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Surface stress waves in a nonhomogeneous elastic half-space Part I. General results based on spectral analysis Existence and analyticity theorems

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EXISTENCE of surface waves in a nonhomogeneous elastic half-space is proved on the basis of the stress elastodynamics formulation (cf. [1]). It is demonstrated that in the case when nonhomogeneity depends on depth of the semi-space, both the velocity and amplitude of a surface wave are analytical functions of the wave number.

1. Introduction

IN 1971 (cf. [1]) J. IGNACZAK showed that the problem of surface wave propagation in nonhomogeneous isotropic elastic half-space can be reduced to the following eigenvalue problem: find a positive number λ and a real-valued symmetric tensor field

$$\alpha_{ij} = \alpha_{ij}(x_2) \qquad (\alpha_{ij} \in C^2[0,\infty), \qquad i,j = 1,2)$$

satisfying the following equation:

$$\mathbf{A}(s)\,\boldsymbol{\alpha} - \lambda \mathbf{B}\,\boldsymbol{\alpha} = \mathbf{0}, \tag{1.1}$$

together with conditions

$$\alpha_{22}(0) = \alpha_{12}(0) = \alpha_{22}(\infty) = \alpha_{12}(\infty) = 0, \tag{1.2}$$

where

$$\boldsymbol{\alpha}(x_2) = [\alpha_{11}(x_2) \quad \alpha_{22}(x_2) \quad \alpha_{12}(x_2)]^T, \tag{1.3}$$

$$\mathbf{A} \equiv \mathbf{A}(s,\varrho) \equiv \begin{bmatrix} \dfrac{s^2}{\varrho} & 0 & \dfrac{s}{\varrho}D \\ 0 & -D\dfrac{1}{\varrho}D & sD\dfrac{1}{\varrho} \\ -sD\dfrac{1}{\varrho} & -\dfrac{s}{\varrho}D & \dfrac{s^2}{\varrho} - D\dfrac{1}{\varrho}D \end{bmatrix}, \tag{1.4}$$

$$(1.5)\qquad \mathbf{B} \equiv B(\mu,\nu) \equiv \begin{bmatrix} \dfrac{1-\nu}{2\mu} & \dfrac{-\nu}{2\mu} & 0 \\ \dfrac{-\nu}{2\mu} & \dfrac{1-\nu}{2\mu} & 0 \\ 0 & 0 & \dfrac{1}{\mu} \end{bmatrix}.$$

Tensor $\boldsymbol{\alpha}$ defines the stress tensor amplitude and symbol D denotes differentiation with respect to x_2 ($D = d/dx_2$). Number s is the wave-number, and $\varrho = \varrho(x_2)$, $\mu = \mu(x_2)$ and $\nu = \nu(x_2)$ are density, shear modulus, and Poisson's ratio, respectively ($0 \le x_2 < \infty$).

The formulation (1.1) – (1.5) is based on a pure stress method of classical elastodynamics.([1])

In an earlier paper [4] J. IGNACZAK showed, that the problem of surface wave propagation in a nonhomogeneous isotropic elastic half-space with shear moduls μ and Poisson's ratio ν depending on x_2, and with constant density, can be reduced to the following one: find a pair $(c_R, \beta(x))$ satisfying the ordinary differential equation of the fourth order

$$(1.6)\qquad \left(\frac{1}{s^2} D \frac{1}{1-\Omega} D - 1\right) \frac{1}{1-\kappa} \frac{\Omega}{2-\Omega} \left[D^2 - s^2(1-\Omega\kappa)\right]\beta \\ +4\left[\frac{1}{2-\Omega}D^2 - D\frac{1}{1-\Omega}D\frac{1-\Omega}{2-\Omega}\right]\beta = 0 \qquad \text{for} \quad x_2 \in (0,\infty),$$

and the boundary conditions

$$(1.7)\qquad \begin{aligned} &\beta(0) = \beta(\infty) = 0, \\ &\frac{1}{s^2(2-\Omega)} D \left\{ \frac{\Omega}{2-\Omega}\frac{1}{1-\kappa}\left[D^2 - s^2(1-\Omega\kappa)\right]\beta - 4s^2\frac{1-\Omega}{2-\Omega}\beta \right\}\Big|_{\substack{x_2=0\\ x_2=\infty}} = 0, \end{aligned}$$

where

$$(1.8)\qquad \kappa(x_2) = \frac{1-2\nu(x_2)}{2-2\nu(x_2)}, \qquad \Omega(x) = \frac{c_R^2}{\mu(x_2)}.$$

([1]) The problem (1.1) – (1.2) can be discussed in a class of square integrable functions, i.e.:

$$\boldsymbol{\alpha} = [\alpha_{11} \quad \alpha_{22} \quad \alpha_{12}]^T \in L^2(0,\infty) \times L^2(0,\infty) \times L^2(0,\infty) = [L^2(0,\infty)]^3 \quad \mathbf{A}, \mathbf{B} \in [L^2(0,\infty)]^3,$$

and it is correctly posed when the condition $R(\mathbf{A}) = R(\mathbf{B})$ is satisfied; $R(\mathbf{A})$, $R(\mathbf{B})$ denote the ranges of operators $\mathbf{A}, \mathbf{B}$ (cf. [2] p. 16). From equality $R(\mathbf{A}) = R(\mathbf{B})$ it follows that:

$$R(\mathbf{A}) = R(\mathbf{B}) = \Big\{ (\alpha_{11}, \alpha_{22}, \alpha_{12}) \in [C^2[0,\infty)]^3 : \\ -\left[\frac{\alpha_{11} - \nu\alpha_{ii}}{2\mu}\right]'' + \frac{s^2(\alpha_{22} - \nu\alpha_{ii})}{2\mu} - s\left[\frac{\alpha_{12}}{\mu}\right]' = 0; \quad i = 1,2 \Big\}.$$

The differential equation ($' = D$) in brackets corresponds to the compatibility condition (cf. [3] p. 345) for the problem.

The surface wave velocity c_R is the eigenvalue of the problem ((1.6)–(1.8)). Function $\beta(x_2)$ describing the variation of normal stress is the eigenfunction associated with eigenvalue c_R, ($\beta(x_2) = \alpha_{22}(x_2)$). In 1967 C.R.A. RAO [5] extended the formulation (1.6)–(1.7) to the case when density ϱ, shear modulus μ, and Poisson's ratio ν are arbitrary functions of x_2. In the particular case, when

$$\varrho(x_2) \equiv 1, \qquad \mu(x_2) \equiv \text{const}, \qquad \varepsilon > 0; \tag{1.9}$$

$$\begin{aligned} \nu_0 &= \nu(0), \qquad \nu_\infty = \nu(\infty), \\ \nu(x_2) &= 1 - (1-\nu_\infty)\left[1 + \frac{\nu_0 - \nu_\infty}{1-\nu_0}(1+\varepsilon x_2)^{-2}\right]^{-1}, \end{aligned} \tag{1.10}$$

J. IGNACZAK (cf. [4]) obtained an analytical closed-form solution. C.R.A. RAO (cf. [6, 7]) investigated the problem in case:

$$\varrho(x_2) \equiv 1, \qquad \nu(x_2) \equiv \nu_0, \qquad \mu(x_2) = \mu_\infty + (\mu_0 - \mu_\infty)e^{-\varepsilon x_2} \tag{1.11}$$

using the power series expansion method.

The problem (1.6)–(1.7) was also investigated by T. ROŻNOWSKI, (cf. [8, 9, 10]).

In [8] a solution was found under the assumptions that density and Poisson's ratio are constant, and shear modulus μ is a "weakly" variable exponential function such that the term

$$4\left(\frac{1}{2-\Omega(x_2)}\frac{d^2}{dx_2^2} - \frac{d}{dx_2}\frac{1}{1-\Omega(x_2)}\frac{d}{dx_2}\frac{1-\Omega(x_2)}{2-\Omega(x_2)}\right)\beta \tag{1.12}$$

can be neglected.

In [9] T. ROŻNOWSKI analysed the equations of motion for a transversely isotropic nonhomogeneous elastic semispace, using the stress motion equations, and formulated the problem of surface wave of the Rayleigh type. He showed that the problem can be also reduced to an ordinary differential equation of the fourth order with variable coefficients. T. ROŻNOWSKI in [10] analysed five particular cases of the wave phenomena:

a) transversely isotropic body with a "small nonhomogeneity",
b) "weakly anisotropic" nonhomogeneous body,
c) "weakly anisotropic" body with a "small nonhomogeneity",
d) transversely isotropic homogeneous body,
e) isotropic nonhomogeneous body.

The surface wave problem can be formulated in an alternative way starting from the displacement equations.

A.G. ALENITSYN (cf. [11, 12, 13, 14]) investigated the equations of motion in the displacement formulation for large wave numbers using asymptotic methods. As a result, he obtained an approximate dispersion relation (cf. [15]).

In this paper some new properties of the surface waves will be presented. The stress formulation will be used. This paper consists of four sections. Sec. 2 is devoted to general formulation of the problem. In Sec. 3 qualitative properties of the solution are discussed. It is demonstrated that for density, shear modulus, and Poisson's ratio being bounded and of class $C^2[0,\infty)$, the wave velocity and stress amplitude are analytical functions of the wave number. In Sec. 4 it is shown that at least one solution exists (and at most a finite number of solutions) under the assumptions, that density and shear modulus are constant and Poisson's ratio is a bounded function from $C^2[0,\infty)$. The obtained results are limited to the surface waves propagating in a nonhomogeneous half-space under isothermal conditions.

2. Stress formulation of a surface wave problem

Let us consider the two-dimensional stress equation of the linear elastodynamics (cf. [1]) for a nonhomogeneous isotropic medium (²)

$$\mu^{-1}(x)\left[\frac{\partial^2}{\partial t^2}\tau_{\alpha\beta}(x,t)-\nu(x)\delta_{\alpha\beta}\frac{\partial^2}{\partial t^2}\tau_{\gamma\gamma}(x,t)\right]-\left[\varrho^{-1}(x)\tau_{\alpha\gamma,\gamma}(x,t)\right]_{,\beta}-\left[\varrho^{-1}(x)\tau_{\beta\gamma,\gamma}(x,t)\right]_{,\alpha}=0, \tag{2.1}$$

where

$$\tau_{\alpha\beta}=\tau_{\alpha\beta}(x,t),\qquad (\alpha,\beta)=(1,2),\qquad [x=(x_1,x_2)]$$

denotes nondimensional stress tensor, $\mu(x)$, $\varrho(x)$ are nondimensional shear modulus and density, $\nu(x)$ is Poisson's ratio. Nondimensional time is defined by the formula

$$t=\frac{\tau\mu_0^{1/2}}{x_0\varrho_0^{1/2}}, \tag{2.2}$$

where τ is real time and μ_0,ϱ_0 and x_0 are units of stress, density and length, respectively. Moreover

$$\dot{\tau}_{\alpha\beta}=\frac{\partial\tau_{\alpha\beta}}{\partial t},\qquad \tau_{\alpha\beta,\gamma}=\frac{\partial\tau_{\alpha\beta}}{\partial x_\gamma}.$$

It is assumed, that the functions $\varrho(x),\mu(x)$ and $\nu(x)$ depend on x_2 $(x_2\in[0,\infty))$ and $\varrho(x_2)$, $\mu(x_2)$, $\nu(x_2)\in C_2[0,\infty)$, and

$$\begin{aligned}&0<\varrho_0\le\varrho(x_2)\le\varrho_1<\infty,\\&0<\mu_0\le\mu(x_2)\le\mu_1<\infty,\\&-1<\nu_0\le\nu(x_2)\le\nu_1<1/2\qquad\text{for}\quad x_2\in[0,\infty).\end{aligned} \tag{2.3}$$

(²) See IGNACZAK [4], RAO [5].

The triplets $(\varrho_0, \mu_0, \nu_0)$ and $(\varrho_1, \mu_1, \nu_1)$ represent minimal and maximal values of (ϱ, μ, ν).

The solution $\tau_{\alpha\beta}$ of Eq. (2.1) will be considered in the half-space

$$U = \{(x_1, x_2): \quad x_2 \geq 0, \quad -\infty < x_1 < \infty\}, \tag{2.4}$$

for every $t \in [0, \infty)$. We shall look for a solution in the form:

$$\begin{aligned} \tau_{11}(x,t) &= \operatorname{Re} \alpha_{11}(x_2) \exp[i(sx_1 - t\sqrt{\lambda})], \\ \tau_{22}(x,t) &= \operatorname{Re} \alpha_{22}(x_2) \exp[i(sx_1 - t\sqrt{\lambda})], \\ \tau_{12}(x,t) &= \operatorname{Re} i\alpha_{12}(x_2) \exp[i(sx_1 - t\sqrt{\lambda})], \end{aligned} \tag{2.5}$$

where $i = \sqrt{-1}$, $s > 0$, $\lambda > 0$ and Re stands for the real part of a complex-valued function. Moreover it is assumed that the solution satisfies the conditions

$$\tau_{22}(x_1, 0, t) = \tau_{12}(x_1, 0, t) = 0 \qquad \text{for} \quad x_1 \in (-\infty, \infty), \quad t \geq 0, \tag{2.6}$$

$$\tau_{22}(x_1, \infty, t) = \tau_{12}(x_1, \infty, t) = \tau_{11}(x_1, \infty, t) = 0 \qquad \text{for} \quad x_1 \in (-\infty, \infty), \quad t \geq 0. \tag{2.7}$$

The wave velocity, wave period and wave length are $c_R = \sqrt{\lambda}/s$, $T = 2\pi/\sqrt{\lambda}$, and $l = 2\pi/s$. The functions $\alpha_{11}(x,t)$, $\alpha_{22}(x,t)$, $\alpha_{12}(x,t)$, and the velocity c_R should be chosen in such a way that tensor field $\boldsymbol{\tau}(x,t)$ defined by (2.5) should satisfy the field equation (2.1) and the conditions (2.6)–(2.7).

Introducing (2.5) to (2.1), (2.6), (2.7) we obtain (cf. [1])

$$\begin{aligned} \varrho^{-1}(s\alpha_{11} + s\dot{\alpha}_{12}) - \lambda(2\mu)^{-1}(\alpha_{11} - \nu\alpha_{\gamma\gamma}) &= 0, \\ -\left[\varrho^{-1}(\dot{\alpha}_{22} - s\alpha_{12})\right]^{\bullet} - \lambda(2\mu)^{-1}(\alpha_{22} - \nu\alpha_{\gamma\gamma}) &= 0, \\ -\left[\varrho^{-1}(s\dot{\alpha}_{12} + s\alpha_{11})\right]^{\bullet} - s\varrho^{-1}(\dot{\alpha}_{22} - s\alpha_{12}) - \lambda(2\mu)^{-1}2\alpha_{12} &= 0 \end{aligned} \tag{2.8}$$

$$\text{for} \quad x_2 \in (0, \infty),$$

and the boundary conditions

$$\alpha_{22}(0) = \alpha_{12}(0) = \alpha_{22}(\infty) = \alpha_{12}(\infty) = 0, \tag{2.9}$$

where

$$\boldsymbol{\alpha} = [\alpha_{11} \quad \alpha_{22} \quad \alpha_{12}]^T \in [C^2[0, \infty)]^3.$$

Starting from Eq. (2.8), the dot over a symbol will denote differentiation with respect to x_2. We shall also use the symbol D for the operator $D = d/dx_2$. C.R.A. RAO showed (cf. [5]) that the linear eigenvalue problem (2.8)–(2.9) can be further reduced, by elimination of α_{11} and α_{12}, to the nonlinear eigenvalue problem

$$\left[\left\{\left[D-\left(H_1-\frac{2h}{2-\Omega}\right)\right]\frac{1}{a^2-e^2}\left[D-(1-2\kappa)H_1\right]-1\right\}\right. \tag{2.10}$$
$$\left.\left\{\frac{\Omega}{2-\Omega}\,\frac{1}{1-\kappa}(D^2+hD-b^2)\right\}\right]\alpha_{22}+4\left\{\frac{1}{2-\Omega}(D^2+hD)\right.$$
$$\left.-\left[D-\left(H_1-\frac{2h}{2-\Omega}\right)\right]\frac{1}{a^2-e^2}\left[D-(1-2\kappa)H_1\right]\frac{a^2}{2-\Omega}\right\}\alpha_{22}=0$$
$$\text{for}\quad x_2\in(0,\infty),$$

$$\alpha_{22}(0)=\alpha_{22}(\infty)=0, \tag{2.11}$$

$$\left\{\frac{1}{a^2-e^2}\left[D-(1-2\kappa)H_1\right]\frac{\Omega}{2-\Omega}\,\frac{1}{1-\kappa}\left[D^2+hD-b^2\right.\right. \tag{2.12}$$
$$\left.\left.-\frac{4a^2(1-\kappa)}{\Omega}\right]\alpha_{22}\right\}\Bigg|_{\substack{x_2=0\\x_2=\infty}}=0,$$

$$\begin{aligned}
\kappa(x_2)&=\frac{1-2\nu(x_2)}{2-2\nu(x_2)}, & \nu(x_2)&=\frac{1-2\kappa(x_2)}{2-2\kappa(x_2)},\\
h&=\varrho D(\varrho^{-1}), & \Omega(x_2)&=\frac{c_R^2\varrho(x_2)}{\mu(x_2)},\\
a^2&=s^2(1-\Omega), & b^2&=s^2(1-\Omega\kappa),\\
H_1&=[\Omega/(2-\Omega)]\cdot[h/(2-2\kappa)], & e^2&=DH_1-(1-2\kappa)H_1^2.
\end{aligned} \tag{2.13}$$

From a solution $(\lambda,\alpha_{22}(x_2))$ of Eqs. (2.10)–(2.12) one can obtain the functions $\alpha_{11}(x_2)$ and $\alpha_{12}(x_2)$ using the formulae

$$\alpha_{11}(x_2)=-\frac{1}{s^2(2-\Omega)}\left\{\left[s^2\Omega+2(D^2+hD)\right]\alpha_{22}\right. \tag{2.14}$$
$$+h\frac{1}{a^2-e^2}\,\frac{1}{1-\kappa}\left[D-(1-2\kappa)H_1\right]\frac{2}{2-\Omega}\,\frac{1}{1-\kappa}\left[D^2+hD-b^2\right.$$
$$\left.\left.-\frac{4a^2(1-\kappa)}{\Omega}\right]\alpha_{22}\right\},$$

$$-2s\alpha_{12}(x_2)=\frac{1}{a^2-e^2}\left[D-(1-2\kappa)H_1\right]\frac{\Omega}{2-\Omega}\,\frac{1}{1-\kappa}\left[D^2+hD-b^2\right. \tag{2.15}$$
$$\left.-\frac{4a^2(1-\kappa)}{\Omega}\right]\alpha_{22}\,.$$

For the special case when the density is constant, $\varrho = 1$, $h = \varrho D(\varrho^{-1}) = 0$ and Eqs. (2.10) – (2.12) reduce to (cf. [4])

$$\left(\frac{1}{s^2}D\frac{1}{1-\Omega}D-1\right)\frac{1}{1-\kappa}\,\frac{\Omega}{2-\Omega}\left[D^2-s^2(1-\Omega\kappa)\right]\alpha_{22} + 4\left[\frac{1}{2-\Omega}D^2-D\frac{1}{1-\Omega}D\frac{1-\Omega}{2-\Omega}\right]\alpha_{22}=0 \qquad \text{for} \quad x_2\in(0,\infty), \tag{2.16}$$

$$\alpha_{22}(0)=\alpha_{22}(\infty)=0, \qquad \left[D\left\{\frac{\Omega}{2-\Omega}\,\frac{1}{1-\kappa}\left[D^2-s^2(1-\Omega\kappa)\right]\alpha_{22}-4s^2\frac{1-\Omega}{2-\Omega}\alpha_{22}\right\}\right]\Bigg|_{\substack{x_2=0\\x_2=\infty}}=0, \tag{2.17}$$

$$\alpha_{11}(x_2)=-\frac{1}{s^2(2-\Omega)}\left(s^2\Omega+2D^2\right)\alpha_{22}, \tag{2.18}$$

$$\alpha_{12}(x_2)=\frac{-1}{s^3(1-\Omega)}D\left\{\frac{\Omega}{2-\Omega}\,\frac{1}{1-\kappa}\left[D^2-s^2(1-\Omega\kappa)\right]\alpha_{22} -4s^2\frac{1-\Omega}{2-\Omega}\alpha_{22}\right\}. \tag{2.19}$$

Clearly, in the eigenvalue problem (2.10) – (2.12) (or (2.16) – (2.17)) the eigenvalue λ enters in a nonlinear way. Also, note that the problem (2.1), (2.6), (2.7) is not a regular one (³). Indeed, writing (2.1) more explicitly, we have:

$$\begin{bmatrix} \dfrac{1-\nu}{\mu} & \dfrac{-\nu}{\mu} & 0 \\ \dfrac{-\nu}{\mu} & \dfrac{1-\nu}{\mu} & 0 \\ 0 & 0 & \dfrac{1}{\mu} \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2}{\partial t^2}\tau_{11} \\ \dfrac{\partial^2}{\partial t^2}\tau_{22} \\ \dfrac{\partial^2}{\partial t^2}\tau_{12} \end{bmatrix} \tag{2.20}$$
$$= \begin{bmatrix} 2\dfrac{\partial}{\partial x_1}\varrho^{-1}\dfrac{\partial}{\partial x_1} & 0 & 2\dfrac{\partial}{\partial x_1}\varrho^{-1}\dfrac{\partial}{\partial x_2} \\ 0 & 2\dfrac{\partial}{\partial x_2}\varrho^{-1}\dfrac{\partial}{\partial x_2} & 2\dfrac{\partial}{\partial x_2}\varrho^{-1}\dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2}\varrho^{-1}\dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_1}\varrho^{-1}\dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_2}\varrho^{-1}\dfrac{\partial}{\partial x_2}+\dfrac{\partial}{\partial x_1}\varrho^{-1}\dfrac{\partial}{\partial x_1} \end{bmatrix} \begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{12} \end{bmatrix}.$$

The characteristic determinant associated with R.H.S of (2.20) takes the form

$$\begin{vmatrix} -2\varrho^{-1}\xi_1^2 & 0 & -2\varrho^{-1}\xi_1\xi_2 \\ 0 & -2\varrho^{-1}\xi_1^2 & -2\varrho^{-1}\xi_2\xi_1 \\ -\varrho^{-1}\xi_2\xi_1 & -\varrho^{-1}\xi_1\xi_2 & -\varrho^{-1}(\xi_1^2+\xi_2^2) \end{vmatrix} \tag{2.21}$$

(³) See [16, 17, 18, 19].

and it is equal to zero for any point (ξ_1, ξ_2). It can be shown that if suitable restrictions are imposed on $\boldsymbol{\tau}$ at $t = 0$, Eq. (2.20) implies the compatibility condition of the two-dimensional elasticity theory (⁴)

$$\text{(2.22)} \qquad \left\{\mu^{-1}[(1-\nu)\tau_{11} - \nu\tau_{22}]\right\}_{,22} + \left\{\mu^{-1}[(1-\nu)\tau_{22} - \nu\tau_{11}]\right\}_{,11} - 2\left\{\mu^{-1}\tau_{12}\right\}_{,12} = 0 \qquad \text{for} \quad (x,t) \in U \times [0,\infty).$$

So, the system (2.20) subject to the condition (2.22) can be considered as a regular one.

The condition (2.22) follows from (2.20), if the stress field $\tau_{\alpha\beta}$ is sufficiently smooth on $U \times [0,\infty)$, and the L.H.S. of Eq. (2.22) together with its first time derivative vanishes for $t = 0$. The last conditions are equivalent to the assumption that deformation and its velocity satisfy the compatibility condition for $t = 0$. Vanishing of the determinant (2.21) implies that the operator

$$\text{(2.23)} \qquad \sum \tau_{\alpha\beta}(x,t) \equiv \begin{bmatrix} 2\dfrac{\partial}{\partial x_1}\varrho^{-1}\dfrac{\partial}{\partial x_1} & 0 & 2\dfrac{\partial}{\partial x_1}\varrho^{-1}\dfrac{\partial}{\partial x_2} \\ 0 & 2\dfrac{\partial}{\partial x_2}\varrho^{-1}\dfrac{\partial}{\partial x_2} & 2\dfrac{\partial}{\partial x_2}\varrho^{-1}\dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2}\varrho^{-1}\dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1}\varrho^{-1}\dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_2}\varrho^{-1}\dfrac{\partial}{\partial x_2} + \dfrac{\partial}{\partial x_1}\varrho^{-1}\dfrac{\partial}{\partial x_1} \end{bmatrix} \begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{12} \end{bmatrix}$$

defined on the domain

$$D_1(\boldsymbol{\Sigma}) = \Big\{(\tau_{11}, \tau_{22}, \tau_{12}) \in [C^2(U \times [0,\infty))]^3 :$$
$$\tau_{22}(x_1;0,t) = \tau_{12}(x_1;0,t) = \tau_{22}(x_1;\infty,t) = \tau_{12}(x_1;\infty,t) = \tau_{11}(x_1;\infty,t) = 0\Big\}$$

or

$$D_2(\boldsymbol{\Sigma}) = \Big\{(\tau_{11}, \tau_{22}, \tau_{12}) \in [L^2(U \times [0,\infty))]^3 :$$
$$\tau_{22}(x_1;0,t) = \tau_{12}(x_1;0,t) = \tau_{22}(x_1;\infty,t) = \tau_{12}(x_1;\infty,t) = \tau_{11}(x_1;\infty,t) = 0\Big\}$$

is not invertible, unless the condition (2.22) is satisfied.

(⁴) The compatibility condition restricted to the field $\boldsymbol{\alpha}$ takes the form:

$$\left\{\mu^{-1}[(1-\nu)\alpha_{11} - \nu\alpha_{22}]\right\}^{\cdot\cdot} - s\left\{\mu^{-1}[(1-\nu)\alpha_{22} - \nu\alpha_{11}]\right\} + 2s\left\{\mu^{-1}\alpha_{12}\right\}^{\cdot} = 0, \qquad \left(\cdot = \frac{d}{dx_2}\right).$$

3. On the analytical dependence of velocity and amplitude of the surface wave on the wave number

In this section we shall analyse the problem (2.8) – (2.9) using B-holomorphic perturbation theory for linear operators proposed by T. KATO (cf. [2]). We will demonstrate that velocity and amplitude of the wave are analytical functions of the wave number s.

In the complex Hilbert space H generated by the scalar product (5)

$$(3.1) \qquad (\boldsymbol{\alpha},\boldsymbol{\beta}) = \int_0^\infty (\overline{\alpha}_{11}\beta_{11} + \overline{\alpha}_{22}\beta_{22} + \overline{\alpha}_{12}\beta_{12})\,dx_2$$

with norm

$$(3.2) \qquad \|\boldsymbol{\alpha}\|^2 = \int_0^\infty (|\alpha_{11}|^2 + |\alpha_{22}|^2 + |\alpha_{12}|^2)\,dx_2 < \infty,$$

Eq. (2.8) can be written in the operator form

$$(3.3) \qquad \mathbf{A}(s)\boldsymbol{\alpha} - \lambda\mathbf{B}\,\boldsymbol{\alpha} = \mathbf{0},$$

or in the expanded form

$$(3.4) \qquad \mathbf{A}(s,\varrho)\boldsymbol{\alpha} - \lambda\mathbf{B}(\mu,\nu)\boldsymbol{\alpha} = \mathbf{0}.$$

The domain of operators **A** and **B** may be defined as follows

$$(3.5) \qquad \mathcal{D}(\mathbf{A}) = \left\{\boldsymbol{\alpha}\colon \alpha_{ij} \in C^2[0,\infty);\ \alpha_{12}(0) = \alpha_{22}(0) = \alpha_{12}(\infty) = \alpha_{22}(\infty) = 0\right\},$$

$$(3.6) \qquad \mathcal{D}(\mathbf{B}) = \left\{\boldsymbol{\alpha}\colon \alpha_{ij} \in C^2[0,\infty)\right\}, \qquad i,j = 1,2.$$

The sets $\mathcal{D}(\mathbf{A})$ and $\mathcal{D}(\mathbf{B})$ are dense in H since the set $C_0^\infty[0,\infty) \times C_0^\infty[0,\infty) \times C_0^\infty[0,\infty)$ is dense in H and is contained in $\mathcal{D}(\mathbf{A})$ and $\mathcal{D}(\mathbf{B})$. We have

PROPOSITION 1. Operators **A** and **B** are symmetric in the Hilbert space H.

The symmetry of operator **A** results from the fact that operators on both sides of the principal diagonal are formally adjoint, e.g. $\frac{s}{\varrho}D$ with $-sD\frac{1}{\varrho}$, $-\frac{s}{\varrho}D$ with $sD\frac{1}{\varrho}$.

(5) In oder to be able to apply Kato's perturbation theory, we have to extend the problem to the complex plane.

For arbitrary $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{D}(\mathbf{A}) \subset H$ we have

$$(\mathbf{A}\boldsymbol{\alpha}, \boldsymbol{\beta}) = \int_0^\infty \{\varrho^{-1}(s^2\overline{\alpha}_{11} + s\dot{\overline{\alpha}}_{12})\beta_{11} - \left[\varrho^{-1}(\dot{\overline{\alpha}}_{22} + s\overline{\alpha}_{12})\right]^{\bullet}\beta_{12}$$
$$- \left[\varrho^{-1}(\dot{\overline{\alpha}}_{12} + s\overline{\alpha}_{11})\right]^{\bullet}\beta_{12} - s\varrho^{-1}(\dot{\overline{\alpha}}_{12} - s\overline{\alpha}_{22})\beta_{12}\}\,dx_2 .$$

Integration by parts with the use of boundary conditions shows that

$$(\mathbf{A}\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\boldsymbol{\alpha}, \mathbf{A}\boldsymbol{\beta}).$$

The symmetry of operator $\mathbf{B}$ is obvious. Matrix $\mathbf{B}$ is positive definite and for every $\boldsymbol{\alpha} \in \mathcal{D}(\mathbf{B}) \subset H$ we have(6)

$$(\mathbf{B}\boldsymbol{\alpha}, \boldsymbol{\alpha}) \geq k(\boldsymbol{\alpha}, \boldsymbol{\alpha}),$$

where

$$k = \min_{x_2 \in [0,\infty)} \left(\frac{1-2\nu}{2\mu}, \frac{1}{\mu}, \frac{1}{2\mu}\right).$$

Let us consider the forms $\mathcal{U}[\boldsymbol{\alpha}] = (\mathbf{A}\boldsymbol{\alpha}, \boldsymbol{\alpha})$, $\mathcal{B}[\boldsymbol{\alpha}] = (\mathbf{B}\boldsymbol{\alpha}, \boldsymbol{\alpha})$ described by the formulae

$$(\mathbf{A}\boldsymbol{\alpha}, \boldsymbol{\alpha}) = \int_0^\infty \frac{1}{\varrho}\left[|\dot{\alpha}_{22} - s\alpha_{12}|^2 + |\dot{\alpha}_{12} + s\alpha_{11}|^2\right] dx_2 ,$$

$$(\mathbf{B}\boldsymbol{\alpha}, \boldsymbol{\alpha}) = \int_0^\infty (2\mu)^{-1}\left[(1-\nu)|\alpha_{22}|^2 + (1-\nu)|\alpha_{22}|^2 + 2|\alpha_{12}|^2 - 2\nu \operatorname{Re}(\alpha_{11}\overline{\alpha}_{22})\right] dx_2 .$$

In view of (2.3) we have $(\mathbf{A}\boldsymbol{\alpha}, \boldsymbol{\alpha}) \geq 0$. Operators $\mathbf{A}$ and $\mathbf{B}$ being symmetric, are closable in the space H. Let $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$ denote the closures of operators $\mathbf{A}$ and $\mathbf{B}$. Let us set in H the form:

$$\mathcal{U}[\boldsymbol{\alpha}] = \sum_{i=0}^{\infty} \mathcal{U}^{(i)}(s_0)[\boldsymbol{\alpha}](z - s_0)^i \tag{3.7}$$

for z belonging to a certain neighbourhood of the real semi-axis s, $s_0 \in (0, \infty)$(7), where

$$\mathcal{U}^{(0)}[\boldsymbol{\alpha}] = (\mathbf{A}(s_0)\boldsymbol{\alpha}, \boldsymbol{\alpha}) = \int_0^\infty \varrho^{-1}\left(|\dot{\alpha}_{22} - s_0\alpha_{12}|^2 + |\dot{\alpha}_{12} + s_0\alpha_{11}|^2\right) dx_2 , \tag{3.8}$$

(6) The eigenvalues of matrix $\mathbf{B}$ are $\frac{1-2\nu}{2\mu}, \frac{1}{2\mu}, \frac{1}{\mu}$. The symmetric matrix $\mathbf{B}$ is positive definite iff all its eigenvalues λ_i are positive and $(\mathbf{B}\boldsymbol{\alpha}, \boldsymbol{\alpha}) \geq \min\limits_i \lambda_i(\boldsymbol{\alpha}, \boldsymbol{\alpha})$ (cf. [20]).

(7) The neighbourhood is a set: $V = \left\{z\colon |z - s_0| < \frac{1}{b+c} \text{ and } z \notin (-\infty, 0]\right\}$ where $b = \frac{1}{\varepsilon}$, $c = \frac{2}{\varepsilon}$, $\varepsilon > 0$. We can expand the region of holomorphicity by choosing a suitable ε. The meaning of b, c, ε will be made clear in the sequel.

$$(3.9)\qquad \mathcal{U}^{(1)}(s_0)[\boldsymbol{\alpha}] = \int_0^\infty \frac{1}{\varrho}\Big\{2s_0|\alpha_{12}|^2 + 2s_0|\alpha_{11}|^2 - 2\mathrm{Re}\,(\alpha_{12}\dot{\overline{\alpha}}_{22}) + 2\mathrm{Re}\,(\alpha_{11}\dot{\overline{\alpha}}_{12})\Big\}dx_2\,,$$

$$(3.10)\qquad \mathcal{U}^{(2)}(s_0)[\boldsymbol{\alpha}] = \int_0^\infty \frac{2}{\varrho}\left(|\alpha_{12}|^2 + |\alpha_{11}|^2\right)dx_2\,,$$

$$(3.11)\qquad \mathcal{U}^{(n)}(s_0)[\boldsymbol{\alpha}] = 0, \qquad n = 3, 4, \ldots .$$

The form $\mathcal{U}^{(1)}(s_0)[\boldsymbol{\alpha}]$ is a derivative of $(\mathbf{A}(s)\boldsymbol{\alpha}, \boldsymbol{\alpha})$ with respect to the real parameter s at $s = s_0$,

$$\mathcal{U}^{(1)}(s_0)[\boldsymbol{\alpha}] = \lim_{s\to s_0} \frac{(\mathbf{A}(s)\boldsymbol{\alpha}, \boldsymbol{\alpha}) - (\mathbf{A}(s_0)\boldsymbol{\alpha}, \boldsymbol{\alpha})}{s - s_0}.$$

Similarly,

$$\mathcal{U}^{(2)}(s_0)[\boldsymbol{\alpha}] = \lim_{s\to s_0} \frac{\mathcal{U}^{(1)}(s)[\boldsymbol{\alpha}] - \mathcal{U}^{(1)}(s_0)[\boldsymbol{\alpha}]}{s - s_0},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\mathcal{U}^{(n)}(s_0)[\boldsymbol{\alpha}] = \lim_{s\to s_0} \frac{\mathcal{U}^{(n-1)}(s)[\boldsymbol{\alpha}] - \mathcal{U}^{(n-1)}(s_0)[\boldsymbol{\alpha}]}{s - s_0}.$$

We shall prove the following lemma:

LEMMA 1. The closure $\widetilde{\mathcal{U}}(z)$ of the form $\mathcal{U}(z)$ generates a family of operators $\widetilde{\mathbf{A}}(z)$ which is B-holomorphic(8).

In order to demonstrate that $\widetilde{\mathbf{A}}(z)$ is a B-holomorphic family of operators we shall use Kato's B-holomorphism criterion(9).

Let $\mathcal{U}^{(n)}(s_0)[\boldsymbol{\alpha}]$ be a sequence of sesquilinear form in H ($n = 0, 1, 2\ldots$), and let the form $\mathcal{U}^{(0)}(s_0)[\boldsymbol{\alpha}]$ be sectorial(10) and closable, and with the domain $D(\mathcal{U}^{(0)}) = D$. Assume that the forms $\mathcal{U}^{(n)}(s_0)[\boldsymbol{\alpha}]$ for $n \geq 1$ are bounded with respect to $\mathcal{U}^{(0)}[\boldsymbol{\alpha}]$, i.e. $D \subset D(\mathcal{U}^{(n)})$, and

$$(*)\qquad \left|\mathcal{U}^{(n)}(s_0)[\boldsymbol{\alpha}]\right| \leq c^{n-1}(a\|\boldsymbol{\alpha}\|^2 + b\mathrm{Re}\,\mathcal{U}^{(0)}(s_0)[\boldsymbol{\alpha}]),$$
$$\boldsymbol{\alpha} \in D, \quad n > 1, \quad a, b \geq 0, \quad c > 0.$$

Then operators $\widetilde{\mathbf{A}}(z)$ corresponding to the forms $\widetilde{\mathcal{U}}(z)[\boldsymbol{\alpha}]$ are a B-holomorphic family of operators for $|z - s_0| < \dfrac{1}{b + c}$.

To show that the assumptions of this criterion are satisfied, let us observe that $\mathcal{U}^{(0)} = \mathcal{U}^{(0)}(s_0)[\boldsymbol{\alpha}] = (A(s_0)\boldsymbol{\alpha}, \boldsymbol{\alpha})$ is a non-negative, symmetric and hence the

(8) (cf. [2] p. 395–397).
(9) (cf. [2] p. 398).
(10) (cf. [2], p. 310).

sectorial form fixed in the dense set D. The density of D results from the fact that the set $\mathcal{D}(\mathbf{A}) \subset D \subset H$ and $\mathcal{D}(\mathbf{A})$ is dense. Thus the form $\mathcal{U}^{(0)}$ is closable.

From the inequalities([11])

$$
\begin{aligned}
(3.12)\qquad \left|\mathcal{U}^{(1)}(s_0)[\boldsymbol{\alpha}]\right| &= \left|\int_0^\infty \frac{1}{\varrho}\Big[-\overline{\alpha}_{12}(\dot{\alpha}_{22} - s_0\alpha_{12}) - (\dot{\overline{\alpha}}_{22} - s_0\overline{\alpha}_{12})\alpha_{12}\right.\\
&\qquad \left. + \overline{\alpha}_{11}(\dot{\alpha}_{22} + s_0\alpha_{11}) + (\dot{\overline{\alpha}}_{12} + s_0\overline{\alpha}_{11})\alpha_{11}\Big]\, dx_2\right|\\
&\leq \left(\int_0^\infty \frac{1}{\varrho}|\alpha_{12}|^2\, dx_2\right)^{1/2}\left(\int_0^\infty \frac{1}{\varrho}\left|\dot{\alpha}_{22} - s_0\alpha_{12}\right|^2\, dx_2\right)^{1/2}\\
&\quad + \left(\int_0^\infty \frac{1}{\varrho}|\alpha_{12}|^2\, dx_2\right)^{1/2}\left(\int_0^\infty \frac{1}{\varrho}\left|\dot{\alpha}_{22} - s_0\alpha_{12}\right|^2\, dx_2\right)^{1/2}\\
&\quad + \left(\int_0^\infty \frac{1}{\varrho}|\alpha_{11}|^2\, dx_2\right)^{1/2}\left(\int_0^\infty \frac{1}{\varrho}|\dot{\alpha}_{12} + s_0\alpha_{11}|^2\, dx_2\right)^{1/2}\\
&\quad + \left(\int_0^\infty \frac{1}{\varrho}|\alpha_{11}|^2\, dx_2\right)^{1/2}\left(\int_0^\infty \frac{1}{\varrho}|\dot{\alpha}_{12} + s_0\alpha_{11}|^2\, dx_2\right)^{1/2}\\
&= 2\left[\left(\int_0^\infty \frac{1}{\varrho}|\alpha_{12}|^2\, dx_2\right)^{1/2}\left(\int_0^\infty \frac{1}{\varrho}|\dot{\alpha}_{22} - s_0\alpha_{12}|^2\, dx_2\right)^{1/2}\right.\\
&\quad \left. + \left(\int_0^\infty \frac{1}{\varrho}|\alpha_{11}|^2\, dx_2\right)^{1/2}\left(\int_0^\infty \frac{1}{\varrho}|\dot{\alpha}_{12} + s_0\alpha_{11}|^2\, dx_2\right)^{1/2}\right]\\
&\leq \varepsilon \max_{x_2\in[0,\infty)} \frac{1}{\varrho}\left(\int_0^\infty \left(|\alpha_{11}|^2 + |\alpha_{22}|^2 + |\alpha_{12}|^2\right) dx_2\right)\\
&+ \frac{1}{\varepsilon}\int_0^\infty \frac{1}{\varrho}\left(|\dot{\alpha}_{22} - s_0\alpha_{12}|^2 + |\dot{\alpha}_{12} + s_0\alpha_{11}|^2\right) dx_2 = \frac{\varepsilon}{\varrho_0}\|\boldsymbol{\alpha}\|^2 + \frac{1}{\varepsilon}\mathcal{U}^{(0)}(s_0)[\boldsymbol{\alpha}]
\end{aligned}
$$

([11]) To prove inequalities (3.12), (3.13) we use the inequalities

$$
\left|\int \sum u_i v_i\, dx\right| \leq \left(\int \sum |u_i|^2\, dx\right)^{1/2}\left(\int \sum |v_i|^2\, dx\right)^{1/2},
$$

$$
2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon}b^2,
$$

where v_i and u_i are complex function, a and b are real functions, and $\varepsilon > 0$.

and

$$(3.13)\qquad \left|\mathcal{U}^{(2)}(s_0)[\boldsymbol{\alpha}]\right| = \int_0^\infty \frac{2}{\varrho}\left(|\alpha_{11}|^2 + |\alpha_{22}|^2\right) dx_2$$

$$\le \max_{x_2\in[0,\infty)} \frac{2}{\varrho}\int_0^\infty \left(|\alpha_{11}|^2 + |\alpha_{22}|^2 + |\alpha_{12}|^2\right) dx_2$$

$$+\frac{2}{\varepsilon^2}\int_0^\infty \frac{1}{\varrho}\left(|\dot{\alpha}_{22} - s_0\alpha_{12}|^2 + |\dot{\alpha}_{12} + s_0\alpha_{11}|^2\right) dx_2$$

$$= \frac{2}{\varrho_0}\|\boldsymbol{\alpha}\|^2 + \frac{2}{\varepsilon^2}\mathrm{Re}\,\mathcal{U}^0(s_0)[\boldsymbol{\alpha}],$$

it follows that $D(\mathcal{U}^{(n)}) \supset D(\mathcal{U}^{(0)})$, $n = 1,2,3,\ldots$, and that there exist $a = \dfrac{\varepsilon}{\varrho_0}$, $b = \dfrac{1}{\varepsilon}$, $c = \dfrac{2}{\varepsilon}$. Thus the operator $\widetilde{A}(z)$ forms a holomorphic family of type (B). From Lemma 1 it follows that the following Proposition is valid.

PROPOSITION 2. The form $\mathcal{U}(z)$ given by (3.7) is defined for $|z - s_0| < \varepsilon/2$, and for $|z - s_0| < \varepsilon/3$ it is sectorial and closable. The closure $\widetilde{\mathcal{U}}(z)$ of the form $\mathcal{U}(z)$ generates a B-holomorphic family of operator $\widetilde{\mathbf{A}}(z)$ where $\widetilde{\mathbf{A}}(z)$ is the maximal and closed operator.

Now we shall consider eigenvalue problem given by

$$(3.14)\qquad \widetilde{\mathbf{A}}(z)\boldsymbol{\alpha} - \lambda\widetilde{\mathbf{B}}\boldsymbol{\alpha} = \mathbf{0},$$

where $\widetilde{\mathbf{A}}(z)$ is the operator defined in Proposition 2 and $\widetilde{\mathbf{B}}$ is the closure of $\mathbf{B}$. From KATO's theorems (cf. [2] p. 416–423) it follows:

THEOREM 1. *If the pair $(\lambda(z), \boldsymbol{\alpha}(z))$ is a solution of the eigenvalue problem* (3.14), *then it is an analytical function with respect to z for $z \in V = \{z : |z - s_0| < \varepsilon/3$ and $z \notin (-\infty, 0]\}$.*

THEOREM 2. *If the pair $(\lambda(s), \boldsymbol{\alpha}(x_2, s))$ is a solution of the eigenvalue problem* (3.3), *then it is an analytical function of the wave-number s.*

It means that

$$(\lambda(s), \boldsymbol{\alpha}(x_2, s)) \equiv \left(\sum_{n=0}^{\infty}\lambda_n(s - s_0)^n,\quad \boldsymbol{\alpha} = \sum_{n=0}^{\infty}\boldsymbol{\alpha}_n(x_2)(s - s_0)^n\right),$$

where

$$\lambda_n = \frac{1}{n!}\left(\frac{d^n\lambda}{ds^n}\right)_{s=s_0},\qquad \boldsymbol{\alpha}_n(x_2) = \frac{1}{n!}\left(\frac{\partial^n\boldsymbol{\alpha}}{\partial s^n}\right)_{s=s_0},\qquad s_0 \in (0,\infty)\ x_2 \ge 0.$$

The proof of Theorem 2 follows directly from Theorem 1 and from the fact that each solution of (3.3) is also a solution of (3.14).

Natural approach to the considered eigenvalue problem

$$\mathbf{A}\boldsymbol{\alpha} - \lambda\mathbf{B}\boldsymbol{\alpha} = \mathbf{0}$$

is investigation of the generalized resolvent

$$(\mathbf{A} - \xi\mathbf{B})^{-1}.$$

Let us introduce the spaces X and Y defined by

$$X = \Big\{(\alpha_{11}, \alpha_{22}, \alpha_{12}) \in [L^2(0,\infty)]^3, \quad \Big[C^2[0,\infty)\Big]^3 : \qquad -\Big[\frac{\alpha_{11} - \nu\alpha_{ii}}{2\mu}\Big]'' + s^2\frac{\alpha_{22} - \nu\alpha_{ii}}{2\mu} - s\Big[\frac{\alpha_{12}}{\mu}\Big]' = 0, \qquad i = 1, 2 \quad \text{for every} \quad x_2 \geq 0\Big\},$$

$$Y = \Big\{(g_{11}, g_{22}, g_{12}) \in [L^2(0,\infty)]^3, \quad [C^2[0,\infty)]^3 : \qquad -\ddot{g}_{11}(x_2) + s^2 g_{22}(x_2) - s\,\dot{g}_{12}(x_2) = 0, \qquad \text{for every} \quad x_2 \geq 0\Big\}.$$

It is easy to check that the spaces X, Y are linear subspaces of $[L^2(0,\infty)]^3$ and $[C^2[0,\infty)]^3$.

Let $\mathcal{C}(X, Y)$ be a space of closed operators from X to Y.

Let $\mathcal{B}(X, Y)$ be a space of bounded operators from X to Y.

Since $\widetilde{\mathbf{A}} \in \mathcal{C}(X, Y)$, $\widetilde{\mathbf{B}} \in \mathcal{B}(X, Y)$ and $\widetilde{\mathbf{B}^{-1}} \in \mathcal{B}(X, Y)$, thus $\widetilde{\mathbf{B}^{-1}\mathbf{A}} \in \mathcal{C}(X, X) = \mathcal{C}(X)$, $\widetilde{\mathbf{A}\,\mathbf{B}^{-1}} \in \mathcal{C}(Y, Y) = \mathcal{C}(Y)$ and the eigenvalue problems

$$\widetilde{\mathbf{A}}\boldsymbol{\alpha} - \lambda\widetilde{\mathbf{B}}\boldsymbol{\alpha} = \mathbf{0}, \qquad \widetilde{\mathbf{B}^{-1}\mathbf{A}}\boldsymbol{\alpha} - \lambda\boldsymbol{\alpha} = \mathbf{0}, \qquad \widetilde{\mathbf{A}\mathbf{B}^{-1}}\boldsymbol{\alpha} - \lambda\boldsymbol{\alpha} = \mathbf{0}$$

are equivalent (cf. [2] p. 417, 418).

To investigate the resolvent $(A - \xi B)^{-1}$, let us take the homogeneous case $\varrho = \text{const}$, $\mu = \text{const}$, $\nu = \text{const}$, as an illustration.

A solution of the equation $\mathbf{A}\boldsymbol{\alpha} - \xi\mathbf{B}\boldsymbol{\alpha} = \mathbf{0}$, $\boldsymbol{\alpha} \in D(\mathbf{A}) \cap D(\mathbf{B}) \subset X$ is $\boldsymbol{\alpha} = [0, 0, 0]^T$ if $\xi \notin \{\omega_1, \omega_2, \omega_3\}$, where ω_1, ω_2, ω_3 are the roots of equation

$$(2 - \omega)^2 - 4\sqrt{(1 - \omega)(1 - \omega\kappa)} = 0, \qquad \kappa = (1 - 2\nu)(2 - 2\nu)^{-1}.$$

To prove this, note that a solution of the equation $\mathbf{A}\boldsymbol{\alpha} - \xi\mathbf{B}\boldsymbol{\alpha} = \mathbf{0}$, takes the form:

$$\begin{aligned}
\alpha_{11} &= -\beta_0\left[e^{-x_2h_2} - \frac{2 + \xi(1 - 2\kappa)}{2 - \xi}e^{-x_2h_1}\right],\\
\alpha_{22} &= \beta_0\left[e^{-x_2h_2} - e^{-x_2h_1}\right],\\
\alpha_{12} &= -\frac{2}{s}\,\frac{\beta_0}{2 - \xi}h_1\left[e^{-x_2h_2} - e^{-x_2h_1}\right],\\
h_1 &= s\sqrt{1 - \xi\kappa}\,, \qquad h_2 = s\sqrt{1 - \xi}\,.
\end{aligned}$$

Introducing such $\boldsymbol{\alpha}$ to the compatibility condition (cf. [6] p. 7) we get

$$\frac{\beta_0 s^2}{2\mu(2-\xi)} e^{-x_2 s\sqrt{1-\xi}} \left[(2-\xi)^2 - 4\sqrt{(1-\xi)(1-\xi\kappa)}\right] + \frac{\beta_0 s^2}{2\mu(2-\xi)(1-\nu)} e^{-x_2 s\sqrt{1-\xi\kappa}} \,[\,0\,] = 0.$$

Therefore if $\xi \notin \{\omega_1, \omega_2, \omega_3\}$ then $(2-\xi)^2 - 4\sqrt{(1-\xi)(1-\xi\kappa)} \neq 0$ and $\beta_0 = 0$. In this case $(\mathbf{A} - \xi\mathbf{B})^{-1}$ exists.

Let us consider the multiplicity of eigenvalue $\lambda = 0$. This problem can be written in the form

$$\mathbf{A}(s)\boldsymbol{\alpha} = \mathbf{0}.$$

As the domain of the operator A we take the set:

$$D(\mathbf{A}) = \Big\{\boldsymbol{\alpha} = [\alpha_{11}\,\alpha_{22}\,\alpha_{12}]^T \in [L^2(0,\infty)]^3, \quad [C^2[0,\infty)]^3 : \alpha_{22}(0) = \alpha_{12}(0) = \alpha_{22}(\infty) = \alpha_{12}(\infty) = \alpha_{11}(\infty) = 0\Big\}.$$

We have

$$\mathbf{A}(s)\boldsymbol{\alpha} = \mathbf{0} \;\Leftrightarrow\; \begin{cases} s\alpha_{11} + \dot{\alpha}_{12} = 0 \\ -s\alpha_{12} + \dot{\alpha}_{22} = 0 \end{cases} \;\Leftrightarrow\; \begin{cases} \alpha_{11}(x_2) = C_1\varphi''(x_2), \\ \alpha_{22}(x_2) = -s^2 C_1 \varphi(x_2), \\ \alpha_{12}(x_2) = -sC_1\varphi'(x_2), \end{cases}$$

where $\varphi = \varphi(x_2)$ is an arbitrary differential function. Selecting $\varphi(x_2)$ in such a way as to meet the boundary conditions, we obtain

$$\ker \mathbf{A}: \qquad \begin{aligned} \alpha_{11}(x_2) &= C_1(2 - 4\alpha_k x_2 + \alpha_k^2 x_2^2)e^{-\alpha_k x_2}, \\ \alpha_{22}(x_2) &= -s^2 x_2^2 C_1 e^{-\alpha_k x_2}, \\ \alpha_{12}(x_2) &= -sC_1(2x_2 - \alpha_k x_2^2)e^{-\alpha_k x_2}, \end{aligned}$$

where

$$C_1 \in R, \qquad x_2 \in [0,\infty), \qquad \alpha_k > 0.$$

It is clear that in this case

$$\dim \ker \mathbf{A} = \infty.$$

Note that in the case, when the domain of the operator is a subspace of the functions satisfying the compatibility condition,

$$\dim \ker \mathbf{A} = 0.$$

4. Existence of surface waves in nonhomogeneous isotropic elastic half-space with arbitrary variation of Poisson's ratio

The problem of propagation of surface waves in a nonhomogeneous isotropic elastic half-space with variable Poissons's ratio can be reduced to the following eigenproblem (cf. [4]): find a nonvanishing pair $(c_R, \alpha_{22}(x_2))$ satisfying the relations:

$$\left[\frac{1}{s^2(1-\Omega_0)}D^2-1\right]\frac{1}{1-\kappa(x_2)}\left[D^2-s^2(1-\Omega_0\kappa(x_2))\right]\alpha_{22}=0 \quad \text{for} \quad x_2\in(0,\infty), \tag{4.1}$$

$$\begin{cases} \alpha_{22}(0)=\alpha_{22}(\infty)=0 \\ D\left\{\dfrac{\Omega_0}{2-\Omega_0}\dfrac{1}{1-\kappa(x_2)}\left[D^2-s^2(1-\Omega_0\kappa(x_2))\right]-4s^2\dfrac{1-\Omega_0}{2-\Omega_0}\alpha_{22}\right\}\Big|_{\substack{x_2=0\\x_2=\infty}}=0. \end{cases} \tag{4.2}$$

Here

$$\kappa(x_2)=\frac{1-2\nu(x_2)}{2-2\nu(x_2)}, \qquad \Omega(x)=\frac{c_R^2}{\mu_0}, \qquad D=\frac{d}{dx_2}, \tag{4.3}$$

$\nu(x_2)$ and μ_0 are the Poisson's ratio and shear modulus, respectively; symbol $c_R=p/s$, where $2\pi/p$ is the wave period and $2\pi/s$ is the wave length, denotes the velocity of surface wave. The eigenvalue c_R corresponding to the eigenfunction α_{22} is to be identified with the Rayleigh velocity.

Now we consider the case

$$\begin{cases} \kappa=\kappa(x_2)\in C^2[0,\infty), & 0<\kappa_0\le\kappa(x_2)\le\kappa_1<3/4, \\ \mu_0=1, & \Omega(x_2)\equiv\Omega_0\equiv c_R^2. \end{cases} \tag{4.4}$$

These hypotheses assure that the elastic energy of the half-space is strictly positive. We shall look for an eigenfunction $\alpha_{22}\in K$, where

$$K:=\left\{\alpha_{22}=\alpha_{22}(x_2)\in C^4[0,\infty), \qquad \alpha_{22}(\infty)=0\right\}.$$

The system (4.1)–(4.2) subject to the conditions (4.4) is equivalent to

$$\frac{1}{1-\kappa(x_2)}\left[D^2-s^2(1-\Omega_0\kappa(x_2))\right]\alpha_{22} = C_1\exp\left(-s\sqrt{1-\Omega_0}x_2\right) \quad \text{for} \quad x_2\in(0,\infty), \tag{4.5}$$

$$\alpha_{22}(0)=0, \tag{4.6}$$

$$D\left\{\frac{\Omega_0}{1-\kappa(x_2)}\left[D^2-s^2(1-\Omega_0\kappa(x_2))\right]\alpha_{22}-4s^2(1-\Omega_0)\alpha_{22}\right\}\Big|_{x_2=0}=0. \tag{4.7}$$

It is shown in [1] that if there exists a solution of eigenproblem (4.1)–(4.2), the eigen-value $\Omega_0 = c_R^2$ is strictly positive. This fact with (4.5)–(4.7) implies that an admissible Ω_0 belongs to the interval $(0, 1)$. Consider now the homogeneous differential equation corresponding to (4.5):

$$\frac{1}{1-\kappa(x_2)}\left[D^2 - s^2(1-\Omega_0\kappa(x_2))\right]\alpha_{22} = 0 \tag{4.8}$$

which, by virtue of (4.4), is equivalent to

$$\left[D^2 - s^2(1-\Omega_0\kappa(x_2))\right]\alpha_{22} = 0. \tag{4.8'}$$

We have the following theorem

THEOREM 3. *Equation* (4.8) *subject to* (4.4) *has two linearly independent solutions:*

$$\alpha_{22}^{(1)}(x_2, \Omega_0, s), \quad \alpha_{22}^{(2)}(x_2, \Omega_0, s)$$

of the form:

$$\alpha_{22}^{(i)} = \alpha_{22}^{(i)}(0, \Omega_0, s)\exp\int_0^{x_2}\xi_i(\tau, s, \Omega_0)\,d\tau \qquad (i = 1, 2), \tag{4.9}$$

where $\xi_1(\tau, \Omega_0, s)$, $\xi_2(\tau, \Omega_0, s)$ *satisfy the inequalities*

$$a \le \xi_1 \le b < c \le \xi_2 \le d \quad \textit{for every} \quad (\tau, \Omega_0, s) \in (0,\infty)\times(0,1)\times(0,\infty). \tag{4.10}$$

Constants a, b, c and d in (4.10) are defined by

$$\begin{aligned} a &= -s\sqrt{1-\Omega_0\kappa_0}, \qquad & b &= -s\sqrt{1-\Omega_0\kappa_1}, \\ c &= s\sqrt{1-\Omega_0\kappa_1}, \qquad & d &= s\sqrt{1-\Omega_0\kappa_0}. \end{aligned} \tag{4.11}$$

The proof of this theorem is based on a theorem due to OLECH (cf. [21], p. 323) and will not be given here.

It follows from Theorem 3 and the conditions (4.6), (4.7) that an admissible solution of Eq. (4.5) takes the form

$$\alpha_{22}(x_2, \Omega_0, s) = A_1\exp\left(\int_0^{x_2}\xi_i(\tau, s, \Omega_0)\,d\tau\right) - \frac{1}{\Omega_0 s^2}C_1\exp\left(-sx_2\sqrt{1-\Omega_0}\right), \tag{4.12}$$

where $(\Omega_0, s) \in (0,1)\times(0,\infty)$. Clearly, this solution belongs to the class $C^4[0,\infty)$.

Therefore, applying the theorem (cf. [21], p. 56) on analytical dependence on the parameters to the equation

$$\ddot{\alpha}_{22} - s^2(1 - \Omega_0\kappa(x_2))\alpha_{22} = 0 \tag{4.13}$$

subject to the conditions

$$\alpha_{22}(0) = 1, \qquad \alpha_{22} \in K,$$

we conclude that the solution of (4.13) given by $\alpha_{22}^{(0)} = \exp\left(\int\limits_0^{x_2} \xi_1(\tau, s, \Omega_0)\, d\tau\right)$ is analytic with respect to $(\Omega_0, s) \in (0,1) \times (0, \infty)$.

Therefore $\xi_1(\tau, \Omega_0, s)$ is also analytic for $(\Omega_0, s) \in (0,1) \times (0, \infty)$.

It is clear that analyticity of α_{22} satisfying (4.13) subject to $\alpha_{22}(0) = 0$, $\alpha_{22} \in C^4[0, \infty)$ implies analyticity of α_{22} satisfying (4.5)–(4.7). Substituting (4.12) into (4.6) and (4.7), and using condition $C_1 \neq 0$, we arrive at the dispersion equation

$$(2 - \Omega_0)^2 + \frac{4\sqrt{1-\Omega_0}\,\xi_1(0, \Omega_0, s)}{s} = 0. \tag{4.14}$$

Since

$$-s\sqrt{1 - \Omega_0\kappa_0} \le \xi_1(0, \Omega_0, s) \le s\sqrt{1 - \Omega_0\kappa_1}\,,$$

for every $(\Omega_0, s) \in \langle 0, 1) \times (0, \infty)$, thus

$$\begin{aligned} -4\sqrt{(1-\Omega_0)(1-\Omega_0\kappa_0)} + (2-\Omega_0)^2 &\le \frac{4\sqrt{1-\Omega_0}\,\xi_1(0, \Omega_0, s)}{s} + (2-\Omega_0) \\ &\le -4\sqrt{(1-\Omega_0)(1-\Omega_0\kappa_1)} + (2-\Omega_0)^2, \end{aligned} \tag{4.15}$$

for every $(\Omega_0, s) \in \langle 0, 1) \times (0, \infty)$.

Now, introducing the notations

$$\begin{aligned} f_0(\Omega_0) &= -4\sqrt{(1-\Omega_0)(1-\Omega_0\kappa_0)} + (2-\Omega_0)^2, \\ f(\Omega_0, s) &= \frac{4\sqrt{1-\Omega_0}\,\xi_1(0, \Omega_0, s)}{s} + (2-\Omega_0), \\ f_1(\Omega_0) &= -4\sqrt{(1-\Omega_0)(1-\Omega_0\kappa_1)} + (2-\Omega_0)^2, \end{aligned}$$

we reduce (4.15) to the form

$$f_0(\Omega_0) \le f(\Omega_0, s) \le f_1(\Omega_0). \tag{4.16}$$

It follows from the definitions of f_0, f and f_1, and from the analyticity of $\xi_1(0, \Omega_0, s)$ that the functions f_0, f and f_1 are analytic for every $(\Omega_0, s) \in (0, 1) \times$

$(0, \infty)$. Moreover, f_0 and f_1 vanish for $\Omega_0 = 0$ and for $\Omega_0 = c_1^2$, $\Omega_0 = c_2^2$, respectively. c_1^2 and c_2^2 are the squares of velocities of surface waves in the semi-space with $\kappa(x) \equiv \kappa_0$, $\mu \equiv 1$ and $\kappa(x) \equiv \kappa_1$, $\mu \equiv 1$, respectively. Therefore, the analyticity of $f(\Omega_0, s)$ for every $(\Omega_0, s) \in (0,1) \times (0, \infty)$ together with the inequalities (4.16) imply that there exists at least one root (or at most, a countable number of roots) of the equation $f(\Omega_0, s) = 0$ for every $(\Omega_0, s) \in [c_1^2, c_2^2] \times (0, \infty)$. This completes the proof of existence of at least one solution to the eigenproblem discussed in the present section. The Fig. 1 shows the graphs of $f_0(\Omega)$ and $f_1(\Omega)$ corresponding to $\kappa_0 = 0.1$ and $\kappa_1 = 0.7$, respectively, as well as a hypothetical graph of f over the interval $0 < \Omega < 1$.

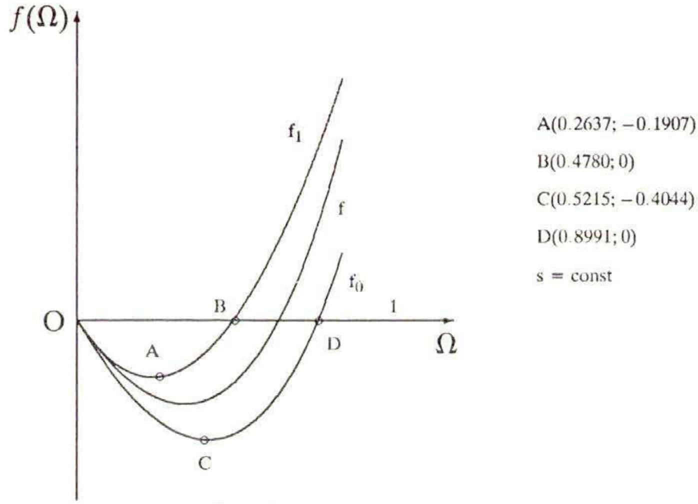

FIG. 1.

We have the following theorem:

THEOREM 4. *For every $s > 0$, the equation $f(\Omega_0, s) = 0$ has at most a finite number of solutions.*

P r o o f. If the number of the solutions of the equation $f(\Omega_0, s) = 0$ for a given $s > 0$ is infinite, then the set $S = \{f(\Omega_0, s) = 0\}$ has an accumulation point in $[c_2^2, c_1^2]$. Since the function $f(\Omega_0, s)$ is analytical in the domain $(\Omega_0, s) \in (0,1) \times (0, \infty)$, f vanishes in the interval $[c_2^2, c_1^2]$ which contradicts the inequality (4.15).

REMARK. If the branches of the dispersion relation (4.14) intersect, then the intersection points are algebraic branch-points (cf. [23] p. 119 part II), (cf. [24] p. 174 – 181).

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