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Symmetrization of a heat conduction model for a rigid medium

W. DOMAŃSKI, T. F. JABŁOŃSKI and W. KOSIŃSKI (WARSZAWA)

THE SYMMETRIZATION of the equations of a heat conduction model for a rigid medium in time and three space dimensions is performed. The general symmetrizability condition is formulated in terms of the entropy function. Examples of particular models (e.g. Debye's model) are discussed.

1. Introduction

MOST OF THE KNOWN DYNAMIC (non-equilibrium) problems in nonlinear continuum mechanics and thermodynamics lead to quasi-linear hyperbolic systems of partial differential equations. The problem of well-posedness, i.e. existence, uniqueness and continuous dependence (stability) of a solution on the initial data, is fundamental for any system of equations. It is well known [1, 2] that Cauchy's initial-value problem for symmetric hyperbolic quasi-linear system is locally well-posed in the Sobolev space H^s, with $s \geq n + 1$, where n is a number of space variables. The quasi-linear systems of continuum mechanics usually are not formulated in symmetric forms. To make use of the above well-posedness result, it is desirable to transform such systems into symmetric forms, by the appropriate change of the unknown variables.

The aim of this paper is to symmetrize the equations describing a non-equilibrium heat conduction problem in a rigid conductor governed by a modified Fourier law. The system of equations is of the second order in the scalar variable β, called internal state variable (or a semi-empirical temperature), and of the first order in the absolute temperature θ. In the general 3D case, this system can be transformed into the first order system in five unknowns. We symmetrize this system with the help of entropy function using some results of FRIEDRICHS, BOILLAT, RUGGERI and STRUMIA [3–5]. Instead of deriving the exact form of the entropy function from thermodynamics, we postulate the family of suitably chosen entropy-like functions that are then used to get the new dependent variables (the main fields).

In order to pick up the entropy from our family of postulated functions we formulate a general symmetrizability condition. It turns out that this condition is in fact the model compatibility condition which, on the other hand, can be obtained from the second law of thermodynamics. This symmetrizability condition can be easily fulfilled not only in the Debye's model, which we analyze in details, but also under some more general assumptions.

2. Model with semi-empirical temperature

Recently in a series of papers [6–9] a thermodynamic, phenomenological theory of heat conduction with finite wave speed has been developed and applied to thermal wave propagation problems, mostly 1D. The theory is based on the concept of a gradient generalization of the internal state variable approach, in which the gradient of a scalar internal state variable β (called a semi-empirical temperature) influences the response of the material at hand. The quantity β cannot be measured directly. Here it is considered as a potential, with the analogy to the classical heat conduction Fourier law. In the new model the heat flux is proportional to the gradient of β, instead of to the gradient of the classical absolute temperature θ.

In the model considered (cf. [7, 9]) we assume that the evolution of β is governed by the following equation:

$$\frac{\partial \beta}{\partial t} = f(\theta, \beta) = f_1^*(\theta) + f_2(\beta) \tag{2.1}$$

(with f_1^*, f_2 being real functions such that $df_2/d\beta < 0$), while the energy balance law reads:

$$\frac{\partial \varrho\varepsilon}{\partial t} + \operatorname{div} \mathbf{q}^* = \varrho\, r, \tag{2.2}$$

where (¹) ϱ is the mass density, ε – the specific internal energy, r – the body heat supply, and $\mathbf{q}^*$ is the heat flux vector. We also assume that the second law of thermodynamics

$$\frac{\partial \varrho\eta^*}{\partial t} + \operatorname{div}\frac{\mathbf{q}^*}{\theta} \geq \frac{\varrho\, r}{\theta} \tag{2.3}$$

is satisfied, with η^* being an entropy. Moreover, in our model we make the two additional simplifying assumptions:

(A.1) $\mathbf{q}^*$ depends linearly on $\nabla\beta$,

(A.2) ε is a function of θ only.

From the second law of thermodynamics (2.3), under the assumption (A.1) we can express the heat flux as:

$$\mathbf{q}^* = -\alpha^*(\theta)\,\nabla\beta, \tag{2.4}$$

where α^* is a positive function of dimension of the thermal conductivity coefficient. Also from (2.3) and from the assumptions (A.1), (A.2) we can derive the

(¹) Throughout this paper we use dimensionless variables. However, the following units have been assumed: temperature (θ and β) in K, length in cm, time in μs, speed in cm/μs, energy in J.

following form of the entropy function:

$$\eta^*(\theta, \nabla\beta) := \eta_e^*(\theta) - \frac{1}{2}c\,|\nabla\beta|^2, \tag{2.5}$$

with c being a positive constant.

3. Basic equations in a quasi-linear form

In order to express the system (2.1), (2.2), (2.4) in the conservative form we introduce the following vector of new dependent variables $\mathbf{u}$:

$$\mathbf{u}(x,t) = [e,\, \mathbf{q},\, \beta], \qquad x \in \mathbb{R}^3, \quad t \in \mathbb{R}, \quad \mathbf{q} = [q_1,\, q_2,\, q_3],$$

where $e = \varrho\varepsilon$ is internal energy and $\mathbf{q} = -\nabla\beta$ is the rescaled heat flux vector (cf. (2.4)). Moreover, we introduce the flux matrix $\mathbf{F}(\mathbf{u})$ and the vector of external influences $\mathbf{b}(\mathbf{u})$ as:

$$\mathbf{F}(\mathbf{u}) = \begin{bmatrix} \alpha(e)\,\mathbf{q} \\ f_1(e)\,\mathbf{I}_3 \\ \mathbf{0} \end{bmatrix}, \qquad \mathbf{b}(\mathbf{u}) = \left[\varrho\, r,\, \frac{df_2}{d\beta}\,\mathbf{q},\, f_1(e) + f_2(\beta)\right],$$

where $\mathbf{I}_3$ is the 3×3 identity matrix, α is a positive function of dimension of the thermal conductivity coefficient, and the function f_1 is f_1^* from (2.1) expressed as a function of e. In what follows we denote:

$$\mathbf{div}\,\mathbf{A} = \nabla\mathbf{A}^T$$

with $\nabla = \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right]$ and $\mathbf{A}$ being an arbitrary 3-column matrix. Now, after some calculation, we can describe the process (2.1), (2.2), (2.4) of the heat conduction in a rigid medium in the form of the following first order system of balance laws:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{div}\,\mathbf{F}(\mathbf{u}) = \mathbf{b}(\mathbf{u}). \tag{3.1}$$

The quasi-linear form of this system is:

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{i=1}^{3} \mathbf{A}_i(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{b}(\mathbf{u}) \tag{3.2}$$

with:

$$\mathbf{A}_i(\mathbf{u}) = \begin{bmatrix} \frac{d\alpha}{de}\,q_i & \alpha(e)\,\boldsymbol{\xi}_i \\ \frac{df_1}{de}\,\boldsymbol{\xi}_i^T & \mathbf{0}_4 \end{bmatrix}, \qquad i = 1, 2, 3,$$

where $\mathbf{0}_4$ is the 4×4 null matrix and $\boldsymbol{\xi}_1 = [1, 0, 0, 0]$, $\boldsymbol{\xi}_2 = [0, 1, 0, 0]$, $\boldsymbol{\xi}_3 = [0, 0, 1, 0]$.

4. New dependent variables

In order to symmetrize our quasi-linear system (3.2) we make use of a well known fact [10, 3, 5] that a system of hyperbolic first order balance laws can be symmetrized, provided that it is equipped with a convex entropy function satisfying supplementary conservation law. More precisely, such a system of balance laws becomes symmetric in the Friedrichs's sense when one takes the gradient components of the entropy function η as the new dependent variables (main fields) $\mathbf{v}$:

$$\mathbf{v} = \mathrm{grad}_{\mathbf{u}}\, \eta.$$

In the case of our system (3.1), having in mind the formula (2.5), we take as the candidate for the entropy η the family of functions that can be expressed in the following form:

$$\eta(e, \mathbf{q}) := \eta_e(e) + \frac{1}{2} c_1\, \mathbf{q} \cdot \mathbf{q}, \tag{4.1}$$

where $c_1 > 0$ and η_e is the so-called equilibrium entropy that will be detailed in the next section. Consequently, we obtain:

$$\mathrm{grad}_{\mathbf{u}}\, \eta = \left[\frac{d\eta_e}{de},\, c_1 q_1,\, c_1 q_2,\, c_1 q_3,\, 0\right].$$

Since semi-empirical temperature β is not involved in the divergence term in the quasi-linear system (3.2), we are free to put an arbitrary function as v_5 (e.g.: $v_5 = c_2\beta$ with $c_2 = \mathrm{const}$). Hence, our main fields $\mathbf{v}$ are:

$$\mathbf{v} = [v_1, \ldots, v_5] = \left[\frac{d\eta_e}{de},\, c_1 q_1,\, c_1 q_2,\, c_1 q_3,\, c_2\beta\right]. \tag{4.2}$$

Using the main fields (4.2) we obtain the symmetrizing matrix $\mathbf{H}$ for our quasi-linear system (3.2) in the form:

$$\mathbf{H} = \mathbf{grad}_{\mathbf{u}}\, \mathbf{v} = \mathrm{diag}\left[\frac{d^2\eta_e}{de^2},\, c_1,\, c_1,\, c_1,\, c_2\right], \tag{4.3}$$

where $\mathbf{grad}_{\mathbf{u}}\, \mathbf{v} = [\mathrm{grad}_{\mathbf{u}} v_1, \mathrm{grad}_{\mathbf{u}} v_2, \ldots, \mathrm{grad}_{\mathbf{u}} v_5]^T$ and diag$[\cdot]$ denotes a diagonal matrix with the diagonal $[\cdot]$. We can choose an appropriate sign of the constant c_2 to make our symmetrizing matrix $\mathbf{H}$ positive definite.

5. The symmetrizability condition

The matrix $\mathbf{H}$ of the form (4.3) symmetrizes our quasi-linear system (3.2) if and only if, by the definition, the following matrices $\mathbf{B}_i$, $i = 1, 2, 3$ are symmetric:

$$\mathbf{B}_i = \mathbf{H}\cdot\mathbf{A}_i = \begin{bmatrix} q_i \dfrac{d\alpha}{de}\dfrac{d^2\eta_e}{de^2} & \alpha \dfrac{d^2\eta_e}{de^2}\boldsymbol{\xi}_i \\ c_1 \dfrac{df_1}{de}\boldsymbol{\xi}_i^T & \mathbf{0} \end{bmatrix}, \qquad i = 1, 2, 3. \tag{5.1}$$

Since the equalities of the corresponding off-diagonal elements of the matrices $\mathbf{B}_i$ do not depend on i, the condition (5.1) is reduced in fact to a single, general symmetrizability condition in the form:

$$c_1 = \alpha(e)\,\frac{d^2\eta_e}{de^2}\Big/\frac{df_1}{de}\,. \tag{5.2}$$

We remind that c_1 is a constant appearing in our family of functions (4.1). It can be shown that c_1 evaluated from (5.2) coincides with the constant c from (2.5) which, on the other hand, is evaluated on the basis of the thermodynamical considerations. It is also worth mentioning that under our assumptions the equilibrium entropy η_e is a convex function of e, provided that $df_1/de > 0$.

6. Specification of the equilibrium entropy

Under our assumptions the equilibrium entropy η_e^*, as a function of the classical temperature θ, is the derivative of the Helmholtz free energy ψ_1:

$$\eta_e^*(\theta) := -\frac{d\psi_1}{d\theta}\,, \tag{6.1}$$

where ψ_1 satisfies the following ordinary differential equation:

$$-\theta\,\frac{d\psi_1}{d\theta} + \psi_1 = \frac{1}{\varrho}\,\widehat{e}(\theta)$$

with $\widehat{e}$ being e as a function ([2]) of θ. Hence, ψ_1 takes the form:

$$\psi_1(\theta) = c_0\,\theta - \frac{\theta}{\varrho}\int_0^\theta \frac{\widehat{e}(s)}{s^2}\,ds, \qquad c_0 = \text{const.}$$

Substituting the solution ψ_1 into our postulate (6.1) we obtain the the equilibrium entropy as the following function of θ:

$$\eta_e^*(\theta) = \frac{\widehat{e}(\theta)}{\varrho\,\theta} + \frac{1}{\varrho}\int_0^\theta \frac{\widehat{e}(s)}{s^2}\,ds - c_0. \tag{6.2}$$

([2]) In order to distinguish between a variable (e.g. e) and the same variable treated as a function of another variable (e.g. e as a function of θ), introduce the symbol $\widehat{\ }$ to denote the function (e.g. $\widehat{e}(\theta)$).

All that we need now is to express the equilibrium entropy as the function of the internal energy e only. To this end we introduce the specific heat c_v that relates, by the definition, e to θ in the following way:

$$c_v = \frac{1}{\varrho}\frac{d\widehat{e}(\theta)}{d\theta}. \tag{6.3}$$

Hence, e as a function of θ reads:

$$\widehat{e}(\theta) = \varrho \int c_v(\theta)\,d\theta.$$

Under the assumption that the specific heat c_v is a positive function of θ, so that $\widehat{e}(\theta)$ is monotonic, the inverse function

$$\widehat{\theta} : e \to \theta, \qquad \widehat{\theta}(e) = \widehat{e}^{-1}(e)$$

exists and the equilibrium entropy η_e as the function of the internal energy e takes the following form (cf. (6.2)):

$$\eta_e(e) = \eta_e^*(\widehat{\theta}(e)) = \frac{e}{\varrho\,\widehat{\theta}(e)} + \frac{1}{\varrho}\int_0^{\widehat{\theta}(e)} \frac{\widehat{e}(s)}{s^2}\,ds - c_0\,. \tag{6.4}$$

In terms of such $\eta_e(e)$, our general symmetrizability condition (5.2) takes the form:

$$c_1 = -\frac{\alpha(e)\,\dfrac{d\widehat{\theta}(e)}{de}}{\varrho\,(\widehat{\theta}(e))^2\,\dfrac{df_1}{de}}, \tag{6.5}$$

and the symmetrized matrices $\mathbf{B}_i$ are:

$$\mathbf{B}_i = -\frac{1}{\varrho\,\widehat{\theta}^2}\frac{d\widehat{\theta}}{de}\begin{bmatrix} q_i\dfrac{d\alpha}{de} & \alpha\,\boldsymbol{\xi}_i \\ \alpha\,\boldsymbol{\xi}_i^T & \mathbf{0}\end{bmatrix}, \qquad i = 1,2,3. \tag{6.6}$$

7. Specification of f_1 for various $\alpha(e)$

We may reformulate the symmetrizability condition (6.5) to obtain, after integration, the general form of the function f_1 such that it allows the symmetrization by our method. The function f_1 in this form is expressed in terms of $\alpha(e)$ and the constant c_1:

$$f_1(e) = \frac{\alpha(e)}{c_1\,\varrho\,\widehat{\theta}(e)} - \frac{1}{c_1\varrho}\int \frac{1}{\widehat{\theta}(e)}\frac{d\alpha(e)}{de}\,de\,. \tag{7.1}$$

Now we specify f_1 for two different functions $\alpha(e)$:

CASE 1

$$\alpha(e) = \alpha_0 (\widehat{\theta}(e))^2, \qquad \alpha_0 > 0. \tag{7.2}$$

Then the function f_1 has the form:

$$f_1(e) = -\frac{\alpha_0 \widehat{\theta}(e)}{c_1 \varrho}.$$

CASE 2

$$\alpha(e) = -\alpha_0 (\widehat{\theta}(e) - \theta_1)(\widehat{\theta}(e) - \theta_2), \qquad \alpha_0 > 0, \quad \theta_1\theta_2 < 0. \tag{7.3}$$

Then the function f_1 is the following:

$$f_1(e) = \frac{\alpha_0 \{\widehat{\theta}(e) - (\theta_1 + \theta_2)\ln(\widehat{\theta}(e))\}}{c_1 \varrho} - \frac{\alpha_0 \theta_1\theta_2}{c_1 \varrho \widehat{\theta}(e)}.$$

8. The example: Debye's model

8.1. Arbitrary $\alpha(e)$

Our general symmetrization formulas can be further specified if the explicit form of the θ-dependence of the specific heat c_v is assumed. For example, in Debye's model with

$$c_v = 4 c_{v0} \theta^3, \qquad c_{v0} > 0, \tag{8.1}$$

the inverse function $\widehat{\theta}$ becomes:

$$\widehat{\theta}(e) = \left(\frac{e}{c_{v0} \varrho}\right)^{1/4}, \tag{8.2}$$

the symmetrizability condition (5.2), (6.5) reads:

$$c_1 = -\frac{\alpha(e)}{4 \dfrac{df_1}{de}} \left(\frac{c_{v0}}{e^5 \varrho^3}\right)^{1/4} \tag{8.3}$$

and the equilibrium entropy $\eta_e(e)$ takes the following form (cf. (6.4)):

$$\eta_e(e) = \left(\frac{4 c_{v0} e^3}{3 \varrho^3}\right)^{1/4} - (c_0 + c_{v0}/3). \tag{8.4}$$

8.2. Specified $\alpha(e)$

Let us recall that α is a positive function of dimension of the thermal conductivity coefficient. Now we specify the symmetrizability condition, symmetrized matrices $\mathbf{B}_i$, $i = 1, 2, 3$, and the function f_1 for two different $\alpha(e)$ taken from Sec. 7.

CASE 1 (cf. (7.2))

$$\alpha(e) = \alpha_0 (\widehat{\theta}(e))^2 = \alpha_0 \sqrt{\frac{e}{\varrho\, c_{v0}}}, \qquad \alpha_0 > 0. \tag{8.5}$$

Then the symmetrizability condition reads (cf. (5.2), (6.5), (8.3)):

$$c_1 = -\frac{\alpha_0}{4\dfrac{df_1}{de}} \left(\frac{1}{c_{v0}\, e^3\, \varrho^5}\right)^{1/4}, \tag{8.6}$$

the symmetrized matrices $\mathbf{B}_i$ are (cf. (6.6)):

$$\mathbf{B}_i = -\frac{\alpha_0}{8\,(c_{v0}\,\varrho^5)^{1/4}} \begin{bmatrix} q_i\, e^{-4/7} & 2\, e^{-4/3} \boldsymbol{\xi}_i \\ 2\, e^{-4/3} \boldsymbol{\xi}_i^T & \mathbf{0} \end{bmatrix}, \qquad i = 1, 2, 3, \tag{8.7}$$

and the function f_1 has the form (cf. (7.1)):

$$\begin{aligned} f_1(e) &= -\alpha_0 \left(\frac{e}{c_{v0}\, c_1\, \varrho^5}\right)^{1/4}, \\ f_1^*(\theta) &= -\frac{\alpha_0\, \theta}{c_1\, \varrho}. \end{aligned} \tag{8.8}$$

CASE 2 (cf. (7.3), (8.2))

$$\alpha(e) = -\alpha_0 \left(\left(\frac{e}{c_{v0}\,\varrho}\right)^{1/4} - \theta_1 \right) \left(\left(\frac{e}{c_{v0}\,\varrho}\right)^{1/4} - \theta_2 \right), \quad \alpha_0 > 0, \quad \theta_1\theta_2 < 0. \tag{8.9}$$

Then the symmetrizability condition reads (cf. (5.2), (6.5), (8.3)):

$$c_1 = \frac{\alpha_0}{4\dfrac{df_1}{de}} \frac{(\theta_1 (c_{v0}\,\varrho)^{1/4} - e^{1/4})\,(\theta_2 (c_{v0}\,\varrho)^{1/4} - e^{1/4})}{(c_{v0}\, e^5\, \varrho^5)^{1/4}}, \tag{8.10}$$

the symmetrized matrices $\mathbf{B}_i$ are (cf. (6.6)):

$$\mathbf{B}_i = \frac{\alpha_0}{16\,(c_{v0}\,\varrho^5)^{1/4}} \begin{bmatrix} b_{11} & b_{12} \boldsymbol{\xi}_i \\ b_{12} \boldsymbol{\xi}_i^T & \mathbf{0} \end{bmatrix}, \qquad i = 1, 2, 3, \tag{8.11}$$

where

$$b_{11} = -q_i e^{-2} \{(c_{v0}\varrho)^{1/4}(\theta_1 + \theta_2) - 2e^{1/4}\},$$
$$b_{12} = 4e^{-5/4}(\theta_1(c_{v0}\varrho)^{1/4} - e^{1/4})(\theta_2(c_{v0}\varrho)^{1/4} - e^{1/4}),$$

and the function f_1 has the form (cf. (7.1)):

$$(8.12)\quad \begin{aligned} f_1(e) &= \frac{\alpha_0}{c_1}\left\{\left(\frac{e}{(c_{v0}\varrho^5)}\right)^{1/4} - \theta_1\theta_2\left(\frac{c_{v0}}{e\varrho^3}\right)^{1/4} - \frac{(\theta_1+\theta_2)\ln(e^{1/4})}{\varrho}\right\},\\ f_1^*(\theta) &= \frac{\alpha_0}{\varrho c_1}\left\{\theta - \frac{\theta_1\theta_2}{\theta} - (\theta_1+\theta_2)\ln(\theta\sqrt[4]{\varrho c_{v0}})\right\}. \end{aligned}$$

9. Conclusions

The equations of a heat conduction model for a rigid medium in time and three space dimensions are analyzed. Using the internal energy, the heat flux vector and the semi-empirical temperature as the dependent variables, we formulate the conservative, and the quasi-linear hyperbolic forms of these equations.

We successfully symmetrize our quasi-linear system by introducing the family of suitably chosen entropy-like functions that are then used to obtain the new dependent variables, and by formulating additionally a general symmetrizability condition that allows us to specify the physically justified entropy function.

It turns out that this symmetrizability condition is in fact the model compatibility condition which, on the other hand, can be obtained from the second law of thermodynamics.

We illustrate our approach on a detailed example of the Debye's model with specified different forms of the thermal conductivity coefficients.

Our approach is effective when the classical temperature is an invertible function of the internal energy. Then we can always symmetrize our system of equations and the symmetrizing matrix is diagonal.

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POLISH ACADEMY OF SCIENCES
INSTITUTE OF FUNDAMENTAL TECHNOLOGICAL RESEARCH.

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