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Non-uniform stagnant motions of materially non-uniform simple fluids

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NON-UNIFORM STAGNANT motions of materially non-uniform (inhomogeneous) incompressible fluids are reconsidered in greater detail. These motions may be used in many practical situations, such as fibre spinning and drawing processes. It is shown that the corresponding constitutive equations are very similar to those describing motions with constant stretch history or, in particular, steady extensional flows.

1. Introduction

THERE ARE AT LEAST three reasons for reconsidering non-uniform motions of materially non-uniform (inhomogeneous) simple fluids. The first reason is connected with pretty weak interest of the researchers involved either in the continuum theories or in the rheology of polymeric liquids. Existing references are rather devoted to what may be called inhomogeneities (dislocations, aeolotropy etc.) in materially uniform simple bodies (cf. [1]). The second reason results from serious needs for such considerations in the rheology of polymers when the material non-uniformity may be caused by a sensitivity of material properties to various temperature, viscosity, structure, etc. variations in the flows considered. The third reason, but not of minor importance, is the fact that the Referees of my previous papers on the necking phenomenon in fibre spinning processes [2, 3] had some doubts about the possibility of applying the constitutive equations in a form very similar to that describing uniform steady elongations of incompressible simple fluids [4].

In 1962 COLEMAN and NOLL discussed the class of substantially stagnant motions [5] or motions with constant stretch history (MCSH) [6].

According to Noll's definition, a motion is called a MCSH if, and only if, relative to a fixed reference configuration at time 0, the deformation gradient at any time τ is given by

$$\mathbf{F}_0(\tau) = \mathbf{Q}(\tau)\exp(\tau\mathbf{M}), \qquad \mathbf{Q}(0) = \mathbf{1}, \tag{1.1}$$

where $\mathbf{Q}(\tau)$ is an orthogonal tensor and $\mathbf{M}$ is a constant tensor such that $\mathbf{M} = \kappa\mathbf{N}_0$, $|\mathbf{N}_0| = 1$, and κ a constant parameter. The above definition shows that in all MCSH, the history of the relative deformation tensor is one and the same function of $t - \tau$ for all current instants t.

Moreover, it results from WANG'S theorem [7] that in all MCSH, the extra-stress tensor can be expressed as an isotropic tensor function of at most first three

Rivlin – Ericksen kinematic tensors, i.e.

$$\mathbf{T}_0(t) = \mathbf{h}(\mathbf{A}_1(t), \mathbf{A}_2(t), \mathbf{A}_3(t),), \qquad \operatorname{tr} \mathbf{T}_E = 0, \tag{1.2}$$

where by definition

$$\mathbf{A}_1 = \mathbf{L}_1^T + \mathbf{L}_1, \qquad \mathbf{A}_{n+1} = \dot{\mathbf{A}}_n + \mathbf{A}_n \mathbf{L}_1 + \mathbf{L}_1^T \mathbf{A}_n, \qquad n \geq 1, \tag{1.3}$$

and the velocity gradient amounts to

$$\mathbf{L}_1(t) = \dot{\mathbf{F}}_0(t)\mathbf{F}_0^{-1}(t) = \dot{\mathbf{Q}}(t)\mathbf{Q}^T(t) + \mathbf{Q}(t)\mathbf{M}\,\mathbf{Q}^T(t). \tag{1.4}$$

In the present paper we generalize the above results for the case of non-uniform stagnant motions (hereafter called NUSM) of materially non-uniform (inhomogeneous) incompressible simple fluids. It is shown that the corresponding constitutive equations are very similar in form to those valid for MCSH.

2. Non-uniform stagnant motions (NUSM)

Consider a more general class of motions for which the deformation gradient at any time τ, relative to a configuration at time 0 is of the form:

$$\mathbf{F}_0(\mathbf{X}, \tau) = \mathbf{Q}(\mathbf{X}, \tau) \exp(\tau \mathbf{M}(\mathbf{X})), \qquad \mathbf{Q}(\mathbf{X}, 0) = \mathbf{1}, \tag{2.1}$$

where $\mathbf{Q}(\mathbf{X}, \tau)$ is an orthogonal tensor, and $\mathbf{M}(\mathbf{X})$ depends only on the position $\mathbf{X}$ of a particle X in an arbitrarily chosen reference configuration $\boldsymbol{\kappa}$ (not necessarily at time 0). Thus, the non-uniformity of the quantities involved can be expressed either by $\mathbf{X}$ or X ($\mathbf{X} = \boldsymbol{\kappa}(X)$).

According to the definition (1.4), we obtain the following velocity gradient:

$$\mathbf{L}_1(\mathbf{X}, t) = \dot{\mathbf{Q}}(\mathbf{X}, t)\mathbf{Q}^T(\mathbf{X}, t) + \mathbf{L}(\mathbf{X}, t), \tag{2.2}$$

where

$$\mathbf{L}(\mathbf{X}, t) = \mathbf{Q}(\mathbf{X}, t)\mathbf{M}(\mathbf{X})\mathbf{Q}^T(\mathbf{X}, t), \tag{2.3}$$

is called the rotated parametric tensor (cf. [8]), and t denotes the current instant of time.

The deformation gradient, relative to a configuration at the current time t, amounts to

$$\begin{gathered}\mathbf{F}_t(\mathbf{X}, t-s) = \mathbf{F}_0(\mathbf{X}, \tau)\mathbf{F}_0^{-1}(\mathbf{X}, t) = \mathbf{Q}(\mathbf{X}, t-s) \exp(-s\mathbf{M}(\mathbf{X}))\mathbf{Q}^T(\mathbf{X}, t), \\ \tau = t - s, \qquad 0 \leq s < \infty,\end{gathered} \tag{2.4}$$

what leads to the following history of the relative deformation tensor (cf. [8]):

$$\mathbf{C}_t^t(\mathbf{X}, s) \equiv \mathbf{C}_t(\mathbf{X}, t - s) = \mathbf{F}_t^T \mathbf{F}_t = \exp(-s\mathbf{L}^T(\mathbf{X}, t)) \exp(-s\mathbf{L}(\mathbf{X}, t)). \tag{2.5}$$

In full analogy to the case of MCSH, we may ask what will happen if $\mathbf{L}_1(\mathbf{X})$ defined through Eq. (2.2) is steady (independent of time t) but non-uniform in space? The answer results from the following differential equation based on Eq. (1.4):

$$\frac{d}{d\tau}\mathbf{F}_0(\mathbf{X}, \tau) = \mathbf{L}_1(\mathbf{X})\mathbf{F}_0(\mathbf{X}, \tau), \tag{2.6}$$

with the initial condition: $\mathbf{F}_0(\mathbf{X}, 0) = \mathbf{1}$. The corresponding solution can be written as

$$\mathbf{F}_0(\mathbf{X}, \tau) = \exp(\tau \mathbf{L}_1(\mathbf{X})). \tag{2.7}$$

The above expression evidently belongs to the class (2.1) with $\mathbf{Q} \equiv \mathbf{1}$. It is obvious that for steady flows in an Eulerian sense

$$\dot{\mathbf{L}}_1(\mathbf{x}) = \mathbf{V}(\mathbf{x}) \cdot \nabla \mathbf{L}_1(\mathbf{x}), \tag{2.8}$$

where $\mathbf{V}$ is the velocity and ∇ denotes the gradient with respect to place $\mathbf{x}$.

It is worthwhile to mention that Noll's classification of MCSH based on the tensor $\mathbf{M}(\mathbf{X})$ (or $\mathbf{L}(\mathbf{X}, t)$) can be generalized to the case of NUSM. Therefore, in certain parts of a fluid, we may have the following classes of flows:

(I) non-uniform viscometric flow

$$\mathbf{M}^2 = \mathbf{0};$$

(II) non-uniform doubly-superposed viscometric flow

$$\mathbf{M}^2 \neq \mathbf{0}, \qquad \mathbf{M}^3 = \mathbf{0};$$

(III) non-uniform triply-superposed viscometric flow and extensional flow

$$\mathbf{M}^n \neq \mathbf{0} \qquad \text{for all} \quad n = 1, 2, \dots .$$

The non-uniform extensional flows, because of their technological validity, will be discussed separately in Sec. 4.

3. Constitutive equations of materially non-uniform (inhomogeneous) simple fluids

As mentioned at the beginning, in many practical situations, instead of solving the usually complex problems, it is more useful to assume *a priori* that unknown temperature, viscosity, structure, etc. distributions lead to a material non-uniformity (inhomogeneity). In other words, such a non-uniformity means that the mechanical properties of a fluid vary from particle to particle.

The constitutive equations of materially non-uniform incompressible simple fluids can be written in the form (cf. [9]):

$$\mathbf{T}_E(\mathbf{X}, t) = \mathop{\mathcal{H}}_{s=0}^{\infty}(\mathbf{C}_t^t(\mathbf{X}, s); \mathbf{X}), \tag{3.1}$$

where $\mathbf{T}_E$ is the non-uniform extra-stress tensor, and $\mathcal{H}$ denotes a constitutive functional. Such a definition is not in contradiction with the principles of determinism and local action. Equations (3.1) also satisfy the principle of objectivity (invariance with respect to the reference frame) since all the tensors involved are objective (cf. [8]).

For non-uniform stagnant motions (NUSM) defined by Eq. (2.10), after introducing Eq. (2.5) into Eq. (3.1) and taking into account the properties of tensor exponentials,

$$\exp \mathbf{A} = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{A}^n, \qquad (\mathbf{Q}\,\mathbf{A}\,\mathbf{Q}^T)^n = \mathbf{Q}\,\mathbf{A}^n\mathbf{Q}^T, \tag{3.2}$$

we arrive at

$$\mathbf{T}_E(\mathbf{X}, t) = \mathbf{h}(\mathbf{L}(\mathbf{X}, t); \mathbf{X}), \tag{3.3}$$

where $\mathbf{h}$ is an isotropic function of the tensor argument. In particular, if the rotated parametric tensor $\mathbf{L}(\mathbf{X})$ is a steady one, the particle position $\mathbf{X}$ may be replaced by its place in space $\mathbf{x}$. This leads to

$$\mathbf{T}_E(\mathbf{x}) = \mathbf{k}(\mathbf{L}(\mathbf{x}); \mathbf{x}). \tag{3.4}$$

Since for the motions considered (NUSM) the following relations are also valid:

$$\mathbf{A}_1 = \mathbf{L}^T + \mathbf{L}, \qquad \mathbf{A}_{n+1} = \mathbf{A}_n\mathbf{L} + \mathbf{L}^T\mathbf{A}_n, \qquad n \geq 1, \tag{3.5}$$

the corresponding representation theorem analogous to that derived by WANG [7] can easily be proved (cf. [8]). Thus, it can be shown that the extra-stress tensor in the most general case amounts to

$$\mathbf{T}_E(\mathbf{X}, t) = \mathbf{f}(\mathbf{A}_1(\mathbf{X}, t), \mathbf{A}_2(\mathbf{X}, t), \mathbf{A}_3(\mathbf{X}, t); \mathbf{X}), \tag{3.6}$$

where all the quantities depend on the particle position $\mathbf{X}$. Similarly to the case of MCSH, a knowledge of the first two kinematic tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ is sufficient to determine $\mathbf{C}_t^t(\mathbf{X}, s)$ uniquely, if either $\mathbf{A}_1$ has three different eigenvalues, or two of them are equal but differ from the third one and, moreover, $[\mathbf{A}_2] = [\mathbf{A}_1^2]$ in the same basis in which $\mathbf{A}_1$ has a diagonal form. Such a generalization is possible since the proof of the theorem is based on the geometry of matrices involved, independently of whether they are functions of $\mathbf{X}$ or not.

4. The case of non-uniform steady extensional flows

The non-uniform steady extensional motions, under the assumption of quasi-elongational approximation (cf. [2, 3]), may be useful as applied to various fibre spinning and drawing processes [10]. For example, any temperature distribution may lead to observable material non-uniformity (inhomogeneity). We will show that the above motions are particular cases of those described by Eq. (2.1) (NUSM).

To this end, consider the following exponential deformation gradient at time τ

$$\mathbf{F}_0(\mathbf{X}, \tau) = \exp(\tau\mathbf{M}(\mathbf{X})), \tag{4.1}$$

where $\mathbf{X}$, like in Sec. 2, denotes the particle position at an arbitrary reference configuration, and the time-independent tensor $\mathbf{M}(\mathbf{X})$ is of a diagonal form. Instead of Eqs. (2.2), (2.4) and (2.5) we arrive at

$$\mathbf{L}_1(\mathbf{X}) = \frac{\partial}{\partial\tau}\mathbf{F}_t(\mathbf{X}, \tau)\Big|_{\tau=t} = \mathbf{L}(\mathbf{X}) = \mathbf{M}(\mathbf{X}), \tag{4.2}$$

$$\mathbf{F}_t(\mathbf{X}, t-s) = \exp(-s\mathbf{M}(\mathbf{X})), \qquad \tau = t-s, \qquad 0 \le s < \infty, \tag{4.3}$$

and

$$\mathbf{C}_t^t(\mathbf{X}, s) \equiv \mathbf{C}_t(\mathbf{X}, t-s) = \exp(-s\mathbf{L}^T(\mathbf{X}, t))\exp(-s\mathbf{L}(\mathbf{X}, t)). \tag{4.4}$$

Therefore, for the flows considered, the velocity gradient $\mathbf{L}_1(\mathbf{X})$ is equal to the parametric tensor $\mathbf{L}(\mathbf{X})$ and also to $\mathbf{M}(\mathbf{X})$.

Now, the constitutive equations (3.1) lead to

$$\mathbf{T}_E(\mathbf{X}) = \mathbf{g}(\mathbf{L}(\mathbf{X}); \mathbf{X}), \tag{4.5}$$

where $\mathbf{g}$ is an isotropic function of the tensor argument, or to Eq. (3.4), if the spatial description of material non-uniformity is used.

Since for general extensional flows with diagonal $\mathbf{A}_1$ we have

$$\mathbf{A}_n = (\mathbf{A}_1)^n = (2\mathbf{L})^n, \qquad n \ge 1, \tag{4.6}$$

we can write instead of Eq. (4.5)

$$\mathbf{T}_E(\mathbf{X}) = \mathbf{k}(\mathbf{A}_1(\mathbf{X}); \mathbf{X}). \tag{4.7}$$

After taking into account the relevant representation of an isotropic tensor function of one symmetric tensor argument (cf. [8, 9]), we finally obtain

$$\mathbf{T}_E(\mathbf{X}) = \beta_1\mathbf{A}_1(\mathbf{X}) + \beta_2\mathbf{A}_1^2(\mathbf{X}), \qquad \operatorname{tr}\mathbf{A}_1 = 0, \tag{4.7}$$

where the material functions β_1 and β_2, depending on the invariants of $\mathbf{A}_1$ are also explicit functions of the position $\mathbf{X}$ (or the place $\mathbf{x}$ in steady flows).

5. Conclusions

Non-uniform stagnant motions (NUSM) are some generalization of the well known motions with constant stretch history (MCSH) defined by Coleman and Noll. In the case of materially non-uniform incompressible simple fluids, the constitutive equations take a form very similar to that valid for MCSH.

In the case of non-uniform steady extensional flows the corresponding constitutive equations simplify considerably and, of course, are independent of time. Those equations may be used in many practically important quasi-elongational flows such as fibre spinning and drawing processes.

References

1. W. NOLL, *Materially uniform simple bodies with inhomogeneities*, Arch. Rat. Mech. Anal., **27**, 1, 1967.
2. S. ZAHORSKI, *Necking in non-isothermal high-speed spinning with radial viscosity variation*, J. Non-Newtonian Fluid Mech., **50**, 65, 1993.
3. S. ZAHORSKI, *Necking in non-isothermal high-speed spinning as a problem of sensitivity to external disturbances*, J. Non-Newtonian Fluid Mech., (to appear).
4. B.D. COLEMAN and W. NOLL, *Steady extension of incompressible simple fluids*, Phys. Fluids, **5**, 840, 1962.
5. B.D. COLEMAN, *Kinematical concepts with applications in the mechanics and thermodynamics of incompressible viscoelastic liquids*, Arch. Rat. Mech. Anal., **9**, 273, 1962.
6. W. NOLL, *Motions with constant stretch history*, Arch. Rat. Mech. Anal., **11**, 97, 1962.
7. C.C. WANG, *A representation theorem for the constitutive equation of a simple material in motions with constant stretch history*, Arch. Rat. Mech. Anal., **20**, 329, 1965.
8. S. ZAHORSKI, *Mechanics of viscoelastic fluids*, Martinus Nijhoff, The Hague 1982.
9. C. TRUESDELL, *A first course in rational continuum mechanics*, The Johns Hopkins University, Baltimore 1982.
10. A. ZIABICKI, *Fundamentals of fibre formation. The science of fibre spinning and drawing*, Wiley, London 1976.

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