{M}^{(1)}_1 = \begin{pmatrix} \chi_+ & -\chi_- \\ -\chi_- & \chi_+ \end{pmatrix},$$

$$\mathbf{N}^{(1)}_2(s) = \frac{1}{2\varpi_\pm} \begin{pmatrix} 0 & 0 \\ 1 & \mp \operatorname{tg}(\pi s/2) \end{pmatrix}, \qquad \mathbf{M}^{(1)}_2 = \frac{1}{2\mu_1 t_+ \varpi_\pm} \begin{pmatrix} \varpi_\pm & 0 \\ 0 & 2\mu_1 t_+ \end{pmatrix};$$

$$\mathbf{N}^{(1)}_3(s) \equiv 0, \qquad \mathbf{M}^{(1)}_3 = \frac{1}{\mu_1(t_+^2 - t_-^2)} \begin{pmatrix} t_+ & -t_- \\ -t_- & t_+ \end{pmatrix}.$$

Then substituting (5.23) in systems (5.20), and applying the inverse Mellin transform, we obtain the systems of integral equations:

$$(5.25) \qquad [\mathcal{N}^{(1)}_j(\mathbf{Y} + \mathbf{Z})](\lambda) + [\mathcal{N}^{(2)}_{j,k}(\mathbf{Y} + \mathbf{Z})](\lambda)$$
$$= [1 + \mu_1 \tau \lambda \mathbf{M}_j]\mathbf{V}(\lambda) + \mu_1 \tau [\mathcal{M}^{(2)}_{j,k}(\xi \mathbf{V}(\xi))](\lambda) + \mathbf{G}_0(\lambda),$$

where the operators $\mathcal{N}^{(1)}_j$, $\mathcal{N}^{(2)}_{j,k}$ are defined analogously to $\mathcal{M}^{(2)}_{j,k}$:

$$[\mathcal{M}^{(2)}_{j,k}u](\lambda) = \int_0^\infty M^{(2)}_{j,k}(\lambda/\xi)u(\xi)d\xi/\xi, \qquad M^{(2)}_{j,k}(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \mathbf{M}^{(2)}_{j,k}(s)t^s\, ds,$$

$$\mathcal{N}^{(1)}_1 = \begin{pmatrix} \chi_+ I & \chi_- \mathcal{S}_1 \\ -\chi_- I & -\chi_+ \mathcal{S}_1 \end{pmatrix}, \qquad \mathcal{N}^{(1)}_2 = \frac{1}{2\varpi_\pm} \begin{pmatrix} 0 & 0 \\ I & \mp \mathcal{S}_1 \end{pmatrix}, \qquad \mathcal{N}^{(1)}_3 = 0.$$

Substituting then in (5.25) the vector-functions $\mathbf{Z}$, $\mathbf{Y}$ from relations (3.5), (5.15) and (5.21), and taking into account the fact that the matrix-function $\mathbf{K}_j(\lambda) = (1 + \lambda)[1 + \mu_1 \tau \lambda \mathbf{M}_j]^{-1}$ is nondegenerate in $\overline{\mathbb{R}}$, we obtain the systems of integral equations for the new vector-function $\mathbf{V}_*(\lambda) = (1 + \lambda)\mathbf{V}(\lambda)$:

$$(5.26) \qquad [\mathcal{E}^{(j)}(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})\mathbf{V}_*](\lambda) = \mathbf{H}_{j,k}(\lambda), \qquad \mathcal{J}^\pm = 1, 2, \quad \mathcal{J} = 1, 3,$$

where the operators $\mathcal{E}^{(j)}(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ and the vector-functions $\mathbf{H}_{j,k}$ are:

$$[\mathcal{E}^{(j)}(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})\mathbf{V}_*](\lambda) = \mathbf{V}_*(\lambda) + \mathbf{K}_j(\lambda)\left[\mathcal{M}^{(2)}_{j,k}(\xi(1 + \xi)^{-1}\mathbf{V}_*(\xi))\right](\lambda)$$
$$-\mathbf{K}_j(\lambda)\left[\left(\mathcal{N}^{(1)}_j + \mathcal{N}^{(2)}_{j,k}\right)\left([1 + \mu_1 \xi M_p(\xi)]\mathcal{R}_{j,k}((1 + t)^{-1}\mathbf{V}_*(t))(\xi)\right)\right](\lambda),$$

$$\mathbf{H}_{j,k}(\lambda) = \mathbf{K}_j(\lambda)\Big([(\mathcal{N}^{(1)}_j + \mathcal{N}^{(2)}_{j,k})H_0](\lambda) - \mathbf{G}_0(\lambda)\Big),$$

$$H_0(\xi) = H_Z(\xi) + \frac{[1 + \mu_1 \xi M_p(\xi)]z_0^* \xi}{(1 + \xi^2)^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\mathbf{G}_0(\lambda) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \mathbf{R}_{j,k}(s)\mathbf{F}^0_Z(s)\lambda^s\, ds.$$

These equations can not be used in the case when the gradients of the solution are prescribed along the most external boundary Γ_n ($\mathcal{J} = 2$) (with respect to the layered part of the domain).

Solutions of equations (5.26) are sought in the spaces (see (5.19)):

$$\text{(5.27)} \qquad \mathbf{V}_* \in \mathbf{L}_2^{p,\alpha,\beta}(\mathbb{R}_+), \qquad -\gamma_1 < \alpha < 0, \qquad 0 < \beta < \delta;$$

besides, for the case $k = 1$ (conditions of the first kind $\mathcal{J}^\pm = 1$ are given along the external boundaries with respect to the wedges), the additional condition (5.20) should be true. Basing on the results known from [10], the symbols of the operators $\mathcal{E}^{(j)}(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ from (5.26) can be calculated:

$$\text{(5.28)} \qquad \begin{aligned} \operatorname{Symb} \mathcal{E}^{(j)}(\mathcal{J}^+, \mathcal{J}^-, 1)_{|_{\mathbf{L}_2^{p,\alpha,\beta}}}(t,\theta) &= \left(\mathbf{M}_j^{(1)}\right)^{-1} \mathbf{M}_{j,k}(\beta - it)\frac{1-\theta}{2} \\ &+ \left[\mathbf{I} - \mathbf{N}_{j,k}(\alpha - it)\mathbf{R}_{j,k}^{-1}(\alpha - it)\right] \frac{1+\theta}{2}, \\ \operatorname{Symb} \mathcal{E}^{(j)}(\mathcal{J}^+, \mathcal{J}^-, 3)_{|_{\mathbf{L}_2^{p,\alpha,\beta}}}(t,\theta) &= \left(\mathbf{M}_j^{(1)}\right)^{-1} \mathbf{M}_{j,k}(\beta - it)\frac{1-\theta}{2} \\ &+ \left[\mathbf{I} - \left(1 - \frac{\mu_1}{\mu_{n+1}}\right) \mathbf{N}_{j,k}(\alpha - it)\mathbf{R}_{j,k}^{-1}(\alpha - it)\right] \frac{1+\theta}{2}, \end{aligned}$$

where the matrix-functions $\mathbf{R}_{j,k}(s)$, $\mathbf{N}_{j,k}(s)$, $\mathbf{M}_{j,k}(s)$ are defined in (5.18), (5.24). As it follows from (5.24), the symbols of operators $\mathcal{E}^{(j)}(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ are not degenerate for the values of $j = 2, 3$, in contrast to the symbols of operators $\mathcal{C}^{(j)}(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ from (5.14). Moreover, one can see that the identities: $\det[\mathbf{I} - y\mathbf{N}_{j,k}(s)\mathbf{R}_{j,k}^{-1}(s)] = \det[\mathbf{I} - y\mathbf{\Phi}_*^{-1}(s)]$, $\det \mathbf{M}_{j,k}(s) = -(s\,\mathrm{ctg}(\pi s/2))^{j-1} x_k^2(s)$ $\det \mathbf{\Phi}_*^{-1}(s)$ are true for any $y \in \mathbb{R}$. Then the indices and pair indices of the corresponding operators $\mathcal{E}^{(j)}(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ in the spaces $\mathbf{L}_2^{p,\alpha,\beta}(\mathbb{R}_+)$ can be calculated:

$$\kappa(\alpha,\beta,\mathcal{J}^+,\mathcal{J}^-,1) = \begin{cases} \operatorname{sign}\alpha - \operatorname{sign}\beta; & (\kappa_1 = \operatorname{sign}\alpha,\ \kappa_2 = -\operatorname{sign}\beta), \quad \mathcal{J}^+\mathcal{J}^- = 1, \\ \operatorname{sign}\alpha; & (\kappa_1 = \operatorname{sign}\alpha,\ \kappa_2 = 0), \quad \mathcal{J}^+\mathcal{J}^- = 2, \\ 0; & (\kappa_1 = \kappa_2 = 0), \quad \mathcal{J}^+\mathcal{J}^- = 4; \end{cases}$$

$$\kappa(\alpha,\beta,\mathcal{J}^+,\mathcal{J}^-,3) = \begin{cases} -\operatorname{sign}\beta; & (\kappa_1 = 0,\ \kappa_2 = -\operatorname{sign}\beta), \quad \mathcal{J}^+\mathcal{J}^- = 1, \\ 0; & (\kappa_1 = \kappa_2 = 0), \quad \mathcal{J}^+\mathcal{J}^- > 1. \end{cases}$$

So, for the values of the parameters $\alpha < 0$, $\beta > 0$ as in (5.27), the indices and the partial indices of the operators are equal to zero or negative. In the last case there exists exactly $|\kappa|$ unknown parameters (z_0^* or (and) z_+^*) which are found from the additional condition (5.20) and the corresponding condition (2.11) together with (5.5). Note here, that only one of the conditions (2.11) is independent, when both external boundary conditions along the wedge surfaces are of the first type ($\mathcal{J}^+ = \mathcal{J}^- = 1$), because $z_*^- = 0$ in (5.5) for this case.

The remaining problems $(\mathcal{J}^+, \mathcal{J}^-, 2)$ for the second and the third combinations of the parameters $\tau_\pm$ ($j = 2, 3$, see Lemma), which have not been considered as yet, can be investigated on the basis of systems (5.11), (5.12). The corresponding degenerate operators $\mathcal{C}^{(2)}(\mathcal{J}^+, \mathcal{J}^-, 2)$, $\mathcal{C}^{(3)}(\mathcal{J}^+, \mathcal{J}^-, 2)$ could be analyzed similarly to operators $\mathcal{B}^{(2)}$ $(\mathcal{J}^+, \mathcal{J}^-, 2)$ and $\mathcal{B}^{(3)}(\mathcal{J}^+, \mathcal{J}^-, 2)$ in Sec. 4.

Finally, the systems of integral equations (5.11), (5.12), (5.26) obtained under the general assumption $\tau > 0$ have been investigated for all problems $(\mathcal{J}^+, \mathcal{J}^-, \mathcal{J})$ $(\mathcal{J}^\pm = 1, 2,\ \mathcal{J} = 1, 2, 3)$ and for all values of the parameters $\tau_\pm \geq 0$. The values of parameters u_*, $v_\pm$ of the class $\mathbf{LW}(\Omega)$ have been found (see (5.5)), $\gamma_0 = 1$, $k = 1$, but the value of $\gamma_\infty = \min\{\gamma_1, \gamma_2\}$ is calculated from the symbols of the corresponding operators (as in the theorems presented in the previous section).

6. Conclusions

We have considered all different combinations of the external boundary conditions, and values of the parameters τ, $\tau_\pm \geq 0$ determining the interfacial conditions near the wedge tip. As it could be expected, the singularity of $\mathbf{grad}\, u$ near the wedge tip depends essentially on the models of the interface. Thus, if the model of interface is of the form: $\left([u] - r\tau_\pm \mu \dfrac{\partial u}{\partial n}\right)\Big|_{\Gamma_\pm} = 0$, $\left[\mu \dfrac{\partial u}{\partial n}\right]\Big|_{\Gamma_\pm} = 0$ (corresponding to the adhesive region represented by two thin wedges only), the main exponent of the singularity is in the interval $(-1, 0)$. It has the value close to that of the case of an "ideal" bimaterial contact for small values of the normed parameters $\mu_1^- \tau_1^-$, $\mu_l^+ \tau_l^+$. Besides, there is a second exponent in the interval $(-1, 0)$, which has the value near zero. Nevertheless, the corresponding term of the asymptotic expression should be also taken into account in the process of fracture mechanics analysis.

When the geometry of the adhesive is assumed to be of the general form $\left[\mu \dfrac{\partial u}{\partial n}\right]\Big|_{\Gamma_\pm} = 0$, $\left([u] - (r\tau_\pm + \tau)\mu \dfrac{\partial u}{\partial n}\right)\Big|_{\Gamma_\pm} = 0$, $(\tau, \tau_\pm > 0)$ or in the case of a thin layer only (where $\tau > 0$, $\tau_\pm = 0$), $\mathbf{grad}\, u$ increases in the neighbourhood of the wedge tip as $\ln r$ inside the domains Ω only. But inside the domains $\Omega_\pm$, the value of $\mathbf{grad}\, u$ is bounded as well as the normal derivative $\partial u / \partial n$ along the interface.

Note that the cases, when at least one of the parameters τ, $\tau_\pm$ is negative, are not considered in this paper. Such situations appear on the declining segment of curve $\varepsilon - \sigma$ and are often connected with a loss of stability of bodies in contact.

Let us remark that all the used functions $M_p(\lambda)$, $m_p^\pm(\lambda)$, $M_q(s)$, $M_r(s)$ can be effectively calculated by the recurrence formulae presented in Appendix A [12], and the asymptotics of these functions have been analytically obtained. Moreover, an effective way of finding the complex zeros of determinants of the matrix-functions $\Phi_*(s)$ (and the symbols of the singular integral operators) has been proposed in [1].

In Appendix it is shown that the method developed makes it possible to solve not only Poisson's equations but also the equations of second order of a general form. It is only necessary that the method of integral (Fourier and Mellin) transforms could be applied to these equations. Hence, the results of [12] and this paper completely solve such problems under arbitrary boundary conditions.

Appendix

Consider similar problems for the following equations:

$$
\text{(A.1)}\qquad
\begin{aligned}
&v_i \frac{\partial^2 u_i}{\partial x_1^2} + \frac{\partial}{\partial x_2}\mu_i \frac{\partial}{\partial x_2} u_i = -W_i, && (x_1, x_2) \in \Omega_i,\\
&v_j^+ \frac{\partial}{r\partial r} r \frac{\partial u_j^+}{\partial r} + \frac{\partial}{r^2\partial\theta}\mu_j^+ \frac{\partial}{\partial\theta} u_j^+ = -W_j^+, && (r,\theta) \in \Omega_j^+,\\
&v_k^- \frac{\partial}{r\partial r} r \frac{\partial u_k^-}{\partial r} + \frac{\partial}{r^2\partial\theta}\mu_k^- \frac{\partial}{\partial\theta} u_k^- = -W_k^-, && (r,\theta) \in \Omega_k^-,
\end{aligned}
$$

instead of the equations (2.1). Here v_i, $\mu_i = v_i$, $\mu_i(x_2)$, $v_j^\pm, \mu_j^\pm = v_j^\pm, \mu_j^\pm(\theta)$, are known bounded positive functions. Without any loss of generality we can assume that:

$$
\text{(A.2)}\qquad v_i, \mu_i \in \mathbf{C}^2(y_{i-1}, y_i), \qquad v_j^+, \mu_j^+ \in \mathbf{C}^2(\theta_{j-1}^+, \theta_j^+), \qquad v_k^-, \mu_k^- \in \mathbf{C}^2(\theta_{k-1}^-, \theta_k^-),
$$

and they can be extended to closed intervals.

All external and internal boundary conditions are prescribed in (2.2)–(2.9). Such problems can be solved by using the mentioned method. We shall find in this Appendix only the necessary conditions which make it possible to use the formulae given in [12] (Appendix A) in order to obtain the equations similar to (3.1).

Applying the Fourier and Mellin transforms in the corresponding regions we obtain:

$$
\text{(A.3)}\qquad
\begin{aligned}
&-\lambda^2 v_i \overline{u}_i + \frac{\partial}{\partial x_2}\mu_i \frac{\partial}{\partial x_2}\overline{u}_i = -\overline{W}_i, && \lambda \in \mathbb{R}, \quad x_2 \in (y_{i-1}, y_i),\\
&v_j^+ s^2 \widetilde{u}_j^+ + \frac{\partial}{\partial\theta}\mu_j^+ \frac{\partial}{\partial\theta}\widetilde{u}_j^+ = -\widetilde{W}_j^+, && 0 < \Re s < \gamma_1, \quad \theta \in (\theta_{j-1}^+, \theta_j^+),\\
&v_k^- s^2 \widetilde{u}_k^- + \frac{\partial}{\partial\theta}\mu_k^- \frac{\partial}{\partial\theta}\widetilde{u}_k^- = -\widetilde{W}_k^-, && 0 < \Re s < \gamma_1, \quad \theta \in (\theta_{k-1}^-, \theta_k^-).
\end{aligned}
$$

Let $p_i^\pm(\lambda, x_2)$, $q_j^\pm(s,\theta)$, $r_k^\pm(s,\theta)$ be the linear independent solutions of the corresponding homogeneous equations (A.3). Besides, these functions can be chosen so that they will be even functions with respect to the new variables (λ, and s). Consider in details the solutions of the first equations.

From the VKB method [5] the behaviour of the functions $p_i^{\pm}(\lambda, x_2)$ for large values of the parameter λ can be justified:

$$\text{(A.4)} \qquad p_i^{\pm}(\lambda, x_2) = \frac{1}{\sqrt[4]{v_i(x_2)\mu_i(x_2)}} \cdot \exp\left[\pm|\lambda| \int\limits_{y_{i-1}}^{x_2} \sqrt{\frac{v_i(\xi)}{\mu_i(\xi)}} d\xi\right] \left[1 + O(|\lambda|^{-1})\right], \qquad \lambda \to \infty,$$

uniformly with respect to $x_2 \in [y_{i-1}, y_i]$. These solutions can be found, for example, from the following initial (Cauchy) conditions:

$$p_i^{\pm}(\lambda, y_{\pm}) = \frac{B_i^{\pm}(\lambda)}{\sqrt[4]{v_i\mu_i}}\Big|_{y_{\pm}}, \qquad \frac{\partial}{\partial x_2} p_i^{\pm}(\lambda, y_{\pm}) = \frac{B_i^{\pm}(\lambda)}{\sqrt[4]{v_i\mu_i}} \left[\pm|\lambda|\sqrt{\frac{v_i}{\mu_i}} - \frac{(v_i\mu_i)'}{4v_i\mu_i}\right]\Big|_{y_{\pm}},$$

where $y_+ = y_{i-1}$, $y_- = y_i$, but

$$B_i^+(\lambda) \equiv 1, \qquad B_i^-(\lambda) = \exp\left[-|\lambda| \int\limits_{y_{i-1}}^{y_i} \sqrt{\frac{v_i(\xi)}{\mu_i(\xi)}} d\xi\right].$$

We can also obtain asymptotic expansions of these functions for small values of λ:

$$\text{(A.5)} \qquad p_i^{\pm}(\lambda, x_2) = \frac{B_i^{\pm}(\lambda)}{\sqrt[4]{v_i\mu_i}}\Big|_{y_{\pm}} \left[1 - \mu_i \left[\frac{(v_i\mu_i)'}{4v_i\mu_i} \mp|\lambda|\sqrt{\frac{v_i}{\mu_i}}\right]\Big|_{y_{\pm}} \int\limits_{y_{\pm}}^{x_2} \frac{d\xi}{\mu_i(\xi)}\right] + O(\lambda^2), \qquad \lambda \to 0.$$

Consequently, the functions $p_i^{\pm}(\lambda, x_2)$ are absolutely continuous near points $(0, x_2)$, and are sufficiently smooth in any other points (λ, x_2) from the corresponding region ($|\lambda| \in \mathbb{R}_+$, $x_2 \in [y_{i-1}, y_i]$).

Now, we can write the solutions $\overline{u}_i(\lambda, x_2)$ of the first equations (A.1):

$$\text{(A.6)} \qquad \overline{u}_i(\lambda, x_2) = A_+ p_i^+(\lambda, x_2) + A_- p_i^-(\lambda, x_2) - p_i^+(\lambda, x_2) \int\limits_{x_2}^{y_i} \frac{p_i^-(\lambda, \xi)\overline{W}_i(\lambda, \xi)}{\mu_i(\xi)W(p_i^+, p_i^-)(\lambda, \xi)} d\xi - p_i^-(\lambda, x_2) \int\limits_{y_{i-1}}^{x_2} \frac{p_i^+(\lambda, \xi)\overline{W}_i(\lambda, \xi)}{\mu_i(\xi)W(p_i^+, p_i^-)(\lambda, \xi)} d\xi,$$

where $W(p_i^+, p_i^-)(\lambda, x_2)$ is the corresponding Wronskian.

Following [12], denote functions

(A.7)
$$\mathbf{p}_t^i(\lambda) = \mu_i \frac{\partial}{\partial x_2} \overline{u}_{i|\Gamma_i}, \qquad \mathbf{u}_t^i(\lambda) = \overline{u}_{i|\Gamma_i},$$
$$\mathbf{p}_b^i(\lambda) = \mu_i \frac{\partial}{\partial x_2} \overline{u}_{i|\Gamma_{i-1}}, \qquad \mathbf{u}_b^i(\lambda) = \overline{u}_{i|\Gamma_{i-1}}, \qquad i = 1, 2, ..., n.$$

Then, substituting (A.6) in (A.7), and eliminating the constants $A_\pm$ from these equations, we obtain the relations between functions $\mathbf{u}_{t(b)}^i$ and $\mathbf{p}_{t(b)}^i$ in the form:

(A.8)
$$\begin{pmatrix} \mathbf{u}_t^i \\ \mathbf{u}_b^i \end{pmatrix} = \mathbf{R}_i(\lambda) \begin{pmatrix} \mathbf{p}_t^i \\ \mathbf{p}_b^i \end{pmatrix} + \begin{pmatrix} u_{t0}^i \\ u_{b0}^i \end{pmatrix}, \qquad \mathbf{R}_i(\lambda) = \begin{pmatrix} R_{tt}^i & R_{tb}^i \\ R_{bt}^i & R_{bb}^i \end{pmatrix},$$

where coefficients are calculated from the equations

$$\begin{pmatrix} u_{t0}^i \\ u_{b0}^i \end{pmatrix} = \left[\mathbf{R}_i(\lambda) \begin{pmatrix} \mu_i \frac{\partial}{\partial x_2} p_i^-(\lambda, y_i) & 0 \\ 0 & \mu_i \frac{\partial}{\partial x_2} p_i^+(\lambda, y_{i-1}) \end{pmatrix} - \begin{pmatrix} p_i^-(\lambda, y_i) & 0 \\ 0 & p_i^+(\lambda, y_{i-1}) \end{pmatrix} \right] \begin{pmatrix} L_i^+ \\ L_i^- \end{pmatrix},$$

(A.9)
$$L_i^\pm(\lambda) = \int_{y_{i-1}}^{y_1} \frac{p_i^\pm(\lambda, \xi) \overline{W}_i(\lambda, \xi)}{\mu_i(\xi) W(p_i^+, p_i^-)(\lambda, \xi)} d\xi,$$
$$R_{bt}^i(\lambda) = -\frac{W(p_i^+, p_i^-)(\lambda, y_{i-1})}{\mu_i(y_i) D_0^{(i)}(\lambda)},$$
$$R_{bb}^i(\lambda) = \frac{D^{(i)}(\lambda, y_{i-1}, y_i)}{\mu_i(y_{i-1}) D_0^{(i)}(\lambda)},$$
$$R_{tt}^i(\lambda) = -\frac{D^{(i)}(\lambda, y_i, y_{i-1})}{\mu_i(y_i)) D_0^{(i)}(\lambda)},$$
$$R_{tb}^i(\lambda) = \frac{W(p_i^+, p_i^-)(\lambda, y_i)}{\mu_i(y_{i-1})) D_0^{(i)}(\lambda)}.$$

Finally, the functions $D^{(i)}(\lambda, a, b)$, $D_0^{(i)}(\lambda)$ are expressed in terms of the solutions $p_i^\pm(\lambda, x_2)$:

$$D^{(i)}(\lambda, a, b) = p_i^+(\lambda, a) \frac{\partial}{\partial x_2} p_i^-(\lambda, b) - p_i^-(\lambda, a) \frac{\partial}{\partial x_2} p_i^+(\lambda, b),$$
$$D_0^{(i)}(\lambda) = \frac{\partial}{\partial x_2} p_i^+(\lambda, y_{i-1}) \frac{\partial}{\partial x_2} p_i^-(\lambda, y_i) - \frac{\partial}{\partial x_2} p_i^-(\lambda, y_{i-1}) \frac{\partial}{\partial x_2} p_i^+(\lambda, y_i).$$

Hence, we can use all the results of Appendix A [12] in order to obtain the functions $M_p(\lambda)$, $m_p^{\pm}(\lambda)$ in the first relation of (3.1). For some functions $v_i(x_2)$, $\mu_i(x_2)$, the mentioned solutions $p_i^{\pm}(\lambda, x_2)$ can be calculated exactly (see for example [17]). Anyway, the functions $p_i^{\pm}(\lambda, x_2)$, and consequently, all functions from (A.8) as well $M_p(\lambda)$, $m_p^{\pm}(\lambda)$ can be numerically calculated. Moreover, their asymptotics at the zero and infinity points with respect to the variable λ, which play an important role in the process of investigation of the systems of functional-difference equations, can be analytically determined.

$$\mathbf{R}_i(\lambda) = \frac{1}{|\lambda|}\begin{pmatrix} [v_i\mu_i(y_i)]^{-1/2} & -2B_i^-(\lambda)[v_i\mu_i(y_{i-1})]^{-1/2} \\ 2B_i^-(\lambda)[v_i\mu_i(y_i)]^{-1/2} & -[v_i\mu_i(y_{i-1})]^{-1/2} \end{pmatrix} \times\left(1 + O\left(\frac{1}{|\lambda|}\right)\right), \qquad \lambda \to \infty,$$

$$u_{t0}^i,\ u_{b0}^i = o(|\lambda|^{-1}B_i^-(\lambda)), \qquad \lambda \to \infty;$$

$$\mathbf{R}_i(\lambda) = \frac{1}{\lambda^2}\left[\int\limits_{y_{i-1}}^{y_i} v_i(\xi)\,d\xi\right]^{-1}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + O(1), \qquad \lambda \to 0,$$

$$u_{t0}^i,\ u_{b0}^i(\lambda) = \frac{1}{\lambda^2}\left[\int\limits_{y_{i-1}}^{y_i} v_i(\xi)d\xi\right]^{-1}\int\limits_{y_{i-1}}^{y_i}\overline{W}_i(0,\xi)\,d\xi + O\left(\frac{1}{|\lambda|}\right), \qquad \lambda \to 0.$$

In conclusion let us note that we can always obtain the solutions $p_i^{\pm}(\lambda, x_2)$ satisfying the relations (A.5), and belonging to the class $\mathbf{C}^{\infty}(\mathbb{R}\times(y_{i-1}, y_i))$ by correcting the Cauchy data (A.4). But this makes no sense, because the matrix-function $\mathbf{R}_i(\lambda)$ has always the singularity in zero point, and does not depend on the choice of the solutions $p_i^{\pm}(\lambda, x_2)$.

In the wedge regions the relations similar to (A.8) between Mellin transformations of the solution and the tractions are constructed in a similar manner. To this end it is sufficient to replace the corresponding functions $v_i(x_2)$, $\mu_i(x_2)$ by $v_j^{\pm}(\theta)$, $\mu_j^{\pm}(\theta)$; to substitute new variable $\lambda = is$; and to consider separately the real and the imaginary parts of the solutions. The corresponding results will not be presented here.

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