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Friction relations for the many-sphere Oseen hydrodynamic interactions

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THE PAPER concerns weak inertia effects arising in the many-sphere hydrodynamic interactions. Rigid spheres are held fixed in an incompressible fluid flowing with uniform velocity **U** at infinity. The friction relations, up to the contributions of the order $O(\mathrm{Re})$, where Re is the Reynolds number, are considered on the basis of the Oseen equations.

1. Introduction

THE MOTIVATION for this work is to analyse the effects of weak convective inertia on the hydrodynamic interactions of a finite number of spheres, immersed in an incompressible, unbounded fluid. The present paper is a continuation of earlier publications on the low Reynolds number hydrodynamic interactions [1]. The $O(\mathrm{Re})$ convective inertia effects, where Re is the Reynolds number of the sphere (based on the radius a of the spheres, the kinematic viscosity ν of the fluid and the uniform velocity **U** of the fluid at infinity), are considered on the basis of the Oseen equations [2]. In particular, we will consider the $O(\mathrm{Re})$ contributions, appearing in the friction tensors, describing the dependence of the forces $\mathbf{F}_j$ and torques $\mathbf{T}_j$, $j = 1, \ldots, N$, exerted on the spheres by the fluid, on the uniform velocity of the fluid. The study of the friction relations enables an insight into the hydrodynamic interactions between the spheres.

To quote the literature, concerning the study of the uniform flow past a single sphere at low Re, we recall the paper by DENNIS and WALKER [3], and by DENNIS, INGHAM and SINGH [4]. The authors have compared the calculated drag force exerted by the fluid on the sphere, with the results of previous investigations and with the experimental data. In author's opinion, the approach to $\mathrm{Re} \to 0$ is via CHESTER and BREACH [5] drag, rather than via the Oseen drag, as suggested by the experimental results of MAXWORTHY [6]. To calculate the drag force up to terms of the order of $O(\mathrm{Re}^3)$, Chester and Breach used the method of matched asymptotic expansions. Recently, an arbitrary time-dependent motion of a rigid particle in a time-dependent flow of a fluid has been examined by LOVALENTI and BRADY [7]. The authors have calculated the hydrodynamic force acting on the particle, including the contributions up to the terms of $O(\mathrm{Re})$.

Referring to the examination of the small inertia effects appearing in the many-sphere hydrodynamic interactions, we recall the early experimental results of JAYAWERRA, MASON and SLACK [16]. The authors have analysed the behaviour of clusters of spheres, falling in a viscous fluid. Their observations have been discussed in the theoretical paper by HOCKING [17]. He has pointed out that some hydrodynamic phenomena, observed by the authors of the paper [16], are not explicable by the Stokes slow motion hydrodynamics. Subsequently, the influence of small nonlinear effects on the hydrodynamic interactions of spheres has been discussed by HAPPEL and BRENNER [18]. For the particular case of two falling spheres, these effects have been observed experimentally for the cases of $\mathrm{Re} > 0.25$.

Recently, the effects of weak inertia on the motion of particles in a viscous fluid have been reported in a review paper by LEAL [19]. He has argued that even small departures from the Stokes flows can have a strong influence on the positions or orientations of the particle. Problems of the motion of a few particles in the presence of the bounding walls at moderate Re have been treated by means of a numerical package that simulates two-phase Navier–Stokes flows [8]. The authors of that package have taken into account full nonlinearity and the fluid–solid coupling. The papers concerns, however, two-dimensional flows. KIM, ELGHOBASHI and SIRIGNANO [9] have performed a three-dimensional numerical simulation of a steady uniform flow past two fixed spheres, at Re reaching up to 150.

In the present paper we regard small inertial effects appearing in the steady uniform flow past N fixed spheres, at $\mathrm{Re} < 1$. The problem is considered on the basis of the Oseen equations. To deduce the friction relations, the velocity field of the fluid is expressed in terms of the integral equation, involving the Green tensor acting on the forces $\mathbf{f}_j$, distributed on the surfaces of the spheres [1]. The properties of the Green tensor have been recently discussed by GALDI [2]. The dependence of the Green tensor on $|\mathbf{U}|/\nu$ leads to a nonlinear dependence of the hydrodynamic interactions between the particles on the value of Re. However, for the particular case of the hydrodynamic interactions characterized by $\mathrm{Re}_m < 1$, where $\mathrm{Re}_m = R|\mathbf{U}|/\nu$, R – typical distance between the centres of the spheres, we are, qualitatively speaking, close to the Stokes hydrodynamics. For that régime, we confine our considerations to the $O(\mathrm{Re})$ convective effects. The hydrodynamic interactions are presented to be due to the multiple scattering processes. In terms of the multiple scattering events, such properties of the hydrodynamic interactions as non-locality and non-addivity can be conveniently discussed. Knowing the dependence of the hydrodynamic interactions on Re and on the spatial configuration of the particles, we obtain the $O(\mathrm{Re})$-friction relations. These relations describe the convective corrections to the respective Stokes friction relations. The obtained relations have the form of series expansions with respect to σ, where $\sigma = a/R$, $\sigma < 1/2$.

As an example, we consider the drag and side forces exerted on three spheres, placed in the transversal and longitudinal directions with respect to the uniform flow of the fluid at infinity. The dependence of the associated hydrodynamic interactions on σ is taken into account up to the terms of order $O(\sigma)$.

2. Multiple scattering representation of the hydrodynamic interactions

We adopt the idea of induced forces $\mathbf{f}_j$, $j = 1, \dots, N$, distributed on the surfaces of the spheres [10], to describe the presence of the spheres in the flow. The dependence of the induced forces on the uniform velocity of the fluid $\mathbf{U}$ can be expressed in terms of the following set of integral equations [1]:

$$\dot{\mathbf{R}}_j(\Omega_j) = \mathbf{v}^0(\mathbf{R}_j(\Omega_j)) + \int d\Omega_j' \mathbf{T}[\mathbf{R}_j(\Omega_j) - \mathbf{R}_j'(\Omega_j')] \cdot \mathbf{f}_j'(\Omega_j')$$

$$(2.1) \qquad + \sum_{k \neq j}^{N} \int d\Omega_k \mathbf{T}[\mathbf{R}_j(\Omega_j) - \mathbf{R}_k(\Omega_k)] \cdot \mathbf{f}_k(\Omega_k);$$

$$\mathbf{V}_j(\Omega_j) = \dot{\mathbf{R}}_j(\Omega_j) - \mathbf{v}^0(R_j(\Omega_j)) = -\mathbf{U}, \qquad \dot{\mathbf{R}}_j \equiv 0,$$

where $\mathbf{V}_j$ is the relative velocity of the j-th sphere with respect to the fluid, $\mathbf{R}_j$ – position coordinates of the points on the surface of the j-th sphere, $\dot{\mathbf{R}}_j$ – velocity of the j-th sphere, $\mathbf{T}(\mathbf{R}_j - \mathbf{R}_k)$ – Green tensor of the problem considered. The first integral on the r.h.s. accounts for the interaction of the j-th sphere with the fluid, the second integral is due to the hydrodynamic interactions between the spheres.

The convolution form of Eq. (2.1) reflects the non-local character of hydrodynamic interactions. For the present purposes it is convenient to work with the Fourier transform of the Green tensor [11]:

$$(2.2) \qquad \mathbf{T}(\mathbf{r}) = \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{\exp(i\mathbf{k} \cdot \mathbf{r})}{\mu(k^2 + i\nu^{-1}\mathbf{U} \cdot \mathbf{k})} (\mathbf{1} - \widehat{\mathbf{k}}\widehat{\mathbf{k}}),$$

where $\widehat{k} = \mathbf{k}/|\mathbf{k}|$, $k = |\mathbf{k}|$, μ – the dynamic viscosity of the fluid.

The dependence of the Green tensor on $|\mathbf{U}|/\nu$ leads to the nonlinear relations between the induced forces $\mathbf{f}_j$ and Re. These relations can be expressed in terms of admissible sequences of the hydrodynamic interactions. To deduce the multiple scatttering representation of the respective interactions, we follow the procedure used by YOSHIZAKI and YAMAKAWA for the case of the Stokes hydrodynamics [12].

After the two steps:

(i) expansions of $\mathbf{V}_j$, $\mathbf{f}_j$ in terms of the normalized spherical harmonics,

(ii) integrations with respect to the angular variables Ω_j,

we arrive at the set of algebraic equations, relating the expansion coefficients $\mathbf{f}_{j,l_1m_1}$ of the induced forces to the expansion coefficients $\mathbf{V}_{j,l_1m_1}$ of the relative velocities:

$$(2.3) \qquad \mathbf{V}_{j,l_1m_1} = \sum_{l_2m_2} \mathbf{T}_{l_1m_1}^{l_2m_2}(\mathbf{O}_j) \cdot \mathbf{f}_{j,l_2m_2} + \sum_{k \neq j}^{N} \sum_{l_2m_2} \mathbf{T}_{l_1m_1}^{l_2m_2}(\mathbf{R}_{jk}) \cdot \mathbf{f}_{k,l_2m_2},$$

where $\mathbf{r}_j = \mathbf{R}_j - \mathbf{R}_j^0$, $\mathbf{r}_j = \mathbf{r}_j(a, \Omega_j)$, $\mathbf{R}_{jk} = \mathbf{R}_k^0 - \mathbf{R}_j^0$, and R_j^0 – position of the centre of the j-th sphere,

$$\mathbf{V}_{j,lm} = \left\{ \begin{array}{ll} -\mathbf{U}, & l = 0 \\ 0, & l \geq 1 \end{array} \right\}. \tag{2.4}$$

Tensors $\mathbf{T}_{l_1m_1}^{l_2m_2}(\mathbf{O}_j)$ are called the self-interaction tensors, representing the particular type (specified by the indices l_1m_1, l_2m_2) of influence of a single sphere on the surrounding fluid; tensors $\mathbf{T}_{l_1m_1}^{l_2m_2}(\mathbf{R}_{jk})$ denote the mutual interaction tensors, describing the interaction between the j-th and k-th spheres, respectively. Dependence of the above tensors on the spatial configuration of the spheres can be presented in the following form:

$$\mathbf{T}_{l_1m_1}^{l_2m_2}(\mathbf{R}_{jk}) = \sum_{l_3m_3} \mathbf{T}_{l_1m_1,l_3m_3}^{l_2m_2}(|\mathbf{R}_{jk}|) Y_{l_3}^{m_3}(\Omega_{jk}), \tag{2.5}$$

where spherical polar coordinates $\mathbf{R}_{jk}(|\mathbf{R}_{jk}|, \Omega_{jk})$ are used.

Next, we formally solve the basic set of the algebraic equations by iteration,

$$\begin{aligned} \mathbf{f}_{j,l_1m_1} = {} & \sum_{l_2m_2} \widetilde{\mathbf{T}}_{l_1m_1}^{l_2m_2}(\mathbf{O}_j) \cdot \mathbf{V}_{j,l_2m_2} \\ & - \sum_{k \neq j}^{N} \sum_{l_2m_2} \sum_{l_3m_3} \sum_{l_4m_4} \widetilde{\mathbf{T}}_{l_1m_1}^{l_2m_2}(\mathbf{O}_j) \cdot \mathbf{T}_{l_2m_2}^{l_3m_3}(\mathbf{R}_{jk}) \cdot \widetilde{\mathbf{T}}_{l_3m_3}^{l_4m_4}(\mathbf{O}_k) \cdot \mathbf{V}_{k,l_4m_4} + \ldots, \end{aligned} \tag{2.6}$$

where $\widetilde{\mathbf{T}}_{l_1m_1}^{l_2m_2}$ is the inverse self-interaction tensor. The inverse tensors can be expressed by the following approximate formula [1]:

$$\widetilde{\mathbf{T}} = \widetilde{\mathbf{T}}_d - \widetilde{\mathbf{T}}_d \cdot \mathbf{T}_{od} \cdot \widetilde{\mathbf{T}}_d + \widetilde{\mathbf{T}}_d \cdot \mathbf{T}_{od} \cdot \widetilde{\mathbf{T}}_d \cdot \mathbf{T}_{od} \cdot \widetilde{\mathbf{T}}_d - \ldots, \tag{2.7}$$

where $\widetilde{\mathbf{T}}_d$ are diagonal, and $\mathbf{T}_{od}$ – off-diagonal in l (it means, they are of the form $\widetilde{\mathbf{T}}_{l_1m_1}^{l_1m_2}(\mathbf{O}_j)$ and $\widetilde{\mathbf{T}}_{l_1m_1}^{l_2m_2}(\mathbf{O}_j)$, where $l_1 \neq l_2$, respectively). Thus the expansion coefficients of the induced forces are expressed in terms of admissible sequences of the hydrodynamic interactions. These sequences, depending on the properties of the interaction tensors involved, present the allowed types of coupling of the spheres to the fluid.

3. Weak inertial effects

Considering the weak inertial effects, we focus our attention on the low Re properties of the hydrodynamic interaction tensors. In paper [1], the dependence of the tensors on Re is expressed in terms of the modified Bessel functions $I_{l_1+1/2}$

and $K_{l_1+1/2}$. From the properties of the Bessel functions for Re $\to 0$ it follows that the Stokes self-interaction tensors are equal to

$$\mathbf{T}^{l_1 m_2}_{l_1 m_1}(\mathbf{O}_j) = \frac{1}{4\sqrt{\pi}\, a\mu} \frac{1}{(l_1 + 1/2)} \mathbf{K}^{l_1 m_2}_{l_1 m_1, 00}, \tag{3.1}$$

where

$$\mathbf{K}^{l_2 m_2}_{l_1 m_1, l_3 m_3} = i^{l_1 - l_2 - l_3} \int d\widehat{\mathbf{k}} (\mathbf{1} - \widehat{\mathbf{k}}\widehat{\mathbf{k}}) Y^{-m_1}_{l_1} Y^{m_2}_{l_2} Y^{-m_3}_{l_3}.$$

It was shown in the paper [12], that $\mathbf{K}^{l_2 m_2}_{l_1 m_1, l_3 m_3} \neq 0$ for the following sets of indices

$$l_1 + l_2 - l_3 \geq -2, \qquad l_1 + l_2 + l_3 = 2n, \qquad n = 0, 1, 2, \ldots . \tag{3.2}$$

We note that the Stokes self-interaction tensors are diagonal in l.

The Stokes mutual interaction tensors, under the assumption of $\mathrm{Re}_m \to 0$, can be obtained in the following form:

$$\begin{aligned} \mathbf{T}^{l_2 m_2}_{l_1 m_1, l_3 m_3} &= \sum_{m=0}^{\infty} \mathbf{T}^{l_2 m_2, m}_{l_1 m_1, l_3 m_3} = \frac{\sqrt{\pi}}{4a\mu \Gamma(l_1 + 3/2)\Gamma(l_2 + 3/2)} \left(\frac{a}{R_{jk}}\right)^{l_1 + l_2 + 1} \\ &\cdot \mathbf{K}^{l_2 m_2}_{l_1 m_1, l_3 m_3} \sum_{m=0} \frac{(l_1 + l_2 + 2m + 1/2)\Gamma(l_1 + l_2 + m + 1/2)}{m!} \frac{\Gamma(\zeta)}{\Gamma(Z+1)} \\ &\cdot F_4 \left[-m, l_1 + l_2 + m + \frac{1}{2};\ l_1 + \frac{3}{2},\ l_2 + \frac{3}{2};\ \left(\frac{a}{R_{jk}}\right)^2, \left(\frac{a}{R_{jk}}\right)^2 \right], \end{aligned} \tag{3.3}$$

where $|l_1 + l_2 + 2m - l_3| = 0$,

$$Z = \max\left(l_1 + l_2 + 2m + \frac{1}{2},\ l_3 + \frac{1}{2}\right), \qquad \zeta = \min\left(l_1 + l_2 + 2m + \frac{1}{2},\ l_3 + \frac{1}{2}\right),$$

and F_4 is the hypergeometric series.

The allowed values of the respective indices read:

$$\begin{aligned} &\text{(i)} \qquad m = 0, \qquad l_1 + l_2 = l_3, \\ &\text{(ii)} \qquad m = 1, \qquad l_1 + l_2 + 2 = l_3. \end{aligned} \tag{3.4}$$

From (3.3) we obtain the following leading order dependence of the interaction tensors on R_{jk}:

$$\mathbf{T}^{l_2 m_2}_{l_1 m_1, l_3 m_3} \sim \left(\frac{a}{R_{jk}}\right)^{l_1 + l_2 + 1}. \tag{3.5}$$

This property is characteristic for the Stokes flow régime [12].

The next step is to consider the quadratic dependence of $\mathbf{F}_j$ and $\mathbf{T}_j$ on $\mathbf{U}$. To that end, we have to take into account the $O(\mathrm{Re})$ contributions to the interaction tensors. First, we have the self-interaction tensors, being of the leading order of $O(\mathrm{Re})$,

$$\begin{aligned}\mathbf{T}^{l_2m_2}_{l_2+1m_1}(\mathbf{O}_j) &= \frac{\mathrm{Re}}{8\sqrt{3}\sqrt{\pi}\,a\mu(l_2+1/2)(l_2+3/2)}\left[\widehat{U}_z\mathbf{K}^{l_2m_2}_{l_2+1m_1,10}\right.\\ \text{(3.6)}\qquad &\left.+\frac{1}{\sqrt{2}}\left(\widehat{U}_x-i\widehat{U}_y\right)\mathbf{K}^{l_2m_2}_{l_2+1m_1,1-1}+\frac{1}{\sqrt{2}}\left(\widehat{U}_x+i\widehat{U}_y\right)\mathbf{K}^{l_2m_2}_{l_2+1m_1,11}\right]+\ldots,\\ \mathbf{T}^{l_2m_2}_{l_2+1m_1}(\mathbf{O}_j) &= -\mathbf{T}^{l_2+1-m_1}_{l_2-m_2}(\mathbf{O}_j).\end{aligned}$$

They are off-diagonal in l. Then, the $O(\mathrm{Re})$ contributions appear in a series expansion of the tensor $\mathbf{T}^{00}_{00}$ with respect to Re [1].

The mutual interaction tensors, being of the leading order of Re, under the assumption $\mathrm{Re}_m < 1$, read:

$$\begin{aligned}\text{(3.7)}\qquad \mathbf{T}^{l_2m_2}_{l_1m_1,l_3m_3} &= \sum_{m=0}^{\infty}\mathbf{T}^{l_2m_2,m}_{l_1m_1,,l_3m_3} = -\frac{i^{1-l_3}\mathrm{Re}\sqrt{2\pi(2l_3+1)}}{16a\mu\Gamma(l_1+3/2)\Gamma(l_2+3/2)}\beta(-m_3)\\ &\cdot\left\{\sum_{rs} i^r\sqrt{2r+1}\,\mathbf{K}^{l_2m_2}_{l_1m_1,rs}(-1)^s\beta(-s)\begin{pmatrix} l_3 & 1 & r\\ 0 & 0 & 0\end{pmatrix}\left[\sqrt{2}\widehat{U}_z\begin{pmatrix} l_3 & 1 & r\\ -m_3 & 0 & s\end{pmatrix}\right.\right.\\ &\left.\left.-\left(\widehat{U}_x-i\widehat{U}_y\right)\begin{pmatrix} l_3 & 1 & r\\ -m_3 & 1 & s\end{pmatrix}+\left(\widehat{U}_x+i\widehat{U}_y\right)\begin{pmatrix} l_3 & 1 & r\\ -m_3 & -1 & s\end{pmatrix}\right]\right\}\\ &\cdot\left(\frac{a}{R_{jk}}\right)^{l_1+l_2}\sum_{m=0}^{\infty}\frac{(l_1+l_2+2m+1/2)\Gamma(l_1+l_2+m+1/2)}{m!}\frac{\Gamma(\zeta)}{\Gamma(Z+1)}\\ &\cdot F_4\left[-m,l_1+l_2+m+\frac{1}{2};\ l_1+\frac{3}{2},\ l_2+\frac{3}{2};\ \left(\frac{a}{R_{jk}}\right)^2,\ \left(\frac{a}{R_{jk}}\right)^2\right]+\ldots,\end{aligned}$$

where $\begin{pmatrix}\cdots\\ \cdots\end{pmatrix}$ is the Wigner $3-j$ symbol [15],

$$\begin{gathered}|l_1+l_2+2m-l_3| = 1;\\ r = 1 \quad \text{for} \quad l_3 = 0; \quad r = (l_3-1, l_3+1) \quad \text{for} \quad l_3 \geq 1;\\ \beta(s) = (-1)^{3s+|s|)/2}.\end{gathered}$$

In contrast to the Stokes régime, we have here the following sets of admissible indices:

$$\begin{array}{llll} & \text{(i)} & m=0, & l_1+l_2-l_3=1, \quad \text{and} \quad l_1+l_2-l_3=-1,\\ \text{(3.8)} & \text{(ii)} & m=1, & l_1+l_2-l_3=-1, \quad \text{and} \quad l_1+l_2-l_3=-3,\\ & \text{(iii)} & m=2, & l_1+l_2-l_3=-3.\end{array}$$

The above tensors exhibit the following leading order dependence on R_{jk}:

$$\mathbf{T}^{l_2 m_2}_{l_1 m_1, l_3 m_3} \sim \left(\frac{a}{R_{jk}} \right)^{l_1+l_2} . \tag{3.9}$$

We have also the second source of the contributions to the mutual interaction tensors, being of the order of $O(\mathrm{Re})$: a series expansion of the tensor $\mathbf{T}^{00}_{00}(\mathbf{R}_{jk})$ with respect to Re_m [1].

4. Friction relations

In the present section we will examine the friction relations which express the forces and the torques, experienced by the spheres, as quadratic functions of the fluid velocity $\mathbf{U}$. To that end, the forces and torques are presented in terms of the respective expansion coefficients of the induced forces:

$$\begin{aligned} \mathbf{F}_j &= -\mathbf{f}_{j,00} , \\ \mathbf{T}_j &= \boldsymbol{\varepsilon} : \sum_{m=-1}^{1} (\mathbf{r}_j)_{-m} \mathbf{f}_{j,1m} , \end{aligned} \tag{4.1}$$

where

$$((\mathbf{r}_j)_m)_k = \frac{a}{\sqrt{3}} \left[\delta_{k1}(\delta_{m1} + \delta_{m-1}) \frac{1}{\sqrt{2}} + \delta_{k2}(-\delta_{m1} + \delta_{m-1}) \frac{i}{\sqrt{2}} + \delta_{k3}\delta_{m0} \right] .$$

Using the result (2.6) for the expansion coefficients, the friction relations can be written in the following form:

$$\begin{aligned} \mathbf{F}_j &= \sum_{k=1}^{N} \boldsymbol{\xi}^{TV}_{jk} \cdot \mathbf{U}, \\ \mathbf{T}_j &= \sum_{k=1}^{N} \xi^{RV}_{jk} \cdot \mathbf{U}, \end{aligned} \tag{4.2}$$

where $\boldsymbol{\xi}^{\cdots}_{jj}$ denote self-, and $\boldsymbol{\xi}^{\cdots}_{jk}$, $j \neq k$, the mutual friction tensors.

The above relations are of a structure similar to that of the respective Stokes friction relations, the difference consisting in the properties of the friction tensors. The tensors involved can be presented as a sum of the Stokes contributions and the corrections due to weak convection. That sum accounts for the influence of the spatial configuration of the spheres on their hydrodynamic interactions.

The translational self-friction tensors are equal to

$$(4.3)\qquad \boldsymbol{\xi}_{jj}^{TV} = \widetilde{\mathbf{T}}_j + \sum_{l\neq j} \widetilde{\mathbf{T}}_j \cdot \mathbf{T}_{jl} \cdot \widetilde{\mathbf{T}}_l \cdot \mathbf{T}_{lj} \cdot \widetilde{\mathbf{T}}_j + \widetilde{\mathbf{T}}_j^1$$
$$+ \widetilde{\mathbf{T}}_j \cdot \sum_{l\neq j} \left[\mathbf{T}_{jl}^1 \cdot \widetilde{\mathbf{T}}_l \cdot \mathbf{T}_{lj} + \mathbf{T}_{jl} \cdot \widetilde{\mathbf{T}}_l \cdot \mathbf{T}_{lj}^1 \right] \cdot \widetilde{\mathbf{T}}_j + \widetilde{\mathbf{T}}_j^1 \cdot \sum_{l\neq j} \mathbf{T}_{jl} \cdot \widetilde{\mathbf{T}}_l \cdot \mathbf{T}_{lj} \cdot \widetilde{\mathbf{T}}_j$$
$$+ \widetilde{\mathbf{T}}_j \cdot \sum_{l\neq j} \mathbf{T}_{jl} \cdot \left[\widetilde{\mathbf{T}}_l^1 \cdot \mathbf{T}_{lj} \cdot \widetilde{\mathbf{T}}_j + \widetilde{\mathbf{T}}_l \cdot \mathbf{T}_{lj} \cdot \mathbf{T}_j^1 \right]$$
$$- \widetilde{\mathbf{T}}_j \cdot \sum_{l\neq j} \sum_{n\neq l} \sum_{n\neq j} \left[\mathbf{T}_{jl} \cdot \widetilde{\mathbf{T}}_l \cdot \mathbf{T}_{ln}^1 \cdot \widetilde{\mathbf{T}}_n \cdot \mathbf{T}_{nj} \right.$$
$$\left. + \mathbf{T}_{jl}^1 \cdot \widetilde{\mathbf{T}}_l \cdot \mathbf{T}_{ln} \cdot \widetilde{\mathbf{T}}_n \cdot \mathbf{T}_{nj} + \mathbf{T}_{jl} \cdot \widetilde{\mathbf{T}}_l \cdot \mathbf{T}_{ln} \cdot \widetilde{\mathbf{T}}_n \cdot \mathbf{T}_{nj}^1 \right] \cdot \widetilde{\mathbf{T}}_j + \ldots .$$

The Stokes dependence of the friction tensor on σ is described by the first two terms, being of $O(\sigma^0)$ and $O(\sigma^2)$, respectively. The Stokes hydrodynamic interactions between two spheres contribute to the above tensors. The remaining terms express the inertial corrections, being of $O(\sigma^0)$, $O(\sigma^1)$ and $O(\sigma^2)$, respectively. They are due to two and three-sphere interactions. The non-additivity of the interactions appears in the inertial corrections through the terms of order $O(\sigma^2)$. The self- and mutual interaction tensors entering the expression for $\boldsymbol{\xi}_{jj}^{TV}$ are equal to

(i) the self-interaction tensors

$$(4.4)\qquad \begin{aligned} \widetilde{\mathbf{T}}_j &= 6\pi\mu a \mathbf{I}, \\ \widetilde{\mathbf{T}}_j^1 &= 6\pi\mu a \left[\frac{3}{16} \left(3\mathbf{I} - \widehat{\mathbf{U}}\widehat{\mathbf{U}} \right) \right] \mathrm{Re}, \end{aligned}$$

(ii) the mutual interaction tensors:

$$\mathbf{T}_{jk} = \frac{1}{6\pi\mu R_{jk}} \left[\frac{3}{4} (\mathbf{1} + \widehat{\mathbf{e}}_{jk}\widehat{\mathbf{e}}_{jk}) \right], \qquad \widehat{\mathbf{e}}_{jk} = \mathbf{R}_{jk}/|\mathbf{R}_{jk}|,$$

$$(4.5)\qquad \mathbf{T}_{jk}^1 = -\frac{1}{6\pi\mu a} \left[\frac{3}{16} \left(3\mathbf{I} - \widehat{\mathbf{U}}\widehat{\mathbf{U}} \right) + \sqrt{\pi} \frac{3}{4} \sqrt{\frac{1}{5}} \sum_{m=-1}^{1} \varepsilon(-m) \mathbf{L}_1(m) Y_1^m(\Omega_{jk}) \right.$$
$$\left. - \sqrt{\pi} \frac{1}{16} \sqrt{\frac{1}{5}} \sum_{m=-3}^{3} \varepsilon(-m) \mathbf{L}_3(m) Y_3^m(\Omega_{jk}) \right] \mathrm{Re},$$

where

$$\mathbf{L}_q(m) = \sum_{rs} i^r \sqrt{2r+1} \mathbf{K}_s (-1)^s \varepsilon(-s) \begin{pmatrix} q & 1 & r \\ 0 & 0 & 0 \end{pmatrix} \left[\sqrt{2} \widehat{U}_z \begin{pmatrix} q & 1 & r \\ -m & 0 & s \end{pmatrix} \right.$$
$$\left. - \left(\widehat{U}_x - i\widehat{U}_y \right) \begin{pmatrix} q & 1 & r \\ -m & 1 & s \end{pmatrix} + \left(\widehat{U}_x + i\widehat{U}_y \right) \begin{pmatrix} q & 1 & r \\ -m & -1 & s \end{pmatrix} \right],$$

and tensors $\mathbf{K}_s$ are given by YOSHIZAKI, and YAMAKAWA [12]:

$$\begin{aligned}
\mathbf{K}_0 &= \sqrt{\frac{2}{3}}(-\mathbf{e}_x\mathbf{e}_x - \mathbf{e}_y\mathbf{e}_y + 2\mathbf{e}_z\mathbf{e}_z),\\
\mathbf{K}_1 &= \mathbf{e}_x\mathbf{e}_z + \mathbf{e}_z\mathbf{e}_x - i\mathbf{e}_y\mathbf{e}_z - i\mathbf{e}_z\mathbf{e}_y\,,\\
\mathbf{K}_2 &= \mathbf{e}_x\mathbf{e}_x - \mathbf{e}_y\mathbf{e}_y - i\mathbf{e}_x\mathbf{e}_y - i\mathbf{e}_y\mathbf{e}_x\,,\\
\mathbf{K}_{-m} &= \mathbf{K}^*_m\,,
\end{aligned}$$

where the complex conjugate is denoted by an asterisk, and $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ is an external Cartesian coordinate system.

The first two tensors describe, for the particular case of a single sphere, the Stokes and Oseen contributions to the drag force, exerted by the fluid on the sphere.

The translational mutual friction tensors read:

$$\begin{aligned}
(4.6)\qquad \boldsymbol{\xi}^{TV}_{jk} &= -\widetilde{\mathbf{T}}_j\cdot\mathbf{T}_{jk}\cdot\widetilde{\mathbf{T}}_k + \sum_{l\neq k}\sum_{l\neq j}\widetilde{\mathbf{T}}_j\cdot\mathbf{T}_{jl}\cdot\widetilde{\mathbf{T}}_l\cdot\mathbf{T}_{lk}\cdot\widetilde{\mathbf{T}}_k\\
&\quad -\widetilde{\mathbf{T}}_j\cdot\mathbf{T}^1_{jk}\cdot\widetilde{\mathbf{T}}_k - \widetilde{\mathbf{T}}_j\cdot\mathbf{T}_{jk}\cdot\widetilde{\mathbf{T}}^1_k - \widetilde{\mathbf{T}}^1_j\cdot\mathbf{T}_{jk}\cdot\widetilde{\mathbf{T}}_k - \widetilde{\mathbf{T}}_j\cdot\mathbf{T}^2_{jk}\cdot\widetilde{\mathbf{T}}_k\\
&\quad -\sum_m \widetilde{\mathbf{T}}^{1m}_{00}(\mathbf{O}_j)\cdot\mathbf{T}^{00}_{1m}(\mathbf{R}_{jk})\cdot\widetilde{\mathbf{T}}_k - \widetilde{\mathbf{T}}_j\cdot\sum_m\mathbf{T}^{1m}_{00}(\mathbf{R}_{jk})\cdot\widetilde{\mathbf{T}}^{00}_{1m}(\mathbf{O}_k)\\
&\quad +\widetilde{\mathbf{T}}_j\cdot\sum_{l\neq k}\sum_{l\neq j}\left[\mathbf{T}^1_{jl}\cdot\widetilde{\mathbf{T}}_l\cdot\mathbf{T}_{lk} + \mathbf{T}_{jl}\cdot\widetilde{\mathbf{T}}_l\cdot\mathbf{T}^1_{lk}\right]\cdot\widetilde{\mathbf{T}}_k + \widetilde{\mathbf{T}}^1_j\cdot\sum_{l\neq k}\sum_{l\neq j}\mathbf{T}_{jl}\cdot\widetilde{\mathbf{T}}_l\cdot\mathbf{T}_{lk}\cdot\widetilde{\mathbf{T}}_k\\
&\quad +\widetilde{\mathbf{T}}_j\cdot\sum_{l\neq k}\sum_{l\neq j}\mathbf{T}_{jl}\cdot\left[\widetilde{\mathbf{T}}^1_l\cdot\mathbf{T}_{lk}\cdot\widetilde{\mathbf{T}}_k + \widetilde{\mathbf{T}}_l\cdot\mathbf{T}_{lk}\cdot\widetilde{\mathbf{T}}^1_k\right]\\
&\quad -\widetilde{\mathbf{T}}_j\cdot\sum_{l\neq k}\sum_{n\neq l}\sum_{n\neq j}\left[\mathbf{T}_{jl}\cdot\widetilde{\mathbf{T}}_l\cdot\mathbf{T}^1_{ln}\cdot\widetilde{\mathbf{T}}_n\cdot\mathbf{T}_{nk}\right.\\
&\quad \left.+\mathbf{T}^1_{jl}\cdot\widetilde{\mathbf{T}}_l\cdot\mathbf{T}_{ln}\cdot\widetilde{\mathbf{T}}_n\cdot\mathbf{T}_{nk} + \mathbf{T}_{jl}\cdot\widetilde{\mathbf{T}}_l\cdot\widetilde{\mathbf{T}}_{ln}\cdot\widetilde{\mathbf{T}}_n\cdot\mathbf{T}^1_{nk}\right]\cdot\widetilde{\mathbf{T}}_k + \ldots\,.
\end{aligned}$$

The Stokes contributions to the friction tensors are described by the first two terms, being of $O(\sigma)$ and $O(\sigma^2)$, respectively. For that régime, the hydrodynamic interactions of two and three spheres occur. The remaining terms are due to the inertial effects. They contain two, three, and four-sphere contributions. In the Stokes régime, the property of non-additivity appears starting from the terms of $O(\sigma^2)$, whereas in the $O(\mathrm{Re})$ régime, the three-body effect enters at $O(\sigma^1)$.

The mutual friction tensors $\boldsymbol{\xi}^{TV}_{jk}$ are built up of the following interaction tensors:

(i) the self-interaction tensors:

$\widetilde{\mathbf{T}}_j$ and $\widetilde{\mathbf{T}}^1_j$, given by the expressions (4.4),

$$\begin{aligned}
(4.7)\qquad \widetilde{\mathbf{T}}^{1m}_{00}(\mathbf{O}_j) &= \sqrt{6}\pi\mu a\mathrm{Re}\left[\sqrt{2}\widehat{U}_z\delta_{m0} + \left(\widehat{U}_x - i\widehat{U}_y\right)\delta_{m(-1)} + \left(\widehat{U}_x + i\widehat{U}_y\right)\delta_{m(1)}\right]\mathbf{I},\\
\widetilde{\mathbf{T}}^{00}_{1m}(\mathbf{O}_j) &= -\sqrt{6}\pi\mu a\mathrm{Re}\left[\sqrt{2}\widehat{U}_z\delta_{m0} + \left(\widehat{U}_x + i\widehat{U}_y\right)\delta_{m(-1)} + \left(\widehat{U}_x - i\widehat{U}_y\right)\delta_{m(1)}\right]\mathbf{I};
\end{aligned}$$

(ii) the mutual interactions tensors:

$$\mathbf{T}_{jk} \text{ and } \mathbf{T}^1_{jk}, \text{ given by the expressions (4.5)},$$
$$\mathbf{T}^{1m}_{00}(\mathbf{R}_{jk}) = \sum_{m_3} \mathbf{T}^{1m}_{00,1m_3}(|\mathbf{R}_{jk}|) Y_1^{m_3}(\Omega_{jk}) + \sum_{m_3} \mathbf{T}^{1m}_{00,3m_3}(|\mathbf{R}_{jk}|) Y_3^{m_3}(\Omega_{jk}),$$

where

$$\mathbf{T}^{1m}_{00,1m_3} = \frac{1}{3a\mu}\left(\frac{a}{R_{jk}}\right)^2 \mathbf{K}^{1m}_{00,1m_3} + \dots,$$
$$\mathbf{T}^{1m}_{00,3m_3} = \frac{1}{2a\mu}\left(\frac{a}{R_{jk}}\right)^2 \mathbf{K}^{1m}_{00,3m_3} + \dots,$$
$$\mathbf{T}^{00}_{1m}(\mathbf{R}_{jk}) = -\mathbf{T}^{1-m}_{00}(\mathbf{R}_{jk}), \tag{4.8}$$
$$\mathbf{T}^2_{jk} = \frac{\mathrm{Re}}{6\pi\mu a}\left(\frac{a}{R_{jk}}\right)^2 \left[\frac{\sqrt{\pi}}{2\sqrt{5}} \sum_{m=-1}^{1} \varepsilon(-m)\mathbf{L}_1(m) Y_1^m(\Omega_{jk}) - \frac{1}{4}\sqrt{\frac{21}{5}} \sum_{m=-3}^{3} \varepsilon(-m)\mathbf{L}_3(m) Y_3^m(\Omega_{jk})\right].$$

In view of the properties of the considered hydrodynamic interaction tensors, the $O(\mathrm{Re})$ contributions to the friction tensors $\boldsymbol{\xi}^{TV}_{jk}$ do not obey the symmetry relations

$$\left(\boldsymbol{\xi}^{TV}_{jk}\right)_{pq} = \left(\xi^{TV}_{kj}\right)_{qp}, \tag{4.9}$$

characteristic for the Stokes contributions [13].

The rotational self-friction tensors are of the following form:

$$\boldsymbol{\xi}^{RV}_{jj} = -\boldsymbol{\varepsilon}\colon \sum_{m=-1}^{1} (\mathbf{r}_j)_{-m} \left[\widetilde{\mathbf{T}}^{00}_{1m}(\mathbf{O}_j) \cdot \sum_{l\neq j} \mathbf{T}_{jl} \cdot \widetilde{\mathbf{T}}_l \cdot \mathbf{T}_{lj} \cdot \widetilde{\mathbf{T}}_j + \sum_{m_1} \widetilde{\mathbf{T}}^{1m_1}_{1m}(\mathbf{O}_j) \cdot \sum_{l\neq j} \mathbf{T}^{00}_{1m_1}(\mathbf{R}_{jl}) \cdot \widetilde{\mathbf{T}}_l \cdot \mathbf{T}^1_{lj} \cdot \widetilde{\mathbf{T}}_j\right] + \dots, \tag{4.10}$$

where

$$\widetilde{\mathbf{T}}^{1m_1}_{1m}(\mathbf{O}_j) = 6\sqrt{\pi}\mu a \widetilde{\mathbf{K}}^{1m_1}_{1m,00} + \dots .$$

There is no Stokes contribution to the approximation considered. The inertial contributions consist of two terms of order $O(\sigma^2)$, due to two-sphere interactions.

It is seen from the properties of the tensors $\tilde{\mathbf{T}}^{00}_{1m}(\mathbf{O}_j)$ that

$$(4.11) \qquad \boldsymbol{\varepsilon}\colon \sum_{m=-1}^{1} (\mathbf{r}_j)_{-m} \tilde{\mathbf{T}}^{00}_{1m}(\mathbf{O}_j)\cdot\mathbf{V}_{j,00} \equiv 0.$$

Hence we have recovered the well known result that, due to the symmetry of the problem, the torque acting on a single sphere vanishes.

The rotational mutual friction tensors read:

$$(4.12) \qquad \boldsymbol{\xi}^{RV}_{jk} = -\boldsymbol{\varepsilon}\colon \sum_{m=-1}^{1} (\mathbf{r}_j)_{-m} \Bigg[-\sum_{m_1} \tilde{\mathbf{T}}^{1m_1}_{1m}(\mathbf{O}_j)\cdot\mathbf{T}^{00}_{1m_1}(\mathbf{R}_{jk})\cdot\tilde{\mathbf{T}}_k - \tilde{\mathbf{T}}^{00}_{1m}(\mathbf{O}_j)\cdot\mathbf{T}_{jk}\cdot\tilde{\mathbf{T}}_k - \sum_{m_1} \tilde{\mathbf{T}}^{1m_1}_{1m}(\mathbf{O}_j)\cdot\mathbf{T}^{00}_{1m_1}(\mathbf{R}_{jk})\cdot\tilde{\mathbf{T}}^1_k + \tilde{\mathbf{T}}^{00}_{1m}(\mathbf{O}_j)\cdot\sum_{l\neq k}\sum_{l\neq j}\mathbf{T}_{jl}\cdot\tilde{\mathbf{T}}_l\cdot\mathbf{T}_{lk}\cdot\tilde{\mathbf{T}}_k + \sum_{m_1} \tilde{\mathbf{T}}^{1m_1}_{1m}(\mathbf{O}_j)\cdot\sum_{l\neq k}\sum_{l\neq j}\mathbf{T}^{00}_{1m_1}(\mathbf{R}_{jl})\cdot\tilde{\mathbf{T}}_l\cdot\mathbf{T}^1_{lk}\cdot\tilde{\mathbf{T}}_k \Bigg] + \ldots .$$

The Stokes contributions to the friction tensors, due to two-sphere interactions, are given by the first term, being of $O(\sigma^2)$. The remaining four terms describe the convective inertia effects, being of $O(\sigma^1)$ and $O(\sigma^2)$, respectively. They contain two- and three-sphere contributions. The non-additivity comes in at $O(\sigma^2)$.

To conclude: the weak convection effects enhance the hydrodynamic coupling of the spheres to the fluid. In the approximation considered, this enhancement consists in the following effects:

(i) The Stokes interactions involve not more than three spheres, the $O(\mathrm{Re})$ interactions – four spheres;

(ii) the non-additivity effects appear at $O(\sigma^2)$ for the Stokes régime, but at $O(\sigma^1)$ for the Oseen régime;

(iii) the tensors $\boldsymbol{\xi}^{RV}_{jk}$, vanishing for the Stokes flows, occur in the Oseen flows;

(iv) the contributions to $\boldsymbol{\xi}^{TV}_{jk}$, dependent on the angular, but not on the radial variables, absent in the case of Stokes interactions, are generated in case of the Oseen interactions.

5. Three-sphere effects

As an example, the forces acting on three rigidly held spheres are calculated for two particular configurations of the spheres. Consider first three spheres with the centreline perpendicular to the flow direction ($|\mathbf{R}_{12}| = |\mathbf{R}_{23}| = R$, $\mathbf{U} = (O, O, U)$). Up to the terms of the order of $O(\sigma)$, the hydrodynamic interaction tensors required read:

(i) the self-interaction tensors

$$(5.1)\qquad \begin{aligned} \widetilde{\mathbf{T}}_j &= 6\pi\mu a \mathbf{1}, \\ \widetilde{\mathbf{T}}_j^1 &= 6\pi\mu a\left[\frac{3}{16}\mathrm{Re}(3\mathbf{e}_x\mathbf{e}_x + 3\mathbf{e}_y\mathbf{e}_y + 2\mathbf{e}_z\mathbf{e}_z)\right], \end{aligned}$$

(ii) the mutual interaction tensors ($\sigma \ll 1/2$, $\mathrm{Re}_m < 1$)

$$\mathbf{T}_{jk} = \frac{1}{6\pi\mu a}\sigma\frac{3}{4}\left[2\mathbf{e}_x\mathbf{e}_x + \mathbf{e}_y\mathbf{e}_y + \mathbf{e}_z\mathbf{e}_z\right], \quad j,k = 1,2,3,$$

$$(5.2)\qquad \begin{aligned} \mathbf{T}_{12}^1 = \mathbf{T}_{13}^1 = \mathbf{T}_{23}^1 &= -\frac{1}{6\pi\mu a}\frac{3}{16}\mathrm{Re}\left[3\mathbf{e}_x\mathbf{e}_x - \mathbf{e}_x\mathbf{e}_z + 3\mathbf{e}_y\mathbf{e}_y - \mathbf{e}_z\mathbf{e}_x + 2\mathbf{e}_z\mathbf{e}_z\right], \\ \mathbf{T}_{21}^1 = \mathbf{T}_{31}^1 = \mathbf{T}_{32}^1 &= -\frac{1}{6\pi\mu a}\frac{3}{16}\mathrm{Re}\left[3\mathbf{e}_x\mathbf{e}_x + \mathbf{e}_x\mathbf{e}_z + 3\mathbf{e}_y\mathbf{e}_y + \mathbf{e}_z\mathbf{e}_x + 2\mathbf{e}_z\mathbf{e}_z)\right], \end{aligned}$$

in the external Cartesian coordinate system $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$.

The obtained drag forces, exerted by the fluid on the spheres, are given by the following formulae:

(5.3)

(i) for the side spheres:

$$(F_1)_z = (F_3)_z = 6\pi\mu a U\left[1 + \frac{3}{8}\mathrm{Re} - \frac{9}{8}\sigma + \frac{3}{4}\mathrm{Re}\left(1 - \frac{57}{16}\sigma\right) + \ldots\right],$$

(ii) for the central sphere:

$$(F_2)_z = 6\pi\mu a U\left[1 + \frac{3}{8}\mathrm{Re} - \frac{3}{2}\sigma + \frac{3}{4}\mathrm{Re}\left(1 - \frac{33}{8}\sigma\right) + \ldots\right].$$

The inertial contributions to $(F_1)_z$ and $(F_3)_z$ are due to the following types of interactions:

(i) self-interaction of a single sphere: $3/8\mathrm{Re}$,

(ii) pair-wise interactions, independent of R: $3/4\mathrm{Re}$,

(iii) R-dependent pair-wise interactions: $-(108/64)\mathrm{Re}\sigma$,

(iv) non-additive interactions: $-(63/64)\mathrm{Re}\sigma$.

For the vector component $(F_2)_z$, the respective terms are qualitatively similar,

$$\frac{3}{8}\mathrm{Re}, \quad \frac{3}{4}\mathrm{Re}, \quad -\left(\frac{72}{32}\right)\mathrm{Re}\sigma \quad \text{and} \quad -\left(\frac{27}{32}\right)\mathrm{Re}\sigma.$$

In the expression (5.3), the first two terms describe the drag force, experienced by a single sphere; the remaining terms describe the decrease of the drag forces due to the hydrodynamic interactions between the spheres. The vector components $(F_i)_x$, $(F_i)_y$, $i = 1,2,3$, representing the side forces, read

$$(5.4)\qquad \begin{aligned} (F_1)_x &= -(F_3)_x = -6\pi\mu a U\left[\frac{3}{8}\mathrm{Re}\left(1 - \frac{15}{16}\sigma\right) + \ldots\right], \\ (F_2)_x &= 0, \\ (F_i)_y &= 0, \qquad i = 1,2,3. \end{aligned}$$

We note that the two side spheres are repelled. The side forces contain pair-wise contributions equal to $-3/8\mathrm{Re}$ ($3/8\mathrm{Re}$, respectively), and non-additive contributions, equal to $(45/128)\mathrm{Re}\sigma$ ($-(45/128)\mathrm{Re}\sigma$, respectively).

Let us now consider three spheres in line with the flow direction ($|\mathbf{R}_{12}| = |\mathbf{R}_{23}| = R$, $\mathbf{U} = (O, O, U)$).

Here the respective interaction tensors read:

(i) self-interaction tensors are given by the formulae (5.1),

(ii) mutual interaction tensors ($\sigma \ll 1/2$, $\mathrm{Re}_m < 1$):

$$
\begin{aligned}
\mathbf{T}_{jk} &= \frac{1}{6\pi\mu a}\sigma\frac{3}{4}\left[\mathbf{e}_x\mathbf{e}_x + \mathbf{e}_y\mathbf{e}_y + 2\mathbf{e}_z\mathbf{e}_z\right],\\
\mathbf{T}^1_{12} &= \mathbf{T}^1_{13} = \mathbf{T}^1_{23} = -\frac{1}{6\pi\mu a}\frac{3}{8}\mathrm{Re}(3\mathbf{e}_x\mathbf{e}_x + 3\mathbf{e}_y\mathbf{e}_y + 2\mathbf{e}_z\mathbf{e}_z),\\
\mathbf{T}^1_{21} &= \mathbf{T}^1_{31} = \mathbf{T}^1_{32} = 0.
\end{aligned}
\tag{5.5}
$$

The obtained drag forces are given by the formulae,

(5.6)

(i) for the leading sphere:

$$(F_1)_z = 6\pi\mu a U\left[1 + \frac{3}{8}\mathrm{Re} - \frac{9}{4}\sigma + \frac{3}{2}\mathrm{Re}\left(1 - \frac{9}{2}\sigma\right) + \ldots\right],$$

(ii) for the central sphere:

$$(F_2)_z = 6\pi\mu a U\left[1 + \frac{3}{8}\mathrm{Re} - 3\sigma + \frac{3}{4}\mathrm{Re}\left(1 - \frac{33}{4}\sigma\right) + \ldots\right],$$

(iii) for the rear sphere:

$$(F_3)_z = 6\pi\mu a U\left[1 + \frac{3}{8}\mathrm{Re} - \frac{9}{4}\sigma - \frac{63}{16}\mathrm{Re}\sigma + \ldots\right].$$

The inertial contributions to the drag forces, quadratic in the fluid velocity, are generated by:

(i) self-interactions of the respective spheres: $3/8\mathrm{Re}$,

(ii) pair-wise interactions, independent of R: $3/2\mathrm{Re}$, $3/4\mathrm{Re}$, 0, respectively,

(iii) R-dependent pair-wise interactions: $-27/8\mathrm{Re}\sigma$, $-9/2\mathrm{Re}\sigma$, $-27/8\mathrm{Re}\sigma$, respectively,

(iv) non-additive interactions: $-27/8\mathrm{Re}\sigma$, $-27/16\mathrm{Re}\sigma$, $-9/16\mathrm{Re}\sigma$, respectively.

Let us note the differentiation of the drag forces, exerted by the fluid on the spheres. The side forces are equal to zero, due to the symmetry of the considered spatial distribution.

The above examples illustrate the properties of the friction tensors $\boldsymbol{\xi}^{TV}_{jj}$ and $\boldsymbol{\xi}^{TV}_{jk}$ up to $O(\sigma^1)$. We note that the multisphere interactions, giving rise to the drag and side forces, cannot be described in a pair-wise additive scheme. The approximations in the Oseen equations can in principle be refined, by using the results presented in a series of papers by FINN [14], for a particular class of flows.

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